

Research Article

Analysis of Caputo Impulsive Fractional Order Differential Equations with Applications

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We use Sadovskii's fixed point method to investigate the existence and uniqueness of solutions of Caputo impulsive fractional differential equations of order $\alpha \in (0, 1)$ with one example of impulsive logistic model and few other examples as well. We also discuss Caputo impulsive fractional differential equations with finite delay. The results proven are new and compliment the existing one.

1. Introduction

Dynamics of many evolutionary processes from various fields such as population dynamics, control theory, physics, biology, and medicine, undergo abrupt changes at certain moments of time like earthquake, harvesting, shock, and so forth. These perturbations can be well approximated as instantaneous change of states or impulses. These processes are modeled by *impulsive differential equations*. In 1960, Milman and Myškis introduced impulsive differential equations in their paper [1]. Based on their work, several monographs have been published by many authors like Samoilenko and Perestyuk [2], Lakshmikantham et al. [3], Bainov and Simeonov [4, 5], Bainov and Covachev [6], and Benchohra et al. [7]. All the authors mentioned perviously have considered impulsive differential equations as ordinary differential equations coupled with impulsive effects. They considered the impulsive effects as difference equations being satisfied at impulses time. So, the solutions are piecewise continuous with discontinuities at impulses time. In the fields like biology, population dynamics, and so forth, problems with hereditary are best modeled by delay differential equations [8]. Problems associated with impulsive effects and hereditary property are modeled by impulsive delay differential equations.

The origin of fractional calculus (derivatives $(d^\alpha/dt^\alpha)f$ and integrals $I^\alpha f$ of arbitrary order $\alpha > 0$) goes back to Newton and Leibniz in the 17th century. In a letter correspondence with Leibniz, L'Hospital asked "What if the order of the derivative is $1/2$?" Leibniz replied, "Thus it follows that will be equal to $x\sqrt{dx} : x$, an apparent paradox, from which one day useful consequences will be drawn." This letter of Leibniz was in September 1695. So, 1695 is considered as the birthday of fractional calculus. Fractional order differential equations are generalizations of classical integer order differential equations and are increasingly used to model problems in fluid dynamics, finance, and other areas of application. Recent investigations have shown that sometimes physical systems can be modeled more accurately using fractional derivative formulations [9]. There are several excellent monographs available on this field [10–15]. In [11], the authors give a recent and up-to-date description of the developments of fractional differential and fractional integro-differential equations including applications. The existence and uniqueness of solutions to fractional differential equations has been considered by many authors [16–21]. Impulsive fractional differential equations represent a real framework for mathematical modeling to real world problems. Significant progress has been made in the theory of impulsive

fractional differential equations [7, 22–24]. Xu et al. in their paper [25] have described an impulsive delay fishing model.

Fractional derivatives arise naturally in mathematical problems, for $\alpha > 0$ and a function $f : [0, T] \rightarrow \mathbb{R}$, recall [10, Definitions 3.1, 2.2]

(a) *the Caputo fractional derivative*

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \quad (1)$$

(b) *the Riemann-Liouville fractional derivative*

$$D^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds, \quad (2)$$

provided that the right-hand sides exist pointwise on $[0, T]$ (Γ denotes the gamma function). Using the *Riemann-Liouville fractional integral* [10, Definition 2.1] $I_0^\alpha f(t) = (1/\Gamma(\alpha)) \int_0^t (t-s)^{\alpha-1} f(s) ds$, we have ${}^C D^\alpha f(t) = I_0^{1-\alpha} (d/dt) f(t)$ and $D^\alpha f(t) = (d/dt) I_0^{1-\alpha} f(t)$. $I_0^\alpha f$ exists, for instance, for all $\alpha > 0$, if $f \in C^0([0, T]) \cap L^1_{loc}([0, T])$; moreover, $I_0^\alpha f(0) = 0$. Throughout the paper, we assume that $I = [0, T]$.

One can see that both of the fractional derivatives are actually nonlocal operator because integral is a nonlocal operator. Moreover, calculating time fractional derivative of a function at some time requires all the past history, and hence fractional derivatives can be used for modeling systems with memory. Fractional differential equations can be formulated using both Caputo and Riemann-Liouville fractional derivatives. A *Riemann-Liouville initial value problem* can be stated as follows:

$$\begin{aligned} D^\alpha x(t) &= f(t, x(t)), \quad t \in I = [0, T], \\ D^{\alpha-1} x(0) &= x_0, \end{aligned} \quad (3)$$

or equivalently, $x(t) = x_0 t^{\alpha-1} / \Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds$ in its integral representation [11, Theorem 3.24]. For a physical interpretation of the initial conditions in (3), see [26–28]. If derivatives of Caputo type are used instead of Riemann-Liouville type, then initial conditions for the corresponding Caputo fractional differential equations can be formulated as for classical ordinary differential equations, namely, $x(0) = x_0$.

