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Research Article

H^1-Random Attractors and Asymptotic Smoothing Effect of Solutions for Stochastic Boussinesq Equations with Fluctuating Dynamical Boundary Conditions

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This work is concerned with the random dynamics of two-dimensional stochastic Boussinesq system with dynamical boundary condition. The white noises affect the system through a dynamical boundary condition. Using a method based on the theory of omega-limit compactness of a random dynamical system, we prove that the L^2-random attractor for the generated random dynamical system is exactly the H^1-random attractor. This improves a recent conclusion derived by Brune et al. on the existence of the L^2-random attractor for the same system.

1. Introduction

The Boussinesq equations are a coupled system of the Navier-Stokes equations and the scalar transport equation for fluid salinity, temperature, or density. These Boussinesq equations models various phenomena in environmental, geophysical and climate system, for example, oceanic density currents and the thermohaline circulation; see, for example, [1–3]. In this paper, the scalar quantity in the considered Boussinesq system is salinity, with dynamical boundary condition for the salinity.

Let $D \subset \mathbb{R}^2$ be a bounded domain with the C^1-smooth boundary $\partial D = \Gamma$, in the vertical plane. Let $W_0(t), W_1(t)$, and $W_2(t)$ be independent two-sided real-valued Wiener processes with values in appropriate function spaces. This paper is mainly concerned with the long time behavior of the solutions to the following Boussinesq equations with general additive noises and fluctuating dynamical boundary conditions [4]:

$$\begin{aligned} \frac{du}{dt} &= \left(\frac{1}{\mathrm{Re}}\Delta u - u\cdot\nabla u - \nabla p - \frac{1}{\mathrm{Fr}^2}Ue_2\right) + \dot{W}_0 \quad \text{on } D\times\mathbb{R}^+, \\ \operatorname{div} u &= 0 \quad \text{on } D\times\mathbb{R}^+, \\ u &= 0 \quad \text{on } \Gamma\times\mathbb{R}^+, \\ u(0) &= u_0, \\ \frac{dU}{dt} &= \left(\frac{1}{\mathrm{RePr}}\Delta U - u\cdot\nabla U\right) + \dot{W}_1 \quad \text{on } D\times\mathbb{R}^+, \\ \frac{dU_\Gamma}{dt} &= \left(\frac{-\partial_n U_\Gamma - cU_\Gamma + f(x)}{\alpha}\right) + \dot{W}_2 \quad \text{on } \Gamma\times\mathbb{R}^+, \\ \gamma U &= U_\Gamma, \\ U(0) &= U_0, \end{aligned} \tag{1}$$

with velocity $u = u(x,t) = (u_1(x,t), u_2(x,t)) \in \mathbb{R}^2$, salinity $U = U(x,t) \in \mathbb{R}$, and pressure $p = p(x,t) \in \mathbb{R}$. In (1), Fr is the Froude number; Re is the Reynolds number; and Pr is the Prandtl number. Δ is the Laplacian operator; ∇ is the gradient operator; div is the divergence operator; γ is the trace operator with respect to the boundary Γ. $e_2 \in \mathbb{R}^2$ is a unit vector in the upward vertical direction (opposite to the gravity). Finally, $f(x), x \in \Gamma$ is a given function describing the mean salinity flux through the boundary; u_0 and U_0 are the initial conditions; $\partial_n U_\Gamma$ is the outer normal derivative.

Without loss of generality, in this paper we take Fr, Re, Pr, c and α to be 1.

Qualitatively, the Boussinesq system (1) emphasizes the random dynamical boundary condition which modes the interaction of boundary and the domain. In particular, if U describes the temperature then the heat exchange between physical domain and its boundary can be modeled; see, for example, [3]. For this Boussinesq system (1), the large deviation principle via a weak convergence approach is studied in [5], recently. In [4], they derived a priori estimates for the existence of random absorbing sets and showed that the random dynamical system (RDS) generated by the solution of (1) had a random attractor in L^2 space. When the random dynamical boundary condition in (1) is replaced by the nonhomogeneous boundary condition, [6, 7] proved the existences of random attractors for this system with multiplicative noise and additive noises in L^2 space, respectively. For the study of the random attractor about other models integrated the Navier-Stokes equations, we refer to [8] for the MHD equations.

In recent years, the theory of random attractors for some concrete dissipative stochastic partial differential equations has been studied by many authors; see, for example, [6, 8–12]; ever since [13, 14] launched their fundamental work on the RDS. Such an attractor, which generalizes nontrivially the global attractors well developed in [15–18] and so forth, is a compact invariant random set which attracts every orbit in the phase space. It is uniquely determined by attracting deterministic compact sets of phase space [19]. In order to obtain the existence of random attractors, one always need to show the existence of a compact random absorbing set in the sense of absorption [20]. This can be achieved by employing the standard Sobolev compact embedding of several functional spaces, when the generated RDSs are defined in some bounded domains; see, for example, [6–10, 20] and references cited there.

In this paper, we are interested in the existence of random attractors of the Boussinesq system (1) in H^1 space which is stronger than L^2 space. It is pointed out that if the initial data belongs to L^2, the solution to the system (1) enters into the space $L^2 \cap H^1$ and has no higher regularity, see [4]. Hence the Sobolev compact embedding cannot be employed in H^1. Here, we try to overcome the obstacle of compact imbedding by using the notion of omega-limits compactness, which was initiated in [12, 21] in the framework of RDS. This type of compactness is equivalent to the asymptotic compactness [22, 23] in some spaces and can be proved by check the flattening condition; see [21]. More precisely, we prove the existence of the (L^2, H^1)-random attractor for this RDS (for the bispaces random attractors; the reader is referred to [11, 16–18]). To this end, a so qualitatively new method is necessary, for instance, to derive a priori estimates of the solutions such that the flattening condition holds in H^1 space and then the necessary omega-limits compactness for the RDS in H^1 space is followed; see Lemma 12 in Section 4. The main advantage of this technique is that we need not to estimate the solutions in functional spaces of higher regularity to demonstrate the existence of compact random absorbing set which does not work in this case. This method has been used recently to obtain the existence of random attractors in H^1 space for the stochastic reaction-diffusion equations [24, 25].

The conclusion in this study shows that the L^2-random attractor is qualitatively an H^1-random attractor. This implies that the solutions to (1) become eventually more smoothing than the initial data.

The outline of this paper is as follows. Section 2 presents some functional settings for the Boussinesq system. Section 3 lists the conditions and the main conclusion of this note. In Section 4, we prove some estimates for the solution orbits in $\mathbb{H}$ and $\mathbb{V}$ and then prove our main conclusion.

2. Functional Settings

We recall some function spaces and operators that we will be used in the following discussion.

Let

$$L^2 := \left(L^2(D)\right)^2 \times L^2(D) \times L^2(\Gamma), \tag{2}$$

endowed with the scalar inner product $(\cdot,\cdot) = (\cdot,\cdot)_{(L^2(D))^2} + (\cdot,\cdot)_L^2(D) + (\cdot,\cdot)_L^2(\Gamma)$ and with norm denoted by $\|\cdot\|$. This notation is also used to denote the norm in $(L^2(D))^2$ and $L^2(D)$ without any confusion.

We define a functional space $\mathscr{V}$ integrated the boundary and also the divergence free condition:

$$\mathscr{V} = \left\{(u, U, U_\Gamma) \in \left(C^\infty(D)\right)^2 \times C^\infty(D) \times C^\infty(\Gamma);\ \operatorname{div} u = 0\right\}. \tag{3}$$

Define

$$H_\Gamma^1 = \left(H_0^1(D)\right)^2 \times H^1(D) \times H^{1/2}(\Gamma), \tag{4}$$

where $(H_0^1(D))^2$ is the usual Sobolev space with equivalent norm $\|\nabla\cdot\|$ and $H^1(D)$ is also the usual Sobolev space with the equivalent norm:

$$\|U\|_{H^1} = \left(\|\nabla U\|_{H^1}^2 + \|U\|_{L^2(\Gamma)}^2\right)^{1/2}, \quad U \in H^1(D); \tag{5}$$

see [15, page 52], and $H^{1/2}(\Gamma)$ is given by $\gamma(H^1(D))$ endowed by a norm $\|\phi\|_{H^{1/2}} = \inf_{\gamma U=\phi}\|U\|_{H^1}$; see [15, page 48].

Let $\mathbb{H}$ the closure of $\mathscr{V}$ in L^2 and $\mathbb{V}$ be the closure of $\mathscr{V}$ in H_Γ^1, and the norm in $\mathscr{V}$ being denoted by $\|\cdot\|_{\mathbb{V}}$. From the above argument, the space $\mathbb{V}$ is equipped with the norm:

$$\|\mathbf{U}\|_{\mathbb{V}} = \left(\|\nabla u\|^2 + \|\nabla U\|^2 + \|U\|_{L^2(\Gamma)}^2\right)^{1/2}, \quad \mathbf{U} = (u, U, U_\Gamma) \in \mathbb{V}, \tag{6}$$

which is equivalent to the natural norm: $(\|\nabla u\|^2 + \|U\|_{H^1(D)}^2 + \|U\|_{H^{1/2}(\Gamma)}^2)^{1/2}$.

