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Research Article

Further Remarks on Fixed-Point Theorems in the Context of Partial Metric Spaces

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New fixed-point theorems on metric spaces are established, and analogous results on partial metric spaces are deduced. This work can be considered as a continuation of the paper Samet et al. (2013).

1. Introduction

In 1994, Matthews [1] introduced the concept of partial metric spaces as a part of the study of denotational semantics of dataflow networks and showed that the Banach's contraction principle can be generalized to the partial metric context for applications in program verification. Later on, many authors studied fixed-point theorems on partial metric spaces (see, e.g., [2–9] and references therein).

We start by recalling some basic definitions and properties of partial metric spaces (see [1, 5] for more details).

Definition 1. A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$, we have

(P1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(P2) $p(x, x) \le p(x, y)$,

(P3) $p(x, y) = p(y, x)$,

(P4) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x, y) = 0$, then from (P1) and (P2), $x = y$; but if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair $([0, +\infty), p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, +\infty)$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [1].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}, \tag{1}$$

for all $x \in X$ and $\varepsilon > 0$.

Definition 2. Let (X, p) be a partial metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to+\infty} p(x, x_n)$. We may write this as $x_n \to x$; $\{x_n\}$ is called a Cauchy sequence if $\lim_{n,m\to+\infty} p(x_n, x_m)$ exists and is finite; (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, such that $p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$.

If p is a partial metric on X, then the function $p^s : X \times X \to [0, +\infty)$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{2}$$

is a metric on X.

Lemma 3. *Let (X, p) be a partial metric space. Then*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s);*

(b) *a partial metric space* (X, p) *is complete if and only if the metric space* (X, p^s) *is complete; furthermore,* $\lim_{n\to+\infty} p^s(x_n, x) = 0$ *if and only if*

$$p(x,x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n,m\to+\infty} p(x_n, x_m). \quad (3)$$

In [1], Matthews extended the Banach contraction principle to the setting of partial metric spaces.

Theorem 4 (see [1]). *Let* (X, p) *be a complete partial metric space and* $T : X \to X$ *a given mapping. Suppose that there exists a constant* $k \in (0,1)$ *such that*

$$p(Tx, Ty) \le kp(x,y), \quad \forall x, y \in X. \quad (4)$$

Then T *has a unique fixed point* $z \in X$*. Moreover, we have* $p(z,z) = 0$.

Very recently, in [10, 11], the authors proved that a large class of fixed-point theorems in partial metric spaces, including Matthews result, are immediate consequences of fixed-point theorems in metric spaces. More precisely, in [11], the authors established the following result.

Theorem 5 (see [11]). *Let* (X, d) *be a complete metric space,* $T : X \to X$*, and* $\varphi : X \to [0, +\infty)$ *a lower semicontinuous function. Suppose that there exists* $k \in (0,1)$ *such that*

$$\begin{aligned} &d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \\ &\quad \le k(d(x,y) + \varphi(x) + \varphi(y)), \end{aligned} \quad (5)$$

for all $x, y \in X$*. Then* T *has a unique fixed point* $z \in X$*. Moreover, one has* $\varphi(z) = 0$.

Now, taking $\varphi(x) = p(x,x)$ in Theorem 4, we obtain that

$$\begin{aligned} &p^s(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \\ &\quad \le k(p^s(x,y) + \varphi(x) + \varphi(y)), \end{aligned} \quad (6)$$

for all $x, y \in X$. Applying Theorem 5, we obtain immediately the result of Matthews. Another technique suggested by Haghi et al. [10] let D be a metric on X defined by

$$D(x,y) := \begin{cases} p(x,y) & \text{if } x \ne y, \\ 0 & \text{if } x = y. \end{cases} \quad (7)$$

It is not difficult to show that if (X, p) is complete, then (X, D) is complete. Now, Matthews' contraction is equivalent to

$$D(Tx, Ty) \le kD(x,y), \quad (8)$$

for all $x, y \in X$. So, the result of Matthews can be deduced immediately from the Banach contraction principle. Observe that in Theorem 5, if we consider the mapping $D_\varphi : X \times X \to [0, +\infty)$ defined by

$$D_\varphi(x,y) := \begin{cases} d(x,y) + \varphi(x) + \varphi(y) & \text{if } x \ne y, \\ 0 & \text{if } x = y, \end{cases} \quad (9)$$

then D_φ is a metric on X and (X, D_φ) is a complete metric space. Moreover, the contractive condition in Theorem 5 is equivalent to

$$D_\varphi(Tx, Ty) \le kD_\varphi(x,y), \quad (10)$$

for all $x, y \in X$. So, Theorem 5 can also be deduced from the Banach contraction principle.