Our main objective is to discuss existence and uniqueness of solutions of the following *impulsive fractional differential equation of Caputo type* in a Banach space X with norm $\|\cdot\|_X$:

$$\begin{aligned} {}^C D^\alpha x(t) &= f(t, x(t)), \quad t \in I = [0, T], \quad t \neq t_k, \\ \Delta x(t) |_{t=t_k} &= I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ x(0) &= x_0, \end{aligned} \quad (4)$$

where $f \in C(I \times X, X)$, $I_k : X \rightarrow X$ and $x_0 \in X$. $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$, $\Delta x(t) |_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h \rightarrow 0^+} x(t+h)$, and $x(t_k^-) = \lim_{h \rightarrow 0^+} x(t-h)$. We break our function f into two components which satisfy different

conditions. We observed that this kind of functions occurs in ecological modeling. We have given the example of logistic equation in the last section. Our main tool is Sadovskii's fixed point theorem.

Now, we define some important spaces and norm which will encounter frequently:

$$\begin{aligned} PC(I, X) &= \left\{ x : [0, T] \rightarrow X \mid x \in C([t_0, t_1], X) \right. \\ &\quad \cup C((t_k, t_{k+1}], X) \quad k = 1, 2, \dots, m, \\ &\quad \left. x(t_k^+) \text{ and } x(t_k^-) \text{ exist, } x(t_k) = x(t_k^-) \right\}, \end{aligned} \quad (5)$$

with sup-norm $\|\cdot\|$, defined by $\|x\| = \sup\{\|x(t)\|_X : t \in I\}$.

Definition 1. A solution of fractional differential equation (4) is a piecewise continuous function $x \in PC([0, T], X)$ which satisfied (4).

Definition 2 ([29], Definition 11.1). Kuratowski noncompactness measure: let M be a bounded set in metric space (X, d) , then Kuratowski noncompactness measure, $\mu(M)$ is defined as $\inf\{\epsilon : M \text{ covered by a finite many sets such that the diameter of each set } \leq \epsilon\}$.

Definition 3 ([29], Definition 11.6). Condensing map: Let $\Phi : X \rightarrow X$ be a bounded and continuous operator on Banach space X such that $\mu(\Phi(B)) < \mu(B)$ for all bounded set $B \subset D(\Phi)$, where μ is the Kuratowski noncompactness measure, then Φ is called condensing map.

Definition 4. Compact map: a map $f : X \rightarrow X$ is said to be compact if the image of every bounded subset of X under f is precompact (closure is compact).

Theorem 5 (see [30]). *Let B be a convex, bounded, and closed subset of a Banach space X and let $\Phi : B \rightarrow B$ be a condensing map. Then, Φ has a fixed point in B .*

Lemma 6 ([29], Example 11.7). *A map $\Phi = \Phi_1 + \Phi_2 : X \rightarrow X$ is k -contraction with $0 \leq k < 1$ if*

- (a) Φ_1 is k -contraction, that is, $\|\Phi_1(x) - \Phi_1(y)\|_X \leq k\|x - y\|_X$;
- (b) Φ_2 is compact,

and hence Φ is a condensing map.

The structure of the paper is as follows. In Section 2, we prove the existence (Theorem 8) and uniqueness of solutions to (4). We show in Section 3 the existence and uniqueness of solutions for a general class of impulsive functional differential equations of fractional order $\alpha \in (0, 1)$. In Section 4, we give some examples in favor of our sufficient conditions.

2. Impulsive Fractional Differential Equation

Consider the initial value problem (4) on the cylinder $R = \{(t, x) \in \mathbb{R} \times X : t \in [0, T], x \in B(0, r)\}$ for some fixed $T > 0, r > 0$, and assume that there exist $p \in (0, \alpha), \alpha \in (0, 1), M_1, M_2, L_1 \in L_{1/p}([0, T], \mathbb{R}^+)$ and functions $f_1, f_2 \in C(R \times X, X)$ such that $f = f_1 + f_2$, and the following assumptions are satisfied:

- (A.1) f_1 is bounded and Lipschitz, in particular, $\|f_1(t, x)\|_X \leq M_1(t)$ and $\|f_1(t, x) - f_1(t, y)\|_X \leq L_1(t)\|x - y\|_X$ for all $(t, x), (t, y) \in R$,
- (A.2) f_2 is compact and bounded, in particular, $\|f_2(t, x)\|_X \leq M_2(t)$ for all $(t, x) \in R$,
- (A.3) $I_k \in C(X, X)$ such that $\|I_k(x)\|_X \leq l_1$ and $\|I_k(x) - I_k(y)\|_X \leq l_2\|x - y\|_X$,

where $C(R \times X, X)$ denotes set of continuous functions from $R \times X$ to X and $L_{1/p}([0, T], \mathbb{R}^+)$ denotes space $1/p$ -Lebesgue measurable functions from $[0, T]$ to \mathbb{R}^+ with norm $\|\cdot\|_{1/p}$.