For $\mathbf{U} = (u, U, U_\Gamma)$, $\mathbf{V} = (v, V, V_\Gamma)$, we define the operators

$$\begin{aligned} A_1 u &= -\Delta u, \\ A_2 (U, U_\Gamma) &= (-\Delta U, \partial_n U_\Gamma + U_\Gamma), \\ B_1 (u, v) &= u \cdot \nabla v, \\ B_2 (u, V) &= u \cdot \nabla V, \\ A\mathbf{U} &= (A_1 u, A_2 (U, U_\Gamma)), \\ B(\mathbf{U}, \mathbf{V}) &= (B_1 (u, v), B_2 (u, V), 0), \end{aligned} \tag{7}$$

$$F(\mathbf{U}) = (-Ue_2, 0, f(x)), \quad f \in L^2(\Gamma). \tag{8}$$

By Lemma 2.2 in [4], the operator A is a positive self-adjoint unbounded operator and has the Poincaré inequality:

$$(A\mathbf{U}, \mathbf{U}) \geq \lambda \|\mathbf{U}\|^2, \quad \lambda > 0. \tag{9}$$

Then A^{-1} is also self-adjoint but compact operator in $\mathbb{H}$, so we can utilize the elementary spectral theory in a Hilbert space. We infer that there exists a compete orthonormal family in $\mathbb{H}$, also in $\mathbb{V}$, $\{\mathbf{e}_j\}_{j=1}^{\infty}$ of eigenvectors of A. The corresponding spectrum of A is discrete and denoted by $\{\lambda_j\}_{j=1}^{\infty}$ which are positive, increasing, and tend to infinity as $j \to \infty$.

In particular, we also can use the spectrum theory to allow us to define the operator A^s, the power of A. For $s > 0$, the operator A^s is also an strictly positive and self-adjoint unbounded operator in $\mathbb{H}$ with a dense domain $D(A^s) \subset \mathbb{H}$. This allows us to introduce the function spaces,

$$D(A^s) = \left\{ \mathbf{V} = \sum_{j=1}^{\infty} (\mathbf{V}, \mathbf{e}_j) \mathbf{e}_j : \|\mathbf{V}\|_{D(A^s)}^2 = \sum_{j=1}^{\infty} (\mathbf{V}, \mathbf{e}_j)^2 \lambda_j^{2s} < +\infty \right\}. \tag{10}$$

This norm $\|\cdot\|_{D(A^s)}$ on $D(A^s)$ is equivalent to the usual norm induced by H^{2s}; see Temam [15] for details. In particular, $D(A^0) = \mathbb{H}$ and $D(A^{1/2}) = \mathbb{V}$.

Based on the orthonormal basis $\{\mathbf{e}_j\}_{j=1}^{\infty}$ of eigenfunctions of A, we define the m-dimensional subspace $\mathbb{V}_m = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\} \subset \mathbb{V}$ and the canonical orthogonal projection $P_m : \mathbb{V} \mapsto \mathbb{V}_m$ such that for every $\mathbf{V} \in \mathbb{V}$, $\mathbf{V}$ has a unique decomposition: $\mathbf{V} = P_m \mathbf{V} + \mathbf{V}_m$, where

$$\begin{aligned} P_m \mathbf{V} &= \sum_{j=1}^{m} (\mathbf{V}, \mathbf{e}_j) \mathbf{e}_j \in \mathbb{V}_m, \\ \mathbf{V}_m &= (I - P_m) \mathbf{V} = \sum_{j=m+1}^{\infty} (\mathbf{V}, \mathbf{e}_j) \mathbf{e}_j \in \mathbb{V}_m^{\perp}; \end{aligned} \tag{11}$$

that is, $\mathbb{V} = \mathbb{V}_m \oplus \mathbb{V}_m^{\perp}$.

When the projection P_m operates on the first component of $\mathbf{V} = (v, V, V_\Gamma)$, one can easily show that

$$\begin{gathered} \|A_1 P_m v\|^2 \leq \lambda_{m+1} \|A_1^{1/2} P_m v\|^2, \\ \forall \mathbf{V} = (v, V, V_\Gamma) \in D(A) \end{gathered} \tag{12}$$

for $m \in N^+$, the nature numbers set.

We also state the well-known Brezis-Gallouet's inequality with Dirichlet boundary condition in two dimension case; see [26] or Proposition 3 in [27]; there exists a positive constant c such that

$$\begin{gathered} \|v\|_{(L_\infty(D))^2} \leq c \|A_1^{1/2} v\| \left(1 + \log \frac{\|A_1 v\|^2}{\lambda_1 \|A_1^{1/2} v\|^2} \right)^{1/2} \\ \forall \mathbf{V} = (v, V, V_\Gamma) \in D(A), \end{gathered} \tag{13}$$

where λ_1 is the first eigenvalue of the Stokes operator A_1. By (12) we have

$$\begin{aligned} \|P_m v\|_{(L_\infty(D))^2} &\leq c \|A_1^{1/2} P_m v\| \left(1 + \log \frac{\|A_1 P_m v\|^2}{\lambda_1 \|A_1^{1/2} P_m v\|^2} \right)^{1/2} \\ &\leq c \|v\|_{H^1} \left(1 + \log \frac{\lambda_{m+1}}{\lambda_1} \right)^{1/2}, \\ &\qquad \forall \mathbf{V} = (v, V, V_\Gamma) \in D(A). \end{aligned} \tag{14}$$

According to the above notations, we can write (1) as the following abstract evolution equation form:

$$\begin{gathered} d\mathbf{U} + A\mathbf{U} dt + B(\mathbf{U}, \mathbf{U}) dt = F(\mathbf{U}) dt + d\mathbf{W}, \\ \mathbf{U}(x, 0) = \mathbf{U}_0(\omega) \in \mathbb{H}, \end{gathered} \tag{15}$$

where $\mathbf{U} = (u, U, U_\Gamma)$.

3. Existence of Random Attractor in $\mathbb{V}$

In order to model the noise in the initial problem (1), we need to define a metric dynamical system (MDS) which is a group of measure preserving transformations on a probability space. For the definition of the MDS we refer to [28, 29] and so forth.

A standard model for a spatially correlated noise is the generalized time derivative of a two-sidedBrownian motion $\omega = \omega(x, t)$, $x \in \mathbb{R}^2$. Let $\mathbb{H}$ be the separable Hilbert space defined in Section 2. As usual, we introduce the spatially valued Brownian motion MDS $\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$, where $\Omega = \{\omega \in C_0(\mathbb{R}, \mathbb{H}) : \omega(0) = 0\}$ with compact open topology. This topology is metrizable by the complete metric:

$$d(\omega_1, \omega_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(\omega_1, \omega_2)}{1 + d_n(\omega_1, \omega_2)}, \tag{16}$$

where $d_n(\omega_1, \omega_2) = \max_{|t| \leq n} |\omega_1 - \omega_2|$ for ω_1 and ω_2 in Ω. $\mathcal{F} = \mathcal{B}(C_0(\mathbb{R}, \mathbb{H}))$ is the Borel-σ-algebra induced by the compact open topology of Ω. Suppose the Wiener process ω has covariance operator Q. Let $\mathbb{P}$ be the Wiener measure with respect to Q. The Wiener shift is defined by

$$\theta_s \omega(t) = \omega(t + s) - \omega(s), \quad \omega \in \Omega, \ t, s \in \mathbb{R}. \tag{17}$$

Then the measure $\mathbb{P}$ is ergodic and invariant with respect to the shift θ.

The associated probability space defines a canonical Wiener process W. We also note that such a Wiener process W generates a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$:

$$\mathcal{F}_t \equiv \{W(\tau) \mid \tau \le t\} \subset \mathcal{F}. \quad (18)$$

We introduce the following stochastic partial differential equation on D:

$$d\mathbf{Z} + A\mathbf{Z}dt = dW. \quad (19)$$

Because A is a positive and self-adjoint operator, there exists a mild solution to this stochastic equation with the form

$$\mathbf{Z}(t) = \mathbf{Z}(0) + \int_0^t e^{-(t-\tau)A} dW, \quad t > 0, \quad (20)$$

which is called an Ornstein-Uhlenbeck process; see [30]. For the Ornstein-Uhlenbeck process we have the regularity hypothesis; see also [4].

Lemma 1. *Suppose that the covariance operator Q of the Wiener process ω has a finite trace; that is, Q satisfies that*

$$\operatorname{tr}_{\mathbb{H}}\left(QA^{2s-1+\delta}\right) = \operatorname{tr}_{\mathbb{H}}\left(A^{s-1/2+\delta/2}QA^{s-1/2+\delta/2}\right) < +\infty, \quad (21)$$

for some $s \ge 0$ and some (arbitrary small) $\delta > 0$, where $\operatorname{tr}_{\mathbb{H}}$ denotes the trace of the covariance operator. Then an $\mathcal{F}_0$-measurable Gaussian variable $\mathbf{Z} = (z, Z) \in D(A^s)$ exists, and the process $(t,\omega) \to \mathbf{Z}(\theta_t\omega)$ is a continuous stationary solution to the stochastic equation (19)*. Furthermore, the random variable $\|\mathbf{Z}(\omega)\|^2_{D(A^s)}$ is tempered and the expectation*

$$\mathbb{E}\|\mathbf{Z}\|^2_{D(A^s)} = \frac{1}{2}\operatorname{tr}_{\mathscr{H}}\left(A^{s-1/2}QA^{s-1/2}\right) < +\infty. \quad (22)$$

Let $\mathbf{V} = \mathbf{U} - \mathbf{Z}(\theta_t\omega)$; by (15) and (19) we have the following deterministic equation with a parameter ω:

$$\begin{aligned} &\frac{d}{dt}\mathbf{V} + A\mathbf{V} + B\left(\mathbf{V} + \mathbf{Z}(\theta_t\omega), \mathbf{V} + \mathbf{Z}(\theta_t\omega)\right) \\ &\quad = F\left(\mathbf{V} + \mathbf{Z}(\theta_t\omega)\right), \\ &\mathbf{V}(x,0) = \mathbf{V}_0(\omega) = \mathbf{U}_0(\omega) - \mathbf{Z}(\omega) \in \mathbb{H}. \end{aligned} \quad (23)$$

Here we denote the solution to (23) by $\mathbf{V}(t,\omega,\mathbf{V}_0(\omega))$, $\mathbf{V}(t,\omega)$ or more briefly $\mathbf{V}(t)$. By Lemma 4.7 in [4], the solution of the evolution equation (23) generates a continuous measurable RDS ψ in $\mathbb{H}$ given by $\psi(t,\omega)\mathbf{V}_0(\omega) = \mathbf{V}(t,\omega,\mathbf{V}_0(\omega))$. Put $\mathbf{U}(t,\omega,\mathbf{U}_0(\omega)) = \mathbf{V}(t,\omega,\mathbf{U}_0(\omega) - \mathbf{Z}(\omega)) + \mathbf{Z}(\theta_t\omega)$. Then $\mathbf{U}(t,\omega,\mathbf{U}_0(\omega))$ or briefly $\mathbf{U}(t)$ is a solution to (23) with initial value $\mathbf{U}_0(\omega)$.