Define the function $F : [0, +\infty)^3 \to [0, +\infty)$ by

$$F(a,b,c) := a + b + c, \quad \forall a, b, c \ge 0. \quad (11)$$

Then, the contractive condition in Theorem 5 is equivalent to

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \le kF(d(x,y), \varphi(x), \varphi(y)), \quad (12)$$

for all $x, y \in X$. In this paper, we establish new fixed-point theorems in metric spaces involving a function $F : [0, +\infty)^3 \to [0, +\infty)$, where the mapping

$$D_\varphi(x,y) := \begin{cases} F(d(x,y), \varphi(x), \varphi(y)) & \text{if } x \ne y, \\ 0 & \text{if } x = y \end{cases} \quad (13)$$

is not a metric on X. Some fixed-point theorems in partial metric spaces are deduced from our main results in metric spaces.

2. Main Result

We denote by $\mathcal{F}$ the set of functions $F : [0, +\infty)^3 \to [0, +\infty)$ satisfying the following conditions:

(i) $\max\{a, b\} \le F(a,b,c)$ for all $a, b, c \in [0, +\infty)$,

(ii) $F(0,0,0) = 0$,

(iii) F is continuous.

As examples, the following functions belong to $\mathcal{F}$:

(a) $F(a,b,c) = (a+b)(c+1)$,

(b) $F(a,b,c) = a(\ln(c+1)+1) + b(e^c + 1)$,

(c) $F(a,b,c) = a + b + c$,

(d) $F(a,b,c) = \max\{a,b,c\}$,

(e) $F(a,b,c) = a + \max\{a,b,c\}$.

We denote by Ψ the set of functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(j) ψ is nondecreasing,

(jj) $\sum_{n\ge1} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

It is not difficult to show that if $\psi \in \Psi$, then $\psi(t) < t$ for every $t > 0$. These functions are known in the literature as (c)-comparison functions.

Definition 6. Let (X, d) be a metric space, $T : X \to X$, $\varphi : X \to [0, +\infty)$, and $F \in \mathcal{F}$. We say that the pair (T, φ) satisfies an F-generalized ψ-contraction if there exists $\psi \in \Psi$ such that

$$\begin{aligned} &F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \\ &\quad \leq \psi\Big(\max\Big\{F(d(x, y), \varphi(x), \varphi(y)), \\ &\qquad (F(d(x, Tx), \varphi(Tx), \varphi(x)) \\ &\qquad + F(d(y, Ty), \varphi(Ty), \varphi(y))) \times 2^{-1}\Big\}\Big), \end{aligned} \tag{14}$$

for every $x, y \in X$.

Our main result is giving by the following theorem.

Theorem 7. *Let (X, d) be a complete metric space, $T : X \to X$, and $\varphi : X \to [0, +\infty)$. Suppose that the following conditions hold:*

(1) *φ is lower semicontinuous,*

(2) *there exist $\psi \in \Psi$ and $F \in \mathcal{F}$ such that the pair (T, φ) satisfies an F-generalized ψ-contraction.*

Then T has a unique fixed point $z \in X$. Moreover, one has $\varphi(z) = 0$.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for all positive integer n. Let n be a positive integer such that $n \geq 1$. Using condition (2) with $x = x_n$ and $y = x_{n-1}$, we obtain that

$$\begin{aligned} &F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) \\ &\quad \leq \psi(\max\{F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})), \\ &\qquad F(d(x_n, x_{n+1}), \varphi(x_{n+1}), \varphi(x_n))\}). \end{aligned} \tag{15}$$

Without restriction of the generality, we can suppose that $x_{m+1} \neq x_m$ for every m. Suppose that

$$\begin{aligned} &\max\{F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})), \\ &\qquad F(d(x_n, x_{n+1}), \varphi(x_{n+1}), \varphi(x_n))\} \\ &\quad = F(d(x_n, x_{n+1}), \varphi(x_{n+1}), \varphi(x_n)). \end{aligned} \tag{16}$$

In this case, we have

$$\begin{aligned} &F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) \\ &\quad \leq \psi(F(d(x_n, x_{n+1}), \varphi(x_{n+1}), \varphi(x_n))). \end{aligned} \tag{17}$$