Lemma 7 (Fečken et al. [24], Lemma 2). *The initial value problem (4) is equivalent to the nonlinear integral equation*

$$\begin{aligned} x(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s, x(s)) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s, x(s)) ds, \quad t \in [0, t_1] \\ &= x_0 + I_1(x(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s, x(s)) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s, x(s)) ds, \quad t \in (t_1, t_2] \\ &= x_0 + \sum_{k=1}^2 I_k(x(t_k^-)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s, x(s)) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s, x(s)) ds, \quad t \in (t_2, t_3] \\ &= x_0 + \sum_{k=1}^m I_k(x(t_k^-)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s, x(s)) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s, x(s)) ds, \quad t \in (t_m, T]. \end{aligned} \tag{6}$$

In other words, every solution of the integral equation (6) is also solution of our original initial value problem (4) and conversely.

Theorem 8 (existence of solutions). *Under the assumptions (A.1)–(A.3), the problem (4) has at least one solution in $[0, T]$, provided that*

$$\gamma_1 = ml_2 + \frac{c\|L_1\|_{1/p} T^{\alpha-p}}{\Gamma(\alpha+1)} < 1, \quad \text{where } c = \left(\frac{1-p}{\alpha-p}\right)^{1-p}. \tag{7}$$

Proof. Let B_λ be the closed bounded and convex subset of $PC([0, T], X)$, where B_λ is defined as $B_\lambda = \{x : \|x\| \leq \lambda\}, \lambda = \max\{\lambda_0, \lambda_1, \dots, \lambda_m\}$, and

$$\lambda_k = \|x_0\|_X + kl_1 + \frac{c(\|M_1\|_{1/p} + \|M_2\|_{1/p})}{\Gamma(\alpha)} T^{\alpha-p}, \tag{8}$$

$$k = 0, 1, 2, \dots, m.$$

Define a map $F : B_\lambda \rightarrow X$ such that

$$\begin{aligned} Fx(t) &= x_0 + \sum_{0 < t_k < t} I_k(x(t_k^-)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s, x(s)) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s, x(s)) ds. \end{aligned} \tag{9}$$

Let us consider that

$$\begin{aligned} F_1x(t) &= x_0 + \sum_{0 < t_k < t} I_k(x(t_k^-)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s, x(s)) ds, \tag{10} \\ F_2x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s, x(s)) ds. \end{aligned}$$

Step 1 (F is self mapping). Now, we show that $F(B_r) \subset B_r$. For $t \in [0, t_1]$

$$\begin{aligned} \|Fx(t)\|_X &\leq \|x_0\|_X + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s))\|_X ds \\ &\leq \|x_0\|_X + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f_1(s, x(s))\|_X ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f_2(s, x(s))\|_X ds \\ &\leq \|x_0\|_X + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_1(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_2(s) ds \\ &\leq \|x_0\|_X + \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \\ &\quad \times \left(\int_0^t M_1^{1/p}(s) ds \right)^p + \frac{1}{\Gamma(\alpha)} \\ &\quad \times \left(\int_0^t (t-s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \left(\int_0^t M_2^{1/p}(s) ds \right)^p \\ &\leq \|x_0\|_X + \frac{c(\|M_1\|_{1/p} + \|M_2\|_{1/p})}{\Gamma(\alpha)} T^{\alpha-p} = \lambda_0. \end{aligned} \tag{11}$$

For $t \in (t_1, t_2]$,

$$\begin{aligned}
\|Fx(t)\|_X &\leq \|x_0\|_X + |I_1(x(t_1^-))| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s))\|_X ds \\
&\leq \|x_0\|_X + l_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f_1(s, x(s))\|_X ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f_2(s, x(s))\|_X ds \\
&\leq \|x_0\|_X + l_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_1(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_2(s) ds \\
&\leq \|x_0\|_X + l_1 + \frac{1}{\Gamma(\alpha)} \\
&\quad \times \left(\int_0^t (t-s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \left(\int_0^t M_1^{1/p}(s) ds \right)^p \\
&\quad + \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \\
&\quad \times \left(\int_0^t M_2^{1/p}(s) ds \right)^p \\
&\leq \|x_0\|_X + l_1 + \frac{c(\|M_1\|_{1/p} + \|M_2\|_{1/p})}{\Gamma(\alpha)} T^{\alpha-p} \\
&= \lambda_1.
\end{aligned} \tag{12}$$