Given

$$\begin{aligned} \varphi(t,\omega)\mathbf{U}_0(\omega) &= \mathbf{U}\left(t,\omega,\mathbf{U}_0(\omega)\right) \\ &= \mathbf{V}\left(t,\omega,\mathbf{U}_0(\omega) - \mathbf{Z}(\omega)\right) \\ &\quad + \mathbf{Z}(\theta_t\omega), \quad \omega \in \Omega, \end{aligned} \quad (24)$$

then φ is also a continuous measurable RDS in $\mathbb{H}$ for the original equation (15), that is, (1).

As for the general theory of random dynamical systems we may refer to [28, 31] for details.

The main conclusion of this study states the following.

Theorem 2. *One supposes that* (21) *holds. Set that*

$$M(\omega) = K^{\epsilon}\|Z(\omega)\|^2_{H^2} + K^{\epsilon}\|z(\omega)\|^2_{H^1} - \lambda, \quad (25)$$

where K^{ϵ} is a generic constant depending on the data of the problem and some ϵ that has to be chosen sufficiently small and λ is the same as in (9)*. Assume that the mathematical expectation of M*

$$\mathbb{E}M < 0. \quad (26)$$

Then the RDS φ generated by (1) *admits a unique random attractor $\{\mathscr{A}_{\mathbb{V}}(\omega)\}_{\omega\in\Omega}$ in $\mathbb{V}$ in the sense that for $\mathbb{P}$-a.s.$\omega \in \Omega$,*

$$\mathscr{A}_{\mathbb{V}}(\omega) \text{ is compact in } \mathbb{V}, \quad (27)$$

$$\varphi(t,\omega)\mathscr{A}_{\mathbb{V}}(\omega) = \mathscr{A}_{\mathbb{V}}(\theta_t\omega), \quad \text{for every } t \ge 0, \quad (28)$$

$$\begin{aligned} &\lim_{t\to\infty} \operatorname{dist}_{\mathbb{V}}\left(\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega), \mathscr{A}_{\mathbb{V}}(\omega)\right) = 0, \\ &\qquad \forall\{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}, \end{aligned} \quad (29)$$

where $\operatorname{dist}_{\mathbb{V}}$ denotes the Hausdorff semiistance in $\mathbb{V}$ and $\mathscr{D}$ is the collection of tempered subsets of $\mathbb{H}$ as in [4]. Furthermore, $\{\mathscr{A}_{\mathbb{V}}(\omega)\}_{\omega\in\Omega}$ is identical with the random attractor $\{\mathscr{A}_{\mathbb{H}}(\omega)\}_{\omega\in\Omega}$ in $\mathbb{H}$.

By the well-known abstract result of Theorem 3.5 in [20], the existence of a random attractor $\{\mathscr{A}_{\mathbb{H}}(\omega)\}_{\omega\in\Omega}$ in the following sense that

$$\begin{aligned} &\mathscr{A}_{\mathbb{H}}(\omega) \text{ is compact in } \mathbb{H}, \\ &\varphi(t,\omega)\mathscr{A}_{\mathbb{H}}(\omega) = \mathscr{A}_{\mathbb{H}}(\theta_t\omega), \quad \text{for every } t \ge 0, \\ &\lim_{t\to\infty} \operatorname{dist}_{\mathbb{H}}\left(\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega), \mathscr{A}_{\mathbb{H}}(\omega)\right) = 0 \\ &\qquad \text{for all } \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}, \end{aligned} \quad (30)$$

has been obtained in [4]. However, Our Theorem 2 shows the existence of random attractor which is compact in the space $\mathbb{V}$, which is stronger than $\mathbb{H}$. Thus the Sobolev compact embedding theorem is not available and it seems impossible to obtain a compact random absorbing set in the space $\mathbb{V}$ when the initial data belongs to $\mathbb{H}$. Then Theorem 3.5 in [20] is unapplicable to the proof of our Theorem 2.

Fortunately, Theorem 2 can be proved by checking the omega-limit compactness in the space $\mathbb{V}$, which is based on the viewpoint of Kuratowski measure of noncompactness. This kind of compactness can be easily obtained by showing the flattening condition, see [21].

For clarification, we state some general concepts used in the sequel. Let X be a Banach space with norm $\|\cdot\|_X$ and $\mathscr{D}$ be the collection of all random subsets of X.

Definition 3 (see [21]). An RDS φ on X over an MDS is said to be omega-limit compact if for every $\varepsilon > 0$ and $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$, there exists $T = T(\varepsilon, B, \omega)$ such that for all $t \ge T$,

$$k\left(\bigcup_{t\ge T}\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)\right) \le \varepsilon, \quad \mathbb{P}\text{-a.s.}\omega \in \Omega, \tag{31}$$

where $k(B)$ is the Kuratowski measure of noncompactness of B defined by

$$k(B) = \inf\{d > 0 : B \text{ admits a finite cover by sets of diameter} \le d\}. \tag{32}$$

Definition 4 (see [21]). An RDS φ on X over an MDS is said to possess the flattening condition if for $\mathbb{P}$-a.s.$\omega \in \Omega$ and every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$, there exist $T = T(B,\omega,\varepsilon) > 0$ and a finite dimensional space X_1 of X such that for a bounded projector $P : X \to X_1$,

$$\begin{gathered} P\left(\bigcup_{t\ge T}\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)\right) \text{ is bounded in } X, \\ \left\|(I-P)\left(\bigcup_{t\ge T}\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)\right)\right\|_X \le \varepsilon. \end{gathered} \tag{33}$$

Definition 5 (see [29]). (i) A random variable $X \in \mathbb{R}^+$ over an MDS is tempered if for $\mathbb{P}$-a.s.$\omega \in \Omega$,

$$\lim_{t\to\pm\infty}\frac{\log^+ X(\theta_t\omega)}{|t|} = 0. \tag{34}$$

(ii) A random set $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ is called tempered if $R(\omega) = \sup_{x\in B(\omega)}\|x\|_X$ is a tempered random variable.

Definition 6. A random set $\{K(\omega)\}_{\omega\in\Omega}$ is an $\mathscr{D}$-random absorbing set for RDS φ over an MDS if for $\mathbb{P}$-a.s.$\omega \in \Omega$ and every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$, there exists $T = T(B,\omega) > 0$ such that for all $t \ge T$,

$$\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega) \subset K(\omega), \tag{35}$$

where $\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega) = \bigcup_{x\in B(\theta_{-t}\omega)}\varphi(t,\theta_{-t}\omega)x$.

For our problem, $X = \mathbb{V}$ and $\mathscr{D}$ is the collection of tempered subsets of $\mathbb{H}$. The finite dimensional space $X_1 = \mathbb{V}_m = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ and the bounded operator $P = P_m$ for a sufficient large m, where $\mathbf{e}_i$ $(i = 1, 2, \ldots, m)$ and P_m are defined in Section 2.

4. The Proofs of Main Result

To prove Theorem 2, we need a series of lemmas. First we give a useful lemma similar to the classic Gronwall lemma (see [15]).

Lemma 7. *Assume that y, y', and h are three locally integrable functions and y, h nonnegative on $[a,\infty)$ such that for $s \ge a$,*

$$y'(s) + by(s) \le h(s). \tag{36}$$

Then for every $t \ge a$ and any positive constant r, one has

$$y(t+r) \le e^{-br}\left(\int_t^{t+r} y(s)e^{b(s-t)}ds + \int_t^{t+r} h(s)e^{b(s-t)}ds\right). \tag{37}$$

Proof. Let $t \le s \le t+r$. We multiply (36) by $e^{b(s-t)}$ and the resulting inequality reads

$$\frac{d}{ds}\left(y(s)e^{b(s-t)}\right) \le h(s)e^{b(s-t)}. \tag{38}$$

Hence, by integration from s to $t+r$,

$$y(t+r)e^{br} \le y(s)e^{b(s-t)} + \int_t^{t+r} h(s)e^{b(s-t)}ds. \tag{39}$$

Integrating the above inequality with respect to s between t and $t+r$, we get the desired inequality. □