Note that $F(d(x_n, x_{n+1}), \varphi(x_{n+1}), \varphi(x_n)) > 0$. Indeed, if $F(d(x_n, x_{n+1}), \varphi(x_{n+1}), \varphi(x_n)) = 0$, from condition (i), we have $d(x_n, x_{n+1}) \leq F(d(x_n, x_{n+1}), \varphi(x_{n+1}), \varphi(x_n)) = 0$, which implies that $x_n = x_{n+1}$, that is a contradiction with the assumption $x_{m+1} \neq x_m$ for every m. Since $\psi(t) < t$ for every $t > 0$, we get that

$$\begin{aligned} &F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) \\ &\quad < F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)), \end{aligned} \tag{18}$$

which is a contradiction. Then, we deduce that

$$\begin{aligned} &F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) \\ &\quad \leq \psi(F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))), \end{aligned} \tag{19}$$

for every $n \geq 1$. Using the property (i) of F and the monotony property of ψ, we obtain that

$$\begin{aligned} &\max\{\varphi(x_{n+1}), d(x_{n+1}, x_n)\} \\ &\quad \leq \psi^n(F(d(x_1, x_0), \varphi(x_1), \varphi(x_0))), \end{aligned} \tag{20}$$

for every $n \geq 1$. Fix $\varepsilon > 0$, and let $h = h(\varepsilon)$ be a positive integer (given by (jj)) such that

$$\sum_{n \geq h} \psi^n(F(d(x_1, x_0), \varphi(x_1), \varphi(x_0))) < \varepsilon. \tag{21}$$

Let $m > n > h$; using the triangular inequality and (20), we obtain

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{n \geq h} \psi^n(F(d(x_1, x_0), \varphi(x_1), \varphi(x_0))) < \varepsilon. \end{aligned} \tag{22}$$

Thus we proved that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d). Since (X, d) is complete, there exists some $z \in X$ such that $x_n \to z$ as $n \to \infty$. We shall prove that z is a fixed point of T. At first, observe that from condition (jj) and (20), we have

$$\lim_{n \to +\infty} \varphi(x_n) = 0, \tag{23}$$

which implies (since φ is lower semicontinuous) that

$$\varphi(z) = 0. \tag{24}$$

Using condition (2) with $x = z$ and $y = x_n$, we get that

$$\begin{aligned} &F(d(Tz, x_{n+1}), \varphi(Tz), \varphi(x_{n+1})) \\ &\quad \leq \psi\Big(\max\Big\{F(d(z, x_n), \varphi(z), \varphi(x_n)), \\ &\qquad (F(d(z, Tz), \varphi(Tz), \varphi(z)) \\ &\qquad + F(d(x_n, x_{n+1}), \varphi(x_{n+1}), \varphi(x_n))) \times 2^{-1}\Big\}\Big), \end{aligned} \tag{25}$$

for every n. Suppose that $z \neq Tz$. Using the continuity of F, (23), and (24), we have

$$\begin{aligned} &\lim_{n \to +\infty} \max\Big\{F(d(z, x_n), \varphi(z), \varphi(x_n)), \\ &\qquad (F(d(z, Tz), \varphi(Tz), \varphi(z)) \\ &\qquad + F(d(x_n, x_{n+1}), \varphi(x_{n+1}), \varphi(x_n))) \times 2^{-1}\Big\} \\ &\quad = \frac{F(d(z, Tz), \varphi(Tz), 0)}{2} > 0. \end{aligned} \tag{26}$$

Then, there exists a positive integer N such that

$$\max\{F(d(z,x_n),\varphi(z),\varphi(x_n)),\\ (F(d(z,Tz),\varphi(Tz),\varphi(z))\\ +F(d(x_n,x_{n+1}),\varphi(x_{n+1}),\varphi(x_n)))\times 2^{-1}\}>0, \tag{27}$$

for every $n\geq N$. Thus, we have

$$F(d(Tz,x_{n+1}),\varphi(Tz),\varphi(x_{n+1}))\\ <\max\{F(d(z,x_n),\varphi(z),\varphi(x_n)),\\ (F(d(z,Tz),\varphi(Tz),\varphi(z))\\ +F(d(x_n,x_{n+1}),\varphi(x_{n+1}),\varphi(x_n)))\times 2^{-1}\}. \tag{28}$$

Letting $n\to+\infty$ in the above inequality, we get that

$$F(d(Tz,z),\varphi(Tz),0)\leq\frac{F(d(z,Tz),\varphi(Tz),0)}{2}, \tag{29}$$

which is a contradiction. Thus, we proved that $Tz=z$. Now, suppose that $z'\in X$ is another fixed point of T. We can observe that

$$\varphi(z')=0. \tag{30}$$

Indeed, applying (14) with $x=y=z'$, we get that

$$F(0,\varphi(z'),\varphi(z'))\leq\psi(F(0,\varphi(z'),\varphi(z'))), \tag{31}$$

which implies (30). Now, applying (14) with $x=z$ and $y=z'$, we obtain

$$F(d(z,z'),\varphi(z),\varphi(z'))\\ \leq\psi\left(\max\left\{F(d(z,z'),\varphi(z),\varphi(z')),\\ \frac{F(0,\varphi(z),\varphi(z))+F(0,\varphi(z'),\varphi(z'))}{2}\right\}\right). \tag{32}$$