For $t \in (t_2, t_3]$,

$$\begin{aligned}
\|Fx(t)\| &\leq \|x_0\|_X + \sum_{k=0}^2 |I_k(x(t_k^-))| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s))\|_X ds \\
&\leq \|x_0\|_X + 2l_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f_1(s, x(s))\|_X ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f_2(s, x(s))\|_X ds \\
&\leq \|x_0\|_X + 2l_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_1(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_2(s) ds \\
&\leq \|x_0\|_X + 2l_1 + \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \\
&\quad \times \left(\int_0^t M_1^{1/p}(s) ds \right)^p + \frac{1}{\Gamma(\alpha)}
\end{aligned}$$

$$\begin{aligned}
&\quad \times \left(\int_0^t (t-s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \left(\int_0^t M_2^{1/p}(s) ds \right)^p \\
&\leq \|x_0\|_X + 2l_1 + \frac{c(\|M_1\|_{1/p} + \|M_2\|_{1/p})}{\Gamma(\alpha)} T^{\alpha-p} \\
&= \lambda_2.
\end{aligned} \tag{13}$$

For $t \in (t_m, T]$,

$$\begin{aligned}
\|Fx(t)\|_X &\leq \|x_0\|_X + \sum_{k=0}^m |I_k(x(t_k^-))| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s))\|_X ds \\
&\leq \|x_0\|_X + ml_1 \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f_1(s, x(s))\|_X ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f_2(s, x(s))\|_X ds \\
&\leq \|x_0\|_X + ml_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_1(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_2(s) ds \\
&\leq \|x_0\|_X + ml_1 \\
&\quad + \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \\
&\quad \times \left(\int_0^t M_1^{1/p}(s) ds \right)^p + \frac{1}{\Gamma(\alpha)} \\
&\quad \times \left(\int_0^t (t-s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \left(\int_0^t M_2^{1/p}(s) ds \right)^p \\
&\leq \|x_0\|_X + ml_1 + \frac{c(\|M_1\|_{1/p} + \|M_2\|_{1/p})}{\Gamma(\alpha)} T^{\alpha-p} \\
&= \lambda_m,
\end{aligned} \tag{14}$$

and thus $F(B_\lambda) \subset B_\lambda$.

Step 2 (F_1 is continuous and γ -contraction). To prove the continuity of F_1 for $t \in [0, T]$, let us consider a sequence x_n converging to x . Taking the norm of $F_1 x_n(t) - F_1 x(t)$, we have

$$\begin{aligned}
&\|F_1 x_n(t) - F_1 x(t)\|_X \\
&\leq \sum_{0 < t_k < t} \|I_1(x_n(t_k^-) - x(t_k^-))\|_X \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L(s) \|x_n(s) - x(s)\|_X ds \\
&\leq \sum_{0 < t_k < t} l_2 \|x_n(t_k^-) - x(t_k^-)\|_X
\end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \\
 & \times \left(\int_0^t L_1^{1/p}(s) ds \right)^p \|x_n - x\| \\
 & \leq \left(ml_2 + \frac{c \|L_1\|_{1/p} T^{\alpha-p}}{\Gamma(\alpha)} \right) \|x_n - x\|.
 \end{aligned} \tag{15}$$

From the pervious analysis, we obtain

$$\|F_1 x_n - F_1 x\| \leq \left(ml_2 + \frac{c \|L_1\|_{1/p} T^{\alpha-p}}{\Gamma(\alpha)} \right) \|x_n - x\|. \tag{16}$$

To prove that F_1 is γ_1 -contraction, let us consider that, for $x, y \in B_r$,

$$\begin{aligned}
 & \|F_1 x(t) - F_1 y(t)\|_X \\
 & \leq \sum_{0 < t_k < t} \|I_k(x(t_k^-) - y(t_k^-))\|_X \\
 & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L_1(s) \|x(s) - y(s)\|_X ds \\
 & \leq \sum_{0 < t_k < t} l_2 \|x(t_k^-) - y(t_k^-)\|_X \\
 & + \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \\
 & \times \left(\int_0^t L_1^{1/p}(s) ds \right)^p \|x - y\| \\
 & \leq \left(ml_2 + \frac{c \|L_1\|_{1/p} T^{\alpha-p}}{\Gamma(\alpha)} \right) \|x - y\|.
 \end{aligned} \tag{17}$$

Thus, for

$$\gamma_1 = \left(ml_2 + \frac{c \|L_1\|_{1/p} T^{\alpha-p}}{\Gamma(\alpha)} \right) < 1, \tag{18}$$

F_1 is γ_1 -contraction.