Lemma 8. *There exist positive constants K and K^ϵ such that the followings hold:*

$$\|\mathbf{V}(t)\|^2 \le e^{\int_0^t M(\theta_\tau\omega)d\tau}K^\epsilon\|\mathbf{V}_0(\omega)\|^2 + K^\epsilon\int_0^t G(\theta_s\omega)e^{\int_s^t M(\theta_\tau\omega)d\tau}ds, \tag{40}$$

$$\begin{aligned} \frac{d}{dt}\|\mathbf{V}\|^2 + K\|\mathbf{V}\|_{\mathbb{V}}^2 &\le K^\epsilon\left(\|Z(\theta_t\omega)\|_{H^2}^2 + \|z(\theta_t\omega)\|_{H^1}^2\right)\|v\|^2 \\ &\quad + K^\epsilon\|V\|^2 + G(\theta_t\omega), \end{aligned} \tag{41}$$

$$\frac{d}{dt}\|\mathbf{V}\|_{\mathbb{V}}^2 \le g(t,\omega)\|\mathbf{V}\|_{\mathbb{V}}^2 + h(t,\omega), \tag{42}$$

where

$$\begin{aligned} M(\omega) &= K^\epsilon\|Z(\omega)\|_{H^2}^2 + K^\epsilon\|z(\omega)\|_{H^1}^2 - \lambda, \\ G(\omega) &= K^\epsilon + K^\epsilon\|z(\omega)\|^2\|Z(\omega)\|_{H^2}^2 \\ &\quad + K^\epsilon\|z(\omega)\|_{H^1}^2\|z(\omega)\|_{H^2}^2 + K^\epsilon\|Z(\omega)\|_{H^2}^2, \end{aligned} \tag{43}$$

$$\begin{aligned} g(t,\omega) &= K\|v(t,\omega)\|^2\|v(t,\omega)\|_{H^1}^2 + H_1(\theta_t\omega), \\ h(t,\omega) &= K\|V(t,\omega)\|^2 + H_2(\theta_t\omega), \\ H_1(\omega) &= K^\epsilon\|z(\omega)\|_{H^2}^2 + K^\epsilon\|Z(\omega)\|_{H^2}^2, \\ H_2(\omega) &= K^\epsilon\|z(\omega)\|_{H^2}^2\|Z(\omega)\|_{H^1}^2 \\ &\quad + K^\epsilon\|z(\omega)\|_{H^1}^2\|z(\omega)\|_{H^2}^2 + K\|Z(\omega)\|^2 + K^\epsilon. \end{aligned} \tag{44}$$

Proof. The inequality (40) is the same as the inequality (18) at page 1112 in [4]. The inequality (41) is a combination of the formulas (15) and (16) at pages 1110-1111 in [4] with a tiny modification. Following a same calculation as page 1114 in [4] we can obtain (42). □

Lemma 9. *Assume that $\mathbb{E}M < 0$. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$. Then for $\mathbb{P}$-a.s.$\omega \in \Omega$, there exist random radii $\varrho_1(\omega)$ and $\varrho_1(\omega)$ and constant $T = T(B,\omega) > 0$ such that for all $t \geq T$, the solution $\mathbf{V}(t,\omega,\mathbf{U}_0(\omega)-\mathbf{Z}(\omega))$ of problem (23) with $\mathbf{U}_0 \in B(\omega)$ satisfies that for every $l \in [t, t+1]$:*

$$\left\|\mathbf{V}\left(l,\theta_{-t-1}\omega,\mathbf{U}_0(\omega)-\mathbf{Z}(\omega)\right)\right\| \leq \varrho_1(\omega), \tag{45}$$

$$\int_t^{t+1}\left\|\mathbf{V}\left(s,\theta_{-t-1}\omega,\mathbf{U}_0(\omega)-\mathbf{Z}(\omega)\right)\right\|_{\mathbb{V}}^2 ds \leq \varrho_2(\omega), \tag{46}$$

where $\mathscr{D}$ is the collection of tempered random subsets of $\mathbb{H}$.

Proof. We replace t with l, and ω with $\theta_{-t-1}\omega$ in (40) to produce that for $l \in [t, t+1]$,

$$\begin{aligned}
&\left\|\mathbf{V}\left(l,\theta_{-t-1}\omega,\mathbf{U}_0\left(\theta_{-t-1}\omega\right)-\mathbf{Z}\left(\theta_{-t-1}\omega\right)\right)\right\|^2\\
&\leq e^{\int_0^l M(\theta_{\tau-t-1}\omega)d\tau}K^\epsilon\left\|\mathbf{V}_0(\theta_{-t-1}\omega)\right\|^2\\
&\quad+K^\epsilon\int_0^l G\left(\theta_{s-t-1}\omega\right)e^{\int_s^l M(\theta_{\tau-t-1}\omega)d\tau}ds\\
&= e^{\int_{-t-1}^{l-t-1} M(\theta_\tau\omega)d\tau}K^\epsilon\left\|\mathbf{V}_0\left(\theta_{-t-1}\omega\right)\right\|^2\\
&\quad+K^\epsilon\int_{-t-1}^{l-t-1} G\left(\theta_s\omega\right)e^{\int_s^{l-t-1} M(\theta_\tau\omega)d\tau}ds\\
&\leq e^{\int_{-t-1}^{l-t-1} M(\theta_\tau\omega)d\tau}K^\epsilon\\
&\quad\times\left(\left\|\mathbf{V}_0\left(\theta_{-t-1}\omega\right)\right\|^2+\left\|\mathbf{Z}_0\left(\theta_{-t-1}\omega\right)\right\|^2\right)\\
&\quad+K^\epsilon\int_{-t-1}^0 G\left(\theta_s\omega\right)e^{\int_s^{l-t-1} Q(\theta_\tau\omega)d\tau}ds.
\end{aligned} \tag{47}$$

By noting that for $l \in [t,t+1]$, $l-t-1 \in [-1,0]$, then we find that for $s \in [-t-1, 0]$,

$$\begin{aligned}
&e^{\int_s^{l-t-1} M(\theta_\tau\omega)d\tau}\\
&= e^{-\lambda_1(l-t-1)+\lambda_1 s}\\
&\quad\times e^{\int_s^{l-t-1}(K^\epsilon\|Z(\theta_\tau\omega)\|_{H^2}^2+K^\epsilon\|z(\theta_\tau\omega)\|_{H^1}^2)d\tau}\\
&\leq e^{\lambda_1+\lambda_1 s}e^{\int_s^0(K^\epsilon\|Z(\theta_\tau\omega)\|_{H^2}^2+K^\epsilon\|z(\theta_\tau\omega)\|_{H^1}^2)d\tau}\\
&\leq e^{\lambda_1}e^{\int_s^0 M(\theta_\tau\omega)d\tau},
\end{aligned} \tag{48}$$

from which and (47) it follows that for every $l \in [t,t+1]$,

$$\begin{aligned}
&\left\|\mathbf{V}\left(l,\theta_{-t-1}\omega,\mathbf{V}_0\left(\theta_{-t-1}\omega\right)-\mathbf{Z}\left(\theta_{-t-1}\omega\right)\right)\right\|^2\\
&\leq e^{\int_{-t-1}^0 M(\theta_s\omega)ds}K^\epsilon\left(\left\|\mathbf{V}_0\left(\theta_{-t-1}\omega\right)\right\|^2+\left\|\mathbf{Z}\left(\theta_{-t-1}\omega\right)\right\|^2\right)\\
&\quad+K^\epsilon\int_{-t-1}^0 G\left(\theta_s\omega\right)e^{\int_s^0 M(\theta_\tau\omega)d\tau}ds.
\end{aligned} \tag{49}$$

By the Birkhoff's ergodic theorem and along with our assumption (26), it yields that

$$\lim_{t\to\pm\infty}\frac{1}{t+1}\int_{-t-1}^0 M\left(\theta_\tau\omega\right)d\tau = \mathbb{E}M(\omega) < 0, \tag{50}$$

and then it implies that

$$e^{\int_{-t-1}^0 M(\theta_\tau\omega)d\tau} \approx e^{(t+1)\mathbb{E}M} \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \tag{51}$$

As $G(\theta_\tau)$ is subexponential growth, so

$$\int_{-\infty}^0 G\left(\theta_s\omega\right)e^{\int_s^0 M(\theta_\tau\omega)d\tau}ds < +\infty. \tag{52}$$

Note that $\|\mathbf{Z}(\omega)\|^2$ is also tempered random variable and $\mathbf{U}_0(\omega) \in B(\omega)$ with $B \in \mathscr{D}$, whence there exists constant $T = T(B,\omega) > 0$ such that for all $t \geq T$ with $l \in [t,t+1]$,

$$\begin{aligned}
&\left\|\mathbf{V}\left(l,\theta_{-t-1}\omega,\mathbf{U}_0\left(\theta_{-t-1}\omega\right)-\mathbf{Z}\left(\theta_{-t-1}\omega\right)\right)\right\|^2\\
&\leq \varrho_1^2(\omega) := K^\epsilon\left(1+\int_{-\infty}^0 G\left(\theta_s\omega\right)e^{\int_s^0 M(\theta_\tau\omega)d\tau}ds\right).
\end{aligned} \tag{53}$$

Furthermore, $\varrho_1(\omega)$ is tempered; see [4]. This completes the proof of (45).