Using (24) and (30), we obtain

$$F(d(z,z'),0,0)\leq\psi(F(d(z,z'),0,0)), \tag{33}$$

which implies that $d(z,z')=0$; that is, $z=z'$. Thus, we proved that T has a unique fixed point $z\in X$ with $\varphi(z)=0$. □

Example 8. Let $T:[0,1]\to[0,1]$ be the mapping defined by

$$Tx=\begin{cases}\dfrac{x}{4} & \text{if } 0\leq x<1,\\ \dfrac{1}{2} & \text{if } x=1.\end{cases} \tag{34}$$

We endow $X=[0,1]$ with the standard metric $d(x,y)=|x-y|$. Let $\varphi(x)=x$ for all $x\in X$, $\psi(t)=(3/4)t$ for all $t\geq 0$, and $F(a,b,c)=a+b+c$ for all $a,b,c\geq 0$. Clearly, φ is continuous, $\psi\in\Psi$, and $F\in\mathcal{F}$. Moreover, we have

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty))\\ \leq\psi(F(d(x,y),\varphi(x),\varphi(y))), \tag{35}$$

for all $x,y\in X$. By Theorem 7, T has a unique fixed point $z\in X$ $(z=0)$ with $\varphi(z)=0$. Note that in this example, the Banach contraction principle cannot be used since the mapping T is not continuous.

3. Particular Cases

In this section, new fixed point results are deduced from our main result given by Theorem 7.

Corollary 9. *Let (X,d) be a complete metric space, $T:X\to X$, and $\varphi:X\to[0,+\infty)$. Suppose that the following conditions hold:*

(1) *φ is lower semicontinuous,*

(2) *there exists $\psi\in\Psi$ such that*

$$d(Tx,Ty)\leq(\psi((d(x,y)+\varphi(x))(\varphi(y)+1))\\ -\varphi(Tx)(\varphi(Ty)+1))\\ \times(\varphi(Ty)+1)^{-1}, \tag{36}$$

for all $x,y\in X$.

Then T has a unique fixed point $z\in X$. Moreover, we have $\varphi(z)=0$.

Proof. It follows from Theorem 7 with $F(a,b,c)=(a+b)(c+1)$. □

Corollary 10. *Let (X,d) be a complete metric space, $T:X\to X$, and $\varphi:X\to[0,+\infty)$. Suppose that the following conditions hold:*

(1) *φ is lower semicontinuous,*

(2) *there exists $\psi\in\Psi$ such that*

$$d(Tx,Ty)\\ \leq\left(\psi\left(\frac{(d(x,Tx)+\varphi(Tx))(\varphi(x)+1)}{2}\right)\\ -\varphi(Tx)(\varphi(Ty)+1)\right)\\ \times(\varphi(Ty)+1)^{-1}, \tag{37}$$

for all $x,y\in X$.

Then T has a unique fixed point $z\in X$. Moreover, we have $\varphi(z)=0$.

Proof. It follows from Theorem 7 with $F(a,b,c) = (a+b)(c+1)$. □

Corollary 11. *Let (X,d) be a complete metric space, $T : X \to X$, and $\varphi : X \to [0,+\infty)$. Suppose that the following conditions hold:*

(1) *φ is lower semicontinuous,*

(2) *there exists $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \left(\psi\left(\frac{(d(y,Ty)+\varphi(Ty))(\varphi(y)+1)}{2}\right) - \varphi(Tx)(\varphi(Ty)+1)\right) \times (\varphi(Ty)+1)^{-1}, \quad (38)$$

for all $x, y \in X$.

Then T has a unique fixed point $z \in X$. Moreover, we have $\varphi(z) = 0$.

Proof. It follows from Theorem 7 with $F(a,b,c) = (a+b)(c+1)$. □

Corollary 12. *Let (X,d) be a complete metric space, $T : X \to X$, and $\varphi : X \to [0,+\infty)$. Suppose that the following conditions hold:*

(1) *φ is lower semicontinuous,*

(2) *there exists $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \left(\psi\left(d(x,y)\left[\ln(\varphi(y)+1)+1\right] + \varphi(x)\left(e^{\varphi(y)}+1\right)\right) - \varphi(Tx)\left(e^{\varphi(Ty)}+1\right)\right) \times \left(\ln(\varphi(Ty)+1)+1\right)^{-1}, \quad (39)$$

for all $x, y \in X$.