Step 3 (F_2 is compact). For $0 \leq \tau_1 \leq \tau_2 \leq T$, we have

$$\begin{aligned}
 & \|F_2 x(\tau_2) - F_2 x(\tau_1)\|_X \\
 & \leq \frac{1}{\Gamma(\alpha)} \left\| \int_0^{\tau_2} (\tau_2 - s)^{\alpha-1} f_2(s, x(s)) ds \right. \\
 & \quad \left. - \int_0^{\tau_1} (\tau_1 - s)^{\alpha-1} f_2(s, x(s)) ds \right\|_X \\
 & \leq \frac{1}{\Gamma(\alpha)} \left\| \int_0^{\tau_1} (\tau_2 - s)^{\alpha-1} f_2(s, x(s)) ds \right. \\
 & \quad + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} f_2(s, x(s)) ds \\
 & \quad \left. - \int_0^{\tau_1} (\tau_1 - s)^{\alpha-1} f_2(s, x(s)) ds \right\|_X
 \end{aligned}$$

$$\begin{aligned}
 & \leq \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \left((\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1} \right) \\
 & \quad \times \|f_2(s, x(s))\|_X ds \\
 & + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} \|f_2(s, x(s))\|_X ds \\
 & \quad \left((\tau - s)^{\alpha-1} \text{ is a decreasing function of } \tau - s. \right) \\
 & \leq \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \left((\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1} \right) M_2(s) ds \\
 & \quad + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} M_2(s) ds \\
 & \leq \frac{1}{\Gamma(\alpha)} \left(\left(\int_0^{\tau_1} ((\tau_1 - s)^{\alpha-1} \right. \right. \\
 & \quad \left. \left. - (\tau_2 - s)^{\alpha-1})^{1/(1-p)} ds \right)^{1-p} \right) \\
 & \quad \times \left(\int_0^{\tau_1} M_2^{1/p}(s) ds \right)^p + \frac{1}{\Gamma(\alpha)} \\
 & \quad \times \left(\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \left(\int_0^{\tau_1} M_2^{1/p}(s) ds \right)^p \\
 & \leq \frac{1}{\Gamma(\alpha)} \left(\left(\int_0^{\tau_1} ((\tau_1 - s)^{(\alpha-1)/(1-p)} \right. \right. \\
 & \quad \left. \left. - (\tau_2 - s)^{(\alpha-1)/(1-p)}) ds \right)^{1-p} \right) \\
 & \quad \times \left(\int_0^{\tau_1} M_2^{1/p}(s) ds \right)^p \\
 & \quad \left(x^z - y^z \geq (x - y)^z, \forall x \geq y \geq 0, z > 1. \right) \\
 & + \frac{1}{\Gamma(\alpha)} \left(\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \\
 & \quad \times \left(\int_0^{\tau_1} M_2^{1/p}(s) ds \right)^p \\
 & \leq \frac{c}{\Gamma(\alpha)} \left(\tau_1^{(\alpha-p)/(1-p)} - \tau_2^{(\alpha-p)/(1-p)} \right. \\
 & \quad \left. + (\tau_2 - \tau_1)^{(\alpha-p)/(1-p)} \right)^{1-p} \\
 & \quad \times \|M_2\|_{1/p} + \frac{c}{\Gamma(\alpha)} (\tau_2 - \tau_1)^{\alpha-p} \|M_2\|_{1/p} \\
 & \leq \frac{c}{\Gamma(\alpha)} \left((\tau_1 - \tau_2)^{(\alpha-p)/(1-p)} \right)^{1-p} \|M_2\|_{1/p} \\
 & \quad + \frac{c}{\Gamma(\alpha)} (\tau_2 - \tau_1)^{\alpha-p} \|M_2\|_{1/p} \quad (\tau_2 > \tau_1.) \\
 & \leq \frac{2c \|M_2\|_{1/p}}{\Gamma(\alpha)} (\tau_2 - \tau_1)^{\alpha-p}.
 \end{aligned} \tag{19}$$

The right-hand side of the pervoiusly expression does not depend on x . Thus using Arzela-Ascoli theorem for equicontinuous functions (Diethelm, Theorem D.10 [10]), we conclude that $F_2(B_r)$ is relatively compact, and hence F_2 is completely continuous on $I - \{t_1, t_2, \dots, t_m\}$. In a similar way, it can be prove the equi-continuity of F on $t = t_k^-$ and $t = t_k^+$, $k = 1, 2, \dots, m$. And thus F_2 is compact on $[0, T]$.

Step 4 (F is condensing). As $F = F_1 + F_2$, F_1 is continuous contraction, F_2 is compact, so by using Lemma 6, F is condensing map on B_r . \square

And hence by using the Theorem 5, we conclude that (4) has a solution in B_r .

Theorem 9. *If f is bounded and Lipschitz, in particular, $\|f(t, x) - f(t, y)\|_X \leq L_1^*(t)\|x - y\|_X$ for all $(t, x), (t, y) \in R$ and $L_1^* \in L_{1/p}([0, T], \mathbb{R}^+)$, then the problem (4) has a unique solution in B_λ , provided that*

$$\gamma_1^* = ml_2 + \frac{c \|L_1^*\| (1/p) T^{\alpha-p}}{\Gamma(\alpha + 1)} < 1, \tag{20}$$

where $c = ((1 - p)/(\alpha - p))^{1-p}$.