Next, we show (46). Integrating (41) from t to $t+1$ with $t \geq T$, where T is in (53), we get

$$\begin{aligned}
&K\int_t^{t+1}\left\|\mathbf{V}(s,\omega,\mathbf{U}_0(\omega)-\mathbf{Z}(\omega))\right\|_{\mathbb{V}}^2 ds\\
&\leq K^\epsilon\int_t^{t+1}\left(\left\|Z\left(\theta_s\omega\right)\right\|_{H^2}^2+\left\|z\left(\theta_s\omega\right)\right\|_{H^1}^2\right)\\
&\qquad\times\left\|v\left(s,\omega,v_0(\omega)-z(\omega)\right)\right\|^2 ds\\
&\quad+K^\epsilon\int_t^{t+1}\left\|V\left(s,\omega,V_0(\omega)-Z(\omega)\right)\right\|^2 ds\\
&\quad+\int_t^{t+1}G\left(\theta_s\omega\right)ds+K\left\|\mathbf{V}\left(t,\omega,V_0(\omega)-Z(\omega)\right)\right\|^2.
\end{aligned} \tag{54}$$

Then by (45) we get that for all $t \geq T$,

$$\begin{aligned}
&\int_t^{t+1}\left\|\mathbf{V}\left(s,\theta_{-t-1}\omega,\mathbf{U}_0\left(\theta_{-t-1}\omega\right)-\mathbf{Z}\left(\theta_{-t-1}\omega\right)\right)\right\|_{\mathbb{V}}^2 ds\\
&\leq K^\epsilon\varrho_1^2(\omega)\int_t^{t+1}\left(\left\|Z\left(\theta_{s-t-1}\omega\right)\right\|_{H^2}^2+\left\|z\left(\theta_{s-t-1}\omega\right)\right\|_{H^1}^2\right)ds\\
&\quad+\int_t^{t+1}G\left(\theta_{s-t-1}\omega\right)ds+K^\epsilon\varrho_1^2(\omega)\\
&= K^\epsilon\varrho_1^2(\omega)\int_{-1}^0\left(\left\|Z\left(\theta_s\omega\right)\right\|_{H^2}^2+\left\|z\left(\theta_s\omega\right)\right\|_{H^1}^2\right)ds\\
&\quad+\int_{-1}^0 G\left(\theta_s\omega\right)ds+K^\epsilon\varrho_1^2(\omega) := \varrho_2(\omega),
\end{aligned} \tag{55}$$

which gives an expression for $\varrho_2(\omega)$. □

Lemma 10. *Assume that $\mathbb{E}M < 0$. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$. Then for $\mathbb{P}$-a.s.$\omega \in \Omega$, there exist random radium $R(\omega)$ and constant $T = T(B,\omega) > 0$ such that for all $t \geq T$, the solution*

$\mathbf{V}(t,\omega,\mathbf{U}_0(\omega)-\mathbf{Z}(\omega))$ of problem (23) with $\mathbf{U}_0 \in B(\omega)$ satisfies that for every $l \in [t,t+1]$,

$$\left\|\mathbf{V}\left(l,\theta_{-t-1}\omega,\mathbf{U}_0\left(\theta_{-t-1}\omega\right)-\mathbf{Z}\left(\theta_{-t-1}\omega\right)\right)\right\|_{\mathbb{V}} \le R(\omega), \quad (56)$$

where $\mathscr{D}$ is the collection of tempered random subsets of $\mathbb{H}$.

Proof. Using the classic Gronwall's lemma (see [15]) to the inequality (42) on the interval $[s,l]$ with $t \le s \le l \le t+1$, we get that

$$\begin{aligned}&\left\|\mathbf{V}\left(l,\theta_{-t-1}\omega,\mathbf{U}_0\left(\theta_{-t-1}\omega\right)-\mathbf{Z}\left(\theta_{-t-1}\omega\right)\right)\right\|_{\mathbb{V}}^2\\&\quad\le e^{\int_t^{t+1} g\left(\tau,\theta_{-t-1}\omega\right)d\tau}\\&\quad\times\left(\int_t^{t+1}\left\|\mathbf{V}\left(s,\theta_{-t-1}\omega,\mathbf{U}_0\left(\theta_{-t-1}\omega\right)\right.\right.\right.\\&\qquad\left.\left.-\mathbf{Z}\left(\theta_{-t-1}\omega\right)\right)\right\|_{\mathbb{V}}^2 ds\\&\qquad\left.+\int_t^{t+1} h\left(s,\theta_{-t-1}\omega\right)ds\right).\end{aligned} \quad (57)$$

Note that by Lemma 9 there exists $T=T(B,\omega)$ such that for all $t \ge T$,

$$\begin{aligned}&\int_t^{t+1} g\left(\tau,\theta_{-t-1}\omega\right)d\tau\\&\quad=\int_t^{t+1}\left(K\left\|v\left(s,\theta_{-t-1}\omega\right)\right\|^2\left\|v\left(s,\theta_{-t-1}\omega\right)\right\|_{H^1}^2\right.\\&\qquad\left.+H_1\left(\theta_{s-t-1}\omega\right)\right)ds\\&\quad\le K\varrho_1^2(\omega)\varrho_2^2(\omega)+\int_{-1}^0 H_1\left(\theta_s\omega\right)ds := C_1(\omega),\\&\int_t^{t+1} h\left(s,\theta_{-t-1}\omega\right)ds\\&\quad=\int_t^{t+1}\left(K\left\|V\left(s,\theta_{-t-1}\omega\right)\right\|^2+H_2\left(\theta_{s-t-1}\omega\right)\right)ds\\&\quad\le K\varrho_1^2(\omega)+\int_{-1}^0 H_2\left(\theta_s\omega\right)ds := C_2(\omega).\end{aligned} \quad (58)$$

Then by (57) and (58) it gives that for all $t \ge T$ and $l \in [t,t+1]$,

$$\begin{aligned}&\left\|\mathbf{V}\left(l,\theta_{-t-1}\omega,\mathbf{U}_0\left(\theta_{-t-1}\omega\right)-\mathbf{Z}\left(\theta_{-t-1}\omega\right)\right)\right\|_{\mathbb{V}}^2\\&\quad\le e^{C_1(\omega)}\left(K\varrho_2(\omega)+C_2(\omega)\right) := R^2(\omega),\end{aligned} \quad (59)$$

which gives an expression for $R(\omega)$. □

Lemma 11. *Assume that $\mathbb{E}M < 0$. Let $B=\{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$. Then for $\mathbb{P}$-a.s.$\omega \in \Omega$ and every $\varepsilon > 0$, there are $N = N(\varepsilon,\omega), K = K(\omega)$, and $T = T(\varepsilon,B,\omega) > 0$ such that for all $t \ge T$ and $m \ge N$, the solution $\mathbf{V}(t,\omega,\mathbf{U}_0(\omega)-\mathbf{Z}(\omega))$ of problem (23) with $\mathbf{U}_0 \in B(\omega)$ satisfies that*

$$\begin{aligned}&\left\|P_m\mathbf{V}\left(t,\theta_{-t}\omega,\mathbf{U}_0\left(\theta_{-t}\omega\right)-\mathbf{Z}\left(\theta_{-t}\omega\right)\right)\right\|_{\mathbb{V}} \le K,\\&\left\|\mathbf{V}_m\left(t,\theta_{-t}\omega,\mathbf{U}_0\left(\theta_{-t}\omega\right)-\mathbf{Z}\left(\theta_{-t}\omega\right)\right)\right\|_{\mathbb{V}} \le \varepsilon,\end{aligned} \quad (60)$$

where $\mathscr{D}$ is the collection of tempered random subsets of $\mathbb{H}$.

Proof. Multiplying (23) by $A\mathbf{V}_m$ with respect to the L^2 inner product leads us to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\mathbf{V}_m\right\|_{\mathbb{V}}^2+\left\|A\mathbf{V}_m\right\|^2\\&\quad=\left(B_1\left(v+z\left(\theta_t\omega\right),v+z\left(\theta_t\omega\right)\right),A_1v_m\right)\\&\qquad+\left(B_2\left(v+z\left(\theta_t\omega\right),V+Z\left(\theta_t\omega\right)\right),A_2V_m\right)\\&\qquad+\left(e_2\left(V+Z\left(\theta_t\omega\right)\right),A_1v_m\right)+\left(f,A_2(V_m)_\Gamma\right)\\&\quad=I_1+I_2+I_3,\end{aligned} \quad (61)$$

where

$$\begin{aligned}I_1&=\left(B_1\left(v+z\left(\theta_t\omega\right),v+z\left(\theta_t\omega\right)\right),A_1v_m\right),\\I_2&=\left(B_2\left(v+z\left(\theta_t\omega\right),V+Z\left(\theta_t\omega\right)\right),A_2V_m\right),\\I_3&=\left(e_2\left(V+Z\left(\theta_t\omega\right)\right),A_1v_m\right)+\left(f,A_2(V_m)_\Gamma\right).\end{aligned} \quad (62)$$

In order to estimate I_1, we rewrite it as

$$\begin{aligned}I_1&=\left(B_1(v,v),A_1v_m\right)+\left(B_1\left(v,z\left(\theta_t\omega\right)\right),A_1v_m\right)\\&\quad+\left(B_1\left(z\left(\theta_t\omega\right),z\left(\theta_t\omega\right)\right),A_1v_m\right)\\&\quad+\left(B_1\left(z\left(\theta_t\omega\right),v\right),A_1v_m\right),\end{aligned} \quad (63)$$

where by utilizing the inequality (14) and Agmon's inequality in $\mathbb{R}^2$ (see [15]), it gives that

$$\begin{aligned}&\left|B_1(v,v),A_1v_m\right|\\&\quad\le\left|B_1\left(P_mv,v\right),A_1v_m\right|+\left|B_1\left(v_m,v\right),A_1v_m\right|\\&\quad\le\left\|P_mv\right\|_{L^\infty}\|v\|_{H^1}\left\|A_1v_m\right\|\\&\qquad+\left\|v_m\right\|_{L^\infty}\|v\|_{H^1}\left\|A_1v_m\right\|\\&\quad\le K\left(1+\log\frac{\lambda_{m+1}}{\lambda_1}\right)^{1/2}\|v\|_{H^1}^2\left\|A_1v_m\right\|\\&\qquad+K^\epsilon\left\|v_m\right\|^{1/2}\left\|A_1v_m\right\|^{3/2}\|v\|_{H^1}\\&\quad\le\frac{\epsilon}{16}\left\|A_1v_m\right\|^2+K^\epsilon\left(1+\log\frac{\lambda_{m+1}}{\lambda_1}\right)\|v\|_{H^1}^4\\&\qquad+K^\epsilon\|v\|^2\|v\|_{H^1}^4,\\&\left|\left(B_1\left(v,z\left(\theta_t\omega\right)\right),A_1v_m\right)\right|\\&\quad\le\left|B_1\left(P_mv,z\left(\theta_t\omega\right)\right),A_1v_m\right|\\&\qquad+\left|\left(B_1\left(v_m,z\left(\theta_t\omega\right)\right),A_1v_m\right)\right|\\&\quad\le\left\|P_mv\right\|_{L^\infty}\left\|z(\theta_t\omega)\right\|_{H^1}\left\|A_1v_m\right\|\\&\qquad+\left\|v_m\right\|_{L^\infty}\left\|z(\theta_t\omega)\right\|_{H^1}\left\|A_1v_m\right\|\end{aligned} \quad (64)$$