Then T has a unique fixed point $z \in X$. Moreover, we have $\varphi(z) = 0$.

Proof. It follows from Theorem 7 with $F(a,b,c) = a(\ln(c+1)+1) + b(e^c+1)$. □

Corollary 13. *Let (X,d) be a complete metric space, $T : X \to X$, and $\varphi : X \to [0,+\infty)$. Suppose that the following conditions hold:*

(1) *φ is lower semicontinuous,*

(2) *there exists $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \left(\psi\left(\frac{d(x,Tx)\left[\ln(\varphi(x)+1)+1\right] + \varphi(Tx)\left(e^{\varphi(x)}+1\right)}{2}\right) - \varphi(Tx)\left(e^{\varphi(Ty)}+1\right)\right) \times \left(\ln(\varphi(Ty)+1)+1\right)^{-1}, \quad (40)$$

for all $x, y \in X$.

Then T has a unique fixed point $z \in X$. Moreover, we have $\varphi(z) = 0$.

Proof. It follows from Theorem 7 with $F(a,b,c) = a(\ln(c+1)+1) + b(e^c+1)$. □

Corollary 14. *Let (X,d) be a complete metric space, $T : X \to X$, and $\varphi : X \to [0,+\infty)$. Suppose that the following conditions hold:*

(1) *φ is lower semicontinuous,*

(2) *there exists $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \left(\psi\left(\frac{d(y,Ty)\left[\ln(\varphi(y)+1)+1\right] + \varphi(Ty)\left(e^{\varphi(y)}+1\right)}{2}\right) - \varphi(Tx)\left(e^{\varphi(Ty)}+1\right)\right) \times \left(\ln(\varphi(Ty)+1)+1\right)^{-1}, \quad (41)$$

for all $x, y \in X$.

Then T has a unique fixed point $z \in X$. Moreover, we have $\varphi(z) = 0$.

Proof. It follows from Theorem 7 with $F(a,b,c) = a(\ln(c+1)+1) + b(e^c+1)$. □

Many other results can be deduced from Theorem 7 by considering different choices of F.

4. Applications to Partial Metric Spaces

We will show that the following fixed-point theorem in partial metric spaces can be deduced from Theorem 7.

Corollary 15. *Let (X, p) be a complete partial metric space, and let $T : X \to X$. Suppose that there exists $\psi \in \Psi$ such that*

$$p(Tx, Ty) \le \psi\left(\max\left\{p(x, y), \frac{p(x, Tx) + p(y, Ty)}{2}\right\}\right), \tag{42}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$. Moreover, one has $p(z, z) = 0$.

Proof. Let $d(x, y) := p^s(x, y)/2$ for all $x, y \in X$ and $\varphi(x) = p(x, x)/2$ for all $x \in X$. We have

$$p(x, y) = F(d(x, y), \varphi(x), \varphi(y)), \quad \forall x, y \in X, \tag{43}$$

where

$$F(a, b, c) = a + b + c, \quad \forall a, b, c \ge 0. \tag{44}$$

Now, inequality (42) is equivalent to

$$\begin{aligned} &F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \\ &\le \psi\big(\max\big\{F(d(x, y), \varphi(x), \varphi(y)), \\ &\qquad (F(d(x, Tx), \varphi(Tx), \varphi(x)) \\ &\qquad\quad + F(d(y, Ty), \varphi(Ty), \varphi(y))) \times 2^{-1}\big\}\big), \end{aligned} \tag{45}$$

for every $x, y \in X$. Applying Theorem 7, we obtain that T has a unique fixed point $z \in X$ with $\varphi(z) = 0$; that is, $p(z, z) = 0$. □

The following results follow immediately from Corollary 15.

Corollary 16. *Let (X, p) be a complete partial metric space, and let $T : X \to X$. Suppose that there exists $\psi \in \Psi$ such that*

$$p(Tx, Ty) \le \psi(p(x, y)), \tag{46}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$. Moreover, one has $p(z, z) = 0$.

Corollary 17. *Let (X, p) be a complete partial metric space, and let $T : X \to X$. Suppose that there exists $\psi \in \Psi$ such that*

$$p(Tx, Ty) \le \psi\left(\frac{p(x, Tx) + p(y, Ty)}{2}\right), \tag{47}$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$. Moreover, one has $p(z, z) = 0$.

Note that Matthews result (see Theorem 4) follows from Corollary 16 by taking $\psi(t) = kt$, where $k \in (0, 1)$.

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