3. Impulsive Fractional Differential Equations with Finite Delay

In this section, we discuss existence and uniqueness of solutions of the following *impulsive fractional differential equations of Caputo type with finite delay* in a Banach space X with norm $|\cdot|$

$$\begin{aligned} {}^C D^\alpha x(t) &= f(t, x_t), \quad t \in I = [0, T], \quad t \neq t_k, \\ \Delta x(t)|_{t=t_k} &= I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ x(t) &= \phi(t), \quad t \in [-r, 0], \end{aligned} \tag{21}$$

where $f : I \times \mathcal{C} \rightarrow X$, $\mathcal{C} = C([-r, 0], X)$ with norm $\|x\|_r = \sup\{\|x(t)\|_X : t \in [-r, 0]\}$, $I_k \in C(X, X)$, ($k = 1, 2, \dots, m$), and X is a Banach space with a norm $|\cdot|_X$. For any $x : [-r, T] \rightarrow X$ and $t \in I$, $x_t \in \mathcal{C}$ and is defined by $x_t(s) = x(t + s)$, $s \in [-r, 0]$. Here, our tools are Banach and Schaefer fixed point theorems.

Define a new Banach space $PC([-r, T], X)$

$$\begin{aligned} PC([-r, T], X) &= \{x : [-r, T] \rightarrow X \mid x \in C((t_k, t_{k+1}], X) \\ &\cup C([-r, 0], X), \quad k = 0, 1, 2, \dots, m, \\ &x(t_k^+) \text{ and } x(t_k^-) \text{ exist,} \\ &x(t_k) = x(t_k^-)\} \end{aligned} \tag{22}$$

with sup-norm $\|\cdot\|$, defined by $\|x\| = \sup\{\|x(t)\|_X : t \in [-r, T]\}$.

Definition 10. A solution of fractional differential equation (21) is a piecewise continuous function $x \in PC([-r, T], X)$ which satisfies (21).

Consider the initial value problem (21) on $I \times \mathcal{C}$ for some fixed $T > 0$ and assume that there exist $p \in (0, \alpha)$, $\alpha \in (0, 1)$, $M_3, M_4, L_2 \in L_{1/p}([0, T], \mathbb{R}^+)$ such that the following assumptions are satisfied:

- (A.4) $f \in C(I \times \mathcal{C}, X)$,
- (A.5) f bounded, in particular, $\|f(t, \phi)\|_X \leq M_3(t)$ for all $(t, \phi) \in I \times \mathcal{C}$,
- (A.6) f is Lipschitz, in particular, $\|f(t, \phi) - f(t, \psi)\|_X \leq L_2(t) \|\phi - \psi\|_r$ for all $(t, \phi), (t, \psi) \in I \times \mathcal{C}$.

Lemma 11 (Fečken et al. [24], Lemma 2). *The initial value problem (21) is equivalent to the nonlinear integral equation*

$$\begin{aligned} x(t) &= \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds, \\ & \quad t \in [0, t_1] \\ &= \phi(0) + I_1(x(t_1^-)) \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds, \quad t \in (t_1, t_2] \\ &= \phi(0) + \sum_{k=1}^2 I_k(x(t_k^-)) \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds, \quad t \in (t_2, t_3] \\ &= \phi(0) + \sum_{k=1}^m I_k(x(t_k^-)) \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds, \quad t \in (t_m, T] \\ &= \phi(t), \quad t \in [-r, 0]. \end{aligned} \tag{23}$$

In other words, every solution of the integral equation (23) is also solution of our original initial value problem (21) and conversely.

Remark 12. Since history part/initial condition $x(t) = \phi(t)$, $t \in [-r, 0]$, is known, so we will investigate the existence and uniqueness of solution in $I = [0, T]$.

Theorem 13 (existence and uniqueness of solution). *Under the assumptions (A.3)–(A.6), the problem (21) has a unique solution in $[0, T]$, provided that*

$$\gamma_2 = ml_2 + \frac{c \|L_2\|_{1/p} T^{\alpha-p}}{\Gamma(\alpha)} < 1, \tag{24}$$

where $c = ((1 - p)/(\alpha - p))^{1-p}$.

Proof. In this case, we define the operator $F : PC(I, X) \rightarrow PC(I, X)$ by

$$Fx(t) = \sum_{0 < t_k < t} I_k(x_n(t_k^-)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds. \tag{25}$$

Step 1. To prove that F is self mapping, we need to prove that for each $x \in PC(I, X)$, $F_x \in PC(I, X)$.

The proof is similar to the proof of continuity of F_1 in Step 2 of Theorem 8, and hence we omit it.

Step 2 (F is continuous and γ_2 -contraction). The proof of this step is also similar to the proof of continuous and γ_1 -contraction of F_1 in Step 2 of Theorem 8.