$$\begin{aligned}
&\leq K\left(1+\log\frac{\lambda_{m+1}}{\lambda_1}\right)^{1/2}\|v\|_{H^1}\|z(\theta_t\omega)\|_{H^1}\|A_1v_m\| \\
&\quad + K\|v_m\|^{1/2}\|A_1v_m\|^{3/2}\|z(\theta_t\omega)\|_{H^1}^2 \\
&\leq \frac{\epsilon}{16}\|A_1v_m\|^2 \\
&\quad + K^\epsilon\left(1+\log\frac{\lambda_{m+1}}{\lambda_1}\right)\|v\|_{H^1}^2\|z(\theta_t\omega)\|_{H^1}^2 \\
&\quad + K^\epsilon\|v\|^2\|z(\theta_t\omega)\|_{H^1}^4.
\end{aligned} \tag{65}$$

Similarly by utilizing the Agmon's inequality, we deduce that

$$\begin{aligned}
\left|B_1(z(\theta_t\omega),z(\theta_t\omega)),A_1v_m\right| &\leq \frac{\epsilon}{16}\|A_1v_m\|^2 \\
&\quad + K^\epsilon\|z(\theta_t\omega)\|_{H^2}^2\|z(\theta_t\omega)\|_{H^1}^2, \\
\left|B_1(z(\theta_t\omega),v),A_1v_m\right| &\leq \frac{\epsilon}{16}\|A_1v_m\|^2 \\
&\quad + K^\epsilon\|z(\theta_t\omega)\|_{H^2}^2\|v\|_{H^1}^2.
\end{aligned} \tag{66}$$

Then by (63)–(66) we get that

$$\begin{aligned}
I_1 &\leq \frac{\epsilon}{4}\|A_1v_m\|^2 + K^\epsilon\left(1+\log\frac{\lambda_{m+1}}{\lambda_1}\right) \\
&\quad \times\left(\|v\|_{H^1}^2\|z(\theta_t\omega)\|_{H^1}^2 + \|v\|_{H^1}^4\right) \\
&\quad + K^\epsilon\left(\|v\|^2\|v\|_{H^1}^4 + \|z(\theta_t\omega)\|_{H^2}^2\|v\|_{H^1}^2\right. \\
&\quad + \|v\|^2\|z(\theta_t\omega)\|_{H^1}^4 \\
&\quad \left. + \|z(\theta_t\omega)\|_{H^2}^2\|z(\theta_t\omega)\|_{H^1}^2\right).
\end{aligned} \tag{67}$$

We then estimate I_2 in (61). We first have

$$\begin{aligned}
I_2 &= \left(B_2(v,Z(\theta_t\omega)),A_2V_m\right) + \left(B_2(v,V),A_2V_m\right) \\
&\quad + \left(B_2(z(\theta_t\omega),Z(\theta_t\omega)),A_2V_m\right) \\
&\quad + \left(B_2(z(\theta_t\omega),V),A_2V_m\right),
\end{aligned} \tag{68}$$

where

$$\begin{aligned}
&\left|B_2(v,V),A_2V_m\right| \\
&\quad \leq \left|B_2(P_mv,V),A_2V_m\right| + \left|B_2(v_m,V),A_2V_m\right| \\
&\quad \leq \|P_mv\|_{L^\infty}\|V\|_{H^1}\|A_2V_m\| \\
&\qquad + \|v_m\|_{L^\infty}\|V\|_{H^1}\|A_2V_m\| \\
&\quad \leq \frac{\epsilon}{16}\|A_2V_m\|^2 + K^\epsilon\left(1+\log\frac{\lambda_{m+1}}{\lambda_1}\right)\|v\|_{H^1}^2\|V\|_{H^1}^2 \\
&\qquad + K^\epsilon\|v\|\|A_1v_m\|\|V\|_{H^1}^2 \\
&\quad \leq \frac{\epsilon}{16}\|A_2V_m\|^2 + K^\epsilon\left(1+\log\frac{\lambda_{m+1}}{\lambda_1}\right)\|v\|_{H^1}^2\|V\|_{H^1}^2 \\
&\qquad + \frac{\epsilon}{16}\|A_1v_m\|^2 + K^\epsilon\|v\|^2\|V\|_{H^1}^4,
\end{aligned}$$

$$\begin{aligned}
&\left|B_2(v,Z(\theta_t\omega)),A_2V_m\right| \\
&\quad \leq \frac{\epsilon}{16}\|A_2V_m\|^2 \\
&\qquad + K^\epsilon\left(1+\log\frac{\lambda_{m+1}}{\lambda_1}\right)\|v\|_{H^1}^2\|Z(\theta_t\omega)\|_{H^1}^2 \\
&\qquad + \frac{\epsilon}{16}\|A_1v_m\|^2 + K^\epsilon\|v\|^2\|Z(\theta_t\omega)\|_{H^1}^4, \\
&\left|B_2(z(\theta_t\omega),Z(\theta_t\omega)),A_2V_m\right| \\
&\quad \leq \frac{\epsilon}{16}\|A_2V_m\|^2 \\
&\qquad + K^\epsilon\|z(\theta_t\omega)\|_{H^2}^2\|Z(\theta_t\omega)\|_{H^1}^2, \\
&\left|B_2(z(\theta_t\omega),V),A_2V_m\right| \\
&\quad \leq \frac{\epsilon}{16}\|A_2V_m\|^2 \\
&\qquad + K^\epsilon\|z(\theta_t\omega)\|_{H^2}^2\|V\|_{H^1}^2.
\end{aligned} \tag{69}$$

Then it follows from (68) and (69) that

$$\begin{aligned}
I_2 &\leq \frac{\epsilon}{4}\|A_2V_m\|^2 \\
&\quad + \frac{\epsilon}{8}\|A_1v_m\|^2 + K^\epsilon\left(1+\log\frac{\lambda_{m+1}}{\lambda_1}\right) \\
&\quad \times\left(\|v\|_{H^1}^2\|V\|_{H^1}^2 + \|v\|_{H^1}^2\|Z(\theta_t\omega)\|_{H^1}^2\right) \\
&\quad + K^\epsilon\left(\|v\|^2\|V\|_{H^1}^4 + \|v\|^2\|Z(\theta_t\omega)\|_{H^1}^4\right. \\
&\quad + K^\epsilon\|z(\theta_t\omega)\|_{H^2}^2\|V\|_{H^1}^2 \\
&\quad \left. + \|z(\theta_t\omega)\|_{H^2}^2\|Z(\theta_t\omega)\|_{H^1}^2\right).
\end{aligned} \tag{70}$$

Moreover,

$$\begin{aligned}
I_3 &\leq \frac{\epsilon}{8}\|A_1v_m\|^2 \\
&\quad + K^\epsilon\|V\|^2 + K^\epsilon\|Z(\theta_t\omega)\|^2 \\
&\quad + \frac{\epsilon}{4}\|A_2(V_m,(V_m)_\Gamma)\|^2 + K^\epsilon\|f\|^2.
\end{aligned} \tag{71}$$

Then by (67), (70), and (71), formula (61) becomes

$$\begin{aligned}
&\frac{d}{dt}\|\mathbf{V}_m\|_{\mathbb{V}}^2 + \lambda_{m+1}\|\mathbf{V}_m\|_{\mathbb{V}}^2 \\
&\quad \leq \left(1+\log\frac{\lambda_{m+1}}{\lambda_1}\right)P(t,\omega) + Q(t,\omega),
\end{aligned} \tag{72}$$

where

$$\begin{aligned}
&P(t,\omega) \\
&\quad = K^\epsilon\left(\|v\|_{H^1}^4 + \|v\|_{H^1}^2\|z(\theta_t\omega)\|_{H^1}^2\right. \\
&\qquad \left. + \|v\|_{H^1}^2\|V\|_{H^1}^2 + \|v\|_{H^1}^2\|Z(\theta_t\omega)\|_{H^1}^2\right),
\end{aligned}$$

$$
\begin{aligned}
&Q(t,\omega)\\
&\quad = K^{\epsilon}\Big(\|v\|^2\|v\|_{H^1}^4 + \|v\|^2\|z(\theta_t\omega)\|_{H^1}^4\\
&\qquad + \|v\|^2\|V\|_{H^1}^4 + \|v\|^2\|Z(\theta_t\omega)\|_{H^1}^4\\
&\qquad + \|z(\theta_t\omega)\|_{H^2}^2\|V\|_{H^1}^2\\
&\qquad + \|z(\theta_t\omega)\|_{H^2}^2\|Z(\theta_t\omega)\|_{H^1}^2\\
&\qquad + \|z(\theta_t\omega)\|_{H^2}^2\|z(\theta_t\omega)\|_{H^1}^2\\
&\qquad + \|z(\theta_t\omega)\|_{H^2}^2\|v\|_{H^1}^2\\
&\qquad + \|Z(\theta_t\omega)\|^2 + \|f\|^2\Big).
\end{aligned}
\tag{73}
$$