Now by applying Banach’s fixed point theorem, we get that the operator F has an unique fixed point in $PC(I, X)$, and hence the problem (21) has a unique solution in $PC([-r, T], X)$. \square

Our next result is based on Schaefer’s fixed point theorem. In this case, we replace assumption (A.3) with the following linear growth condition:

$$(A.3') \quad I_k \text{ bounded, in particular, } \|I_k(x)\|_X \leq I_1^*,$$

$$(A.5') \quad f \text{ bounded, in particular, } \|f(t, \phi)\|_X \leq M_4(t)(1 + \|\phi\|_r) \text{ for all } (t, \phi) \in I \times \mathcal{C}.$$

Theorem 14. *Under the assumptions (A.3') and (A.5'), problem (21) has at least one solution.*

Proof. We transform the problem into a fixed point problem. For this purpose, consider the operator $F : PC(I, X) \rightarrow PC(I, X)$ defined by

$$Fx(t) = \sum_{0 < t_k < t}^m I_k(x_n(t_k^-)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s) ds. \tag{26}$$

Step 1 (F is continuous). Let $\{x^n\}$ be a sequence such that $x^n \rightarrow x$ in $PC(I, X)$. Then, for every $t \in I$, we have

$$\begin{aligned} & \|Fx^n(t) - Fx(t)\|_X \\ & \leq \sum_{0 < t_k < t} \|I_k(x^n(t_k^-)) - I_k(x(t_k^-))\| \\ & \quad + \frac{1}{\Gamma\alpha} \int_0^t (t-s)^{\alpha-1} \|f(s, x_s^n) - f(s, x_s)\|_X ds \\ & \leq \sum_{0 < t_k < t} \|I_k(x^n(t_k^-)) - I_k(x(t_k^-))\|_X \\ & \quad + \|f(\cdot, x^n) - f(\cdot, x)\| \frac{1}{\Gamma\alpha} \left(\int_0^t (t-s)^{\alpha-1} \right)^{1-p} ds \\ & \leq \sum_{k=1}^m \|I_k(x^n(\cdot)) - I_k(x(\cdot))\| \\ & \quad + \frac{T^\alpha}{\Gamma(\alpha+1)} \|f(\cdot, x^n) - f(\cdot, x)\|. \end{aligned} \tag{27}$$

We can see that if $n \rightarrow \infty$, $Fx^n \rightarrow Fx$, as $x^n \rightarrow x$ and I_k and both f are continuous. Hence, F is continuous.

Step 2 (F maps bounded sets into bounded sets in $PC(I, X)$). It is enough to show that for any $\delta > 0$, there exists an

$l = \|\phi(0)\|_r + ml_1^* + (c(1 + \delta)(\|M_4\|_{1/p})/\Gamma(\alpha))T^{\alpha-p} > 0$ such that $x \in B_\delta = \{x \in PC(I, X) \mid \|x\| \leq \delta\}$, and we have $\|Fx\| \leq l$.

For $t \in [0, T]$, we have

$$\begin{aligned} \|Fx(t)\|_X & \leq \|\phi(0)\|_r + \sum_{0 < t_k < t} |I_k(x(t_k^-))| \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x_s)\|_X ds \\ & \leq \|\phi(0)\|_r + ml_1^* \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_4(s)(1 + \|x_s\|_r) ds \\ & \leq \|\phi(0)\|_r + \frac{(1 + \|x\|)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_4(s) ds \\ & \leq \|\phi(0)\|_r + ml_1^* + \frac{(1 + \|x\|)}{\Gamma(\alpha)} \\ & \quad \times \left(\int_0^t (t-s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \left(\int_0^t M_4^{1/p}(s) ds \right)^p \\ & \leq \|\phi(0)\|_r + ml_1^* + \frac{c(1 + \delta)(\|M_4\|_{1/p})}{\Gamma(\alpha)} T^{\alpha-p}. \end{aligned} \tag{28}$$

Step 3 (F maps bounded sets into equicontinuous sets in $PC(I, X)$). The proof of this step is similar to the proof of compactness of F_2 in Step 3 of Theorem 8.

As a consequence of Steps 1–3 together with PC-type Arzela-Ascoli theorem (Fečken et al. [24], Theorem 2.11), the map $F : PC(I, X) \rightarrow PC(I, X)$ is completely continuous.