Then by utilizing Lemma 7 to (72) we deduce that

$$
\begin{aligned}
&\|\mathbf{V}_m(t+1,\theta_{-t-1}\omega,\mathbf{U}_0(\theta_{-t-1}\omega)-\mathbf{Z}(\theta_{-t-1}\omega))\|_{\mathbb{V}}^2\\
&\quad\le\left(1+\log\frac{\lambda_{m+1}}{\lambda_1}\right)\\
&\qquad\times e^{-\lambda_{m+1}}\int_t^{t+1}e^{\lambda_{m+1}(s-t)}P(s,\theta_{-t-1}\omega)\,ds\\
&\qquad+e^{-\lambda_{m+1}}\int_t^{t+1}e^{\lambda_{m+1}(s-t)}Q(s,\theta_{-t-1}\omega)\,ds\\
&\qquad+e^{-\lambda_{m+1}}\\
&\qquad\times\int_t^{t+1}e^{\lambda_{m+1}(s-t)}\|\mathbf{V}_m(s,\theta_{-t-1}\omega,\mathbf{U}_0(\theta_{-t-1}\omega)\\
&\qquad\qquad-\mathbf{Z}(\theta_{-t-1}\omega))\|_{\mathbb{V}}^2ds.
\end{aligned}
\tag{74}
$$

By Lemma 11, there exists $T = T(B,\omega) > 0$ such that for all $t \ge T$,

$$
\begin{aligned}
&\int_t^{t+1}e^{\lambda_{m+1}s}P(s,\theta_{-t-1}\omega)\,ds\\
&\quad\le K^{\epsilon}\int_t^{t+1}e^{\lambda_{m+1}(s-t)}\\
&\qquad\times\Big(R^4(\omega)+R^2(\omega)\|z(\theta_{s-t-1}\omega)\|_{H^1}^2\\
&\qquad+R^4(\omega)+R^2(\omega)\\
&\qquad\times\|Z(\theta_{s-t-1}\omega)\|_{H^1}^2\Big)\,ds\\
&\quad\le\frac{K^{\epsilon}}{\lambda_{m+1}}\\
&\qquad\times e^{\lambda_{m+1}}\Big(2R^4(\omega)\\
&\qquad+R^2(\omega)\sup_{-1\le s\le 0}\|z(\theta_s\omega)\|_{H^1}^2\\
&\qquad+R^2(\omega)\sup_{-1\le s\le 0}\|Z(\theta_s\omega)\|_{H^1}^2\Big):=\frac{\widehat{R}(\omega)}{\lambda_{m+1}}e^{\lambda_{m+1}},
\end{aligned}
\tag{75}
$$

where

$$
\begin{aligned}
\widehat{R}(\omega)=K^{\epsilon}\Big(&2R^4(\omega)+R^2(\omega)\sup_{-1\le s\le 0}\|z(\theta_s\omega)\|_{H^1}^2\\
&+R^2(\omega)\sup_{-1\le s\le 0}\|Z(\theta_s\omega)\|_{H^1}^2\Big)
\end{aligned}
\tag{76}
$$

is independent of λ_{m+1}. By a similar calculation as (75), we find that there exist an random variable $\widehat{\widehat{R}}(\omega)$ such that for all $t \ge T$,

$$
\begin{aligned}
&\int_t^{t+1}e^{\lambda_{m+1}(s-t)}Q(s,\theta_{-t-1}\omega)\,ds\le\frac{\widehat{\widehat{R}}(\omega)}{\lambda_{m+1}}e^{\lambda_{m+1}},\\
&\int_t^{t+1}e^{\lambda_{m+1}(s-t)}\\
&\quad\times\|\mathbf{V}_m(s,\theta_{-t-1}\omega,\mathbf{U}_0(\theta_{-t-1}\omega)\\
&\qquad-\mathbf{Z}(\theta_{-t-1}\omega))\|_{\mathbb{V}}^2ds\\
&\quad\le\frac{R(\omega)^2}{\lambda_{m+1}}e^{\lambda_{m+1}},
\end{aligned}
\tag{77}
$$

where $R(\omega)$ is in Lemma 9. Then (74) together with (75) and (77) implies that for all $t \ge T$,

$$
\begin{aligned}
&\|\mathbf{V}_m(t+1,\theta_{-t-1}\omega,\mathbf{U}_0(\theta_{-t-1}\omega)-\mathbf{Z}(\theta_{-t-1}\omega))\|_{\mathbb{V}}^2\\
&\quad\le\frac{1}{\lambda_{m+1}}\left(1+\log\frac{\lambda_{m+1}}{\lambda_1}\right)\widehat{R}(\omega)\\
&\qquad+\frac{1}{\lambda_{m+1}}\left(\widehat{\widehat{R}}(\omega)+R(\omega)^2\right)\longrightarrow 0,
\end{aligned}
\tag{78}
$$

as $m \to +\infty$. Consequently, for every $\varepsilon > 0$, there exists a integer N and positive constants $K = K(\omega)$ such that for all $t \ge T+1$ and $m \ge N$,

$$
\begin{aligned}
&\|(I-P_m)\mathbf{V}(t,\theta_{-t}\omega,\mathbf{U}_0(\theta_{-t}\omega)-\mathbf{Z}(\theta_{-t}\omega))\|_{\mathbb{V}}\le\varepsilon,\\
&\|P_m\mathbf{V}(t,\theta_{-t}\omega,\mathbf{U}_0(\theta_{-t}\omega)-\mathbf{Z}(\theta_{-t}\omega))\|_{\mathbb{V}}\le K.
\end{aligned}
\tag{79}
$$

This leads to the desirable conclusion. □

Lemma 12. *Assume that* $\mathbb{E}M < 0$. *Then the RDS* φ *corresponding to the Boussinesq system* (1) *is omega-limit compact in*

$\mathbb{V}$; that is, for every $\varepsilon > 0$ and an arbitrary $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$, there is an $T = T(\varepsilon, B, \omega) > 0$ such that for $\mathbb{P}$-a.s.$\omega \in \Omega$,

$$k\left(\bigcup_{t\geq T}\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)\right) \leq \varepsilon, \tag{80}$$

where $\mathscr{D}$ is the collection of tempered random subsets of $\mathbb{H}$.

Proof. By Lemma 11, for every $\mathbf{U}_0(\omega) \in B(\omega)$, there exist constants $\widehat{K}(\omega)$ and $T = T(\varepsilon, B, \omega)$ and $N_1 \in \mathbb{N}$ such that for all $t \geq T$ and $m \geq N_1$, $\|P_m\mathbf{V}(t,\theta_{-t}\omega,\mathbf{U}_0(\theta_{-t}\omega)-\mathbf{Z}(\theta_{-t}\omega))\|_{\mathbb{V}} \leq \widehat{K}(\omega)$ and

$$\left\|(I-P_m)\mathbf{V}(t,\theta_{-t}\omega,\mathbf{U}_0(\theta_{-t}\omega)-\mathbf{Z}(\theta_{-t}\omega))\right\|_{\mathbb{V}} \leq \frac{\varepsilon}{2}. \tag{81}$$

Note that

$$\begin{gathered}\left\|P_m\mathbf{Z}(\omega)\right\|_{\mathbb{V}} \leq \|\mathbf{Z}(\omega)\|_{\mathbb{V}}, \\ \left\|(I-P_m)\mathbf{Z}(\omega)\right\|_{\mathbb{V}} \leq \frac{1}{\sqrt{\lambda_{m+1}}}\|\mathbf{Z}(\omega)\|_{H^2} \longrightarrow 0\end{gathered} \tag{82}$$

as $m \to \infty$, and then there exists $N_2 \in \mathbb{N}$ such that for every $m \geq N_2$,

$$\begin{gathered}\left\|P_m\mathbf{Z}(\omega)\right\|_{\mathbb{V}} \leq \|\mathbf{Z}(\omega)\|_{\mathbb{V}}, \\ \left\|(I-P_m)\mathbf{Z}(\omega)\right\|_{\mathbb{V}} \leq \frac{\varepsilon}{2}.\end{gathered} \tag{83}$$

Put $N = \max\{N_1, N_2\}$. By the definition of the RDS φ, along with (81) and (82), we find that there exist $K(\omega) = \widehat{K}(\omega) + \|\mathbf{Z}(\omega)\|_{\mathbb{V}}$ such that for all $t \geq T$

$$\begin{gathered}\left\|P_N\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)\right\|_{\mathbb{V}} \leq K(\omega), \\ \left\|(I-P_N)\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)\right\|_{\mathbb{V}} \leq \varepsilon,\end{gathered} \tag{84}$$

where $T = T(\varepsilon, B, \omega) > 0$ is the same as (81). That is to say, the RDS φ satisfies the flattening conditions in $\mathbb{V}$. By utilizing the additive property of Kuratowski measure of noncompactness; see Lemma 2.5 (iii) in [12]; it follows from (84) that for $\mathbb{P}$-a.s.$\omega \in \Omega$,

$$\begin{aligned}&k\left(\bigcup_{t\geq T}\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)\right) \\ &\quad\leq k\left(P_N\left(\bigcup_{t\geq T}\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)\right)\right) \\ &\qquad+k\left((I-P_N)\left(\bigcup_{t\geq T}\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)\right)\right) \\ &\quad\leq 0 + k(B_{\mathbb{V}}(0,\varepsilon)) = 2\varepsilon,\end{aligned} \tag{85}$$

where $B_{\mathbb{V}}(0,\varepsilon)$ is the ε-neighborhood at centre 0 in $\mathbb{V}$. This completes the proof. □

Proof of Theorem 2. By Theorem 5.1 in [4], the RDS φ associated with the Boussinesq system (1) admits a unique compact random attractor $\{\mathscr{A}_H(\omega)\}_{\omega\in\Omega}$ in $\mathbb{H}$. Furthermore $\{\mathscr{A}_H(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$. Put

$$\begin{aligned}K_0(\omega) = \{&\mathbf{U} = (u, U, U_\Gamma) \in \mathbb{H};\ \|\mathbf{U}\| \leq \varrho_1(\omega) + \|\mathbf{Z}(\omega)\|, \\ &\omega \in \Omega\},\end{aligned} \tag{86}$$

where $\varrho_1(\omega)$ is in (42). Observe that $\|\mathbf{Z}(\omega)\|$ is tempered, and then Lemma 9 implies that $\{K_0(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for the RDS φ in $\mathscr{D}$. By Theorem 3.5 in [20], we know that $\{\mathscr{A}_H(\omega)\}_{\omega\in\Omega}$ is the $\mathbf{\Omega}$-limit set of $\{K_0(\omega)\}_{\omega\in\Omega}$ (see [19]); that is,

$$\mathscr{A}_{\mathbb{H}}(\omega) = \bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\varphi(t,\theta_{-t}\omega)K_0(\theta_{-t}\omega)}^{\mathbb{H}}, \quad \omega \in \Omega. \tag{87}$$

For $\omega \in \Omega$, the following is given:

$$\mathscr{A}_{\mathbb{V}}(\omega) = \bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\varphi(t,\theta_{-t}\omega)K_0(\theta_{-t}\omega)}^{\mathbb{V}}. \tag{88}$$

In the sequel we will show that $\{\mathscr{A}_{\mathbb{V}}(\omega)\}_{\omega\in\Omega}$ is a random attractor in $\mathbb{V}$ in the sense that $\{\mathscr{A}_{\mathbb{V}}(\omega)\}_{\omega\in\Omega}$ satisfies (27)–(29).

We divide the proof into three steps.

Step 1 (compactness). By Lemma 4.5 (v) in [12] and the omega-limit compactness of φ in $\mathbb{V}$ (from Lemma 12), we have

$$\begin{aligned}&k\left(\overline{\bigcup_{t\geq T}\varphi(t,\theta_{-t}\omega)K_0(\theta_{-t}\omega)}^{\mathbb{V}}\right) \\ &\quad= k\left(\bigcup_{t\geq T}\varphi(t,\theta_{-t}\omega)K_0(\theta_{-t}\omega)\right) \longrightarrow 0, \quad \text{as } T \longrightarrow \infty.\end{aligned} \tag{89}$$

Since $\overline{\bigcup_{t\geq T}\varphi(t,\theta_{-t}\omega)K_0(\theta_{-t}\omega)}^{\mathbb{V}}$ is norm-closed in $\mathbb{V}$, thanks to the nested property of the Kuratowski measure of noncompactness (see again Lemma 2.5 (iv) in [12]), we know that for $\omega \in \Omega$, $\mathscr{A}_{\mathbb{V}}(\omega)$ is nonempty and compact as required, which shows (27).

Step 2 (invariant property). By (87) and (88) it is easy to see that for $\omega \in \Omega$,

$$\begin{aligned}x \in \mathscr{A}_{\mathbb{H}}(\omega) \Longleftrightarrow \exists t_n \longrightarrow \infty, &\qquad x_n \in K_0(\theta_{-t}\omega) \\ &\text{such that } \varphi(t_n,\theta_{-t_n}\omega)x_n \xrightarrow{\|\cdot\|_{\mathbb{H}}} x,\end{aligned} \tag{90}$$

$$\begin{aligned}y \in \mathscr{A}_{\mathbb{V}}(\omega) \Longleftrightarrow \exists t_n \longrightarrow \infty, &\qquad x_n \in K_0(\theta_{-t}\omega) \\ &\text{such that } \varphi(t_n,\theta_{-t_n}\omega)x_n \xrightarrow{\|\cdot\|_{\mathbb{V}}} y.\end{aligned} \tag{91}$$

It is obvious that $\mathscr{A}_{\mathbb{V}}(\omega) \subseteq \mathscr{A}_{\mathbb{H}}(\omega)$ for $\omega \in \Omega$. Conversely, if $x \in \mathscr{A}_{\mathbb{H}}(\omega)$, by the equivalent regime (90), there exist two sequences $t_n \to \infty$ and $x_n \in K_0(\theta_{-t}\omega)$ such that

$$\varphi(t_n,\theta_{-t_n}\omega)x_n \xrightarrow{\|\cdot\|_{\mathbb{H}}} x. \tag{92}$$

Note that $\varphi(t_n, \theta_{-t_n}\omega)x_n \in \mathbb{V}$ and φ is omega-limit compact in $\mathbb{V}$ (from Lemma 12). Then there exist an $y \in \mathbb{V}$ and a subsequence $\{n_k\}$ with $t_{n_k} \to \infty$ such that

$$\varphi\left(t_{n_k}, \theta_{-t_{n_k}}\omega\right) x_{n_k} \longrightarrow y \quad \text{in the space } \mathbb{V}, \tag{93}$$

and by the nested relation of $\mathbb{V}$ and $\mathbb{H}$, we have $x = y$, whence connection with (91) and (93) we have $x \in \mathcal{A}_{\mathbb{V}}(\omega)$. Then $\mathcal{A}_{\mathbb{H}}(\omega) \subseteq \mathcal{A}_{\mathbb{V}}(\omega)$ for $\omega \in \Omega$, which indicates that $\{\mathcal{A}_{\mathbb{V}}(\omega)\}_{\omega\in\Omega}$ possesses the invariance property (28).

Step 3 (attracting property). Suppose that (29) does not hold. Then there exists $\delta > 0$, $x_n \in B(\theta_{-t_n}\omega)$ and $t_n \to \infty$ such that

$$\operatorname{dist}_{\mathbb{V}}\left(\varphi\left(t_n, \theta_{-t_n}\omega\right) x_n, \mathcal{A}_{\mathbb{V}}(\omega)\right) \geq \delta. \tag{94}$$

According to Lemma 12, φ is omega-limit compact in $\mathbb{V}$. Then we can extract a subsequence from the sequence $\{\varphi(t_n, \theta_{-t_n}\omega)x_n\}_n$ (denoted by its original form) satisfying that there exists an $y \in \mathbb{V}$ such that when $t_n \to \infty$,

$$\varphi\left(t_n, \theta_{-t_n}\omega\right) x_n \longrightarrow y \quad \text{in the space } \mathbb{V}. \tag{95}$$

We then need to show that $y \in \mathcal{A}_{\mathbb{V}}(\omega)$ for $\omega \in \Omega$. Indeed, since $\{K_0(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for φ in $\mathbb{H}$, then for $x_n \in B(\theta_{-t_n}\omega)$, there exists $T = T(B, \omega) > 0$ such that for all $t \geq T$,

$$\varphi\left(t, \theta_{-t}\theta_{-(t_n-t)}\omega\right) x_n\left(\theta_{-t}\theta_{-(t_n-t)}\omega\right) \in K_0\left(\theta_{-(t_n-t)}\omega\right). \tag{96}$$

At the same time, by the cocycle property of the RDS φ, for $t_n \geq t$ we have that

$$\begin{aligned}
&\varphi\left(t_n, \theta_{-t_n}\omega\right) x_n\left(\theta_{-t_n}\omega\right) \\
&\quad = \varphi\left(t_n - t + t, \theta_{-t_n}\omega\right) x_n\left(\theta_{-t_n}\omega\right) \\
&\quad = \varphi\left(t_n - t, \theta_{-(t_n-t)}\omega\right) \varphi\left(t, \theta_{-t_n}\omega\right) x_n\left(\theta_{-t_n}\omega\right) \\
&\quad = \varphi\left(t_n - t, \theta_{-(t_n-t)}\omega\right) \varphi\left(t, \theta_{-t}\theta_{-(t_n-t)}\omega\right) \\
&\qquad \times x_n\left(\theta_{-t}\theta_{-(t_n-t)}\omega\right).
\end{aligned} \tag{97}$$

We now choose fixed $\bar{t} \geq T$ such that $t_n \geq \bar{t}$. Denote $t'_n = t_n - \bar{t}$ and $y_n = \varphi(\bar{t}, \theta_{-\bar{t}}\theta_{-(t'_n)}\omega)x_n(\theta_{-\bar{t}}\theta_{-(t'_n)}\omega)$. Then (96) indicates that $y_n \in K_0(\theta_{-t'_n}\omega)$. It is obvious that if $t_n \to \infty$ then $t'_n \to \infty$. Thus it follows from (97) that the limit in (95) can be rewritten as

$$\varphi\left(t'_n, \theta_{-t'_n}\omega\right) y_n\left(\theta_{-t'_n}\omega\right) \longrightarrow y \quad \text{in the space } \mathbb{V}, \tag{98}$$

as $t'_n \to \infty$, whereas by (91) we get that $y \in \mathcal{A}_{\mathbb{V}}(\omega)$ for $\omega \in \Omega$, which contradicts (94). This proves that $\{\mathcal{A}_{\mathbb{V}}(\omega)\}_{\omega\in\Omega}$ possesses attracting property (29).

Note that the collection $\mathcal{D}$ considered also includes all deterministic bounded subsets in $\mathbb{V}$. Then the uniqueness for $\{\mathcal{A}_{\mathbb{V}}(\omega)\}_{\omega\in\Omega}$ is followed by Theorem 4.3 in [19]. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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