Step 4 (A priori bounds). Now, we prove that the set

$$E(F) = \{x \in PC(I, X) \mid x = \lambda Fx, \text{ for some } \lambda \in (0, 1)\} \text{ is bounded.}$$

We observe that for $t \in [0, T]$ and $x \in E(F)$, $x(t) = \lambda Fx(t)$ and

$$\begin{aligned} \|x(t)\|_X & \leq \|Fx(t)\|_X \\ & \leq \|\phi(0)\|_r + \sum_{0 < t_k < t} \|I_k(x(t_k^-))\|_X \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x_s)\|_X ds \\ & \leq \|\phi(0)\|_r + ml_1^* \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_4(s)(1 + \|x_s\|) ds \\ & \leq \|\phi(0)\|_r + ml_1^* + \frac{(1 + \|x\|)}{\Gamma(\alpha)} \end{aligned}$$

$$\begin{aligned} & \times \left(\int_0^t (t-s)^{(\alpha-1)/(1-p)} ds \right)^{1-p} \left(\int_0^t M_4^{1/p}(s) ds \right)^p \\ & \leq \|\phi(0)\|_r + ml_1^* + \frac{c(1+\|x\|)(\|M_4\|_{1/p})}{\Gamma(\alpha)} T^{\alpha-p}, \end{aligned} \tag{29}$$

and hence

$$\|x\| \leq \frac{\|\phi(0)\|_r + ml_1^* + (c(\|M_4\|_{1/p})/\Gamma(\alpha)) T^{\alpha-p}}{1 - (c(\|M_4\|_{1/p})/\Gamma(\alpha)) T^{\alpha-p}}. \tag{30}$$

This shows that $E(F)$ is bounded.

As a consequence of Schaefer’s fixed point theorem, the problem (21) has at least one solution in $PC([-r, T], X)$. \square

Theorem 15. *If f is bounded and Lipschitz, in particular, $\|f(t, \phi) - f(t, \psi)\|_X \leq L_2^*(t) \|\phi - \psi\|_r$, for all $(t, \phi), (t, \psi) \in I \times \mathcal{C}$ and $L_2^* \in L_{1/p}([0, T], \mathbb{R}^+)$, then problem (21) has a unique solution in $PC([-r, T], X)$, provided that*

$$\gamma_2^* = ml_2 + \frac{c\|L_2^*\|_{1/p} T^{\alpha-p}}{\Gamma(\alpha + 1)} < 1, \tag{31}$$

where $c = ((1 - p)/(\alpha - p))^{1-p}$.

Further, we consider the following more general Caputo fractional differential equation

$$\begin{aligned} {}^C D^\alpha x(t) &= f(t, x(t), x_t), \quad t \in [0, T], \quad t \neq t_k, \\ \Delta x(t)|_{t=t_k} &= I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ x(t) &= \phi(t), \quad t \in [-r, 0], \end{aligned} \tag{32}$$

where $f : I \times X \times \mathcal{C} \rightarrow X$ and $\mathcal{C} = C([-r, 0], X)$ with norm $\|x\|_r = \sup\{\|x(t)\|_X : t \in [-r, 0]\}$, $I_k \in C(X, X)$, ($k = 1, 2, \dots, m$), and X is a separable real Banach space with the norm $\|\cdot\|_X$. Here, our tools will be Banach and Schaefer fixed point theorems.

Definition 16. A solution of fractional differential equation (32) is a piecewise continuous function $x \in PC([-r, T], X)$ which satisfies (32).

Consider the initial value problem (32) on $I \times \mathcal{C} \times X$ for some fixed $T > 0$ and assume that there exist $p \in (0, \alpha)$, $M_5, M_6, L_3, L_4 \in L_{1/p}([0, T], \mathbb{R}^+)$ such that the following assumptions are satisfied:

- (A.7) $f \in C(I \times X \times \mathcal{C}, X)$,
- (A.8) f bounded, in particular, $\|f(t, x, \phi)\|_X \leq M_5(t)$ for all $(t, x, \phi) \in I \times X \times \mathcal{C}$,
- (A.9) f is Lipschitz, in particular, $\|f(t, x, \phi) - f(t, y, \psi)\|_X \leq L_3(t)\|x - y\|_X + L_4(t)\|\phi - \psi\|_r$ for all $(t, x, \phi), (t, y, \psi) \in I \times X \times \mathcal{C}$,
- (A.8') $\|f(t, x, \phi)\|_X \leq M_6(t)(1 + \|x\|_X + \|\phi\|_r)$.

Lemma 17 (Fečken et al. [24], Lemma 2). *The initial value problem (32) is equivalent to the following nonlinear integral equation*

$$\begin{aligned} x(t) &= \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x_s) ds, \\ & \quad t \in [0, t_1] \\ &= \phi(0) + I_1(x(t_1^-)) \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x_s) ds, \quad t \in (t_1, t_2] \\ &= \phi(0) + \sum_{k=1}^2 I_k(x(t_k^-)) \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x_s) ds, \quad t \in (t_2, t_3] \\ &= \phi(0) + \sum_{k=1}^m I_k(x(t_k^-)) \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x_s) ds, \quad t \in (t_m, T] \\ &= \phi(t), \quad t \in [-r, 0]. \end{aligned} \tag{33}$$

In other words, every solution of the integral equation (33) is also solution of our original initial value problem (32) and conversely.

Theorem 18. *Under the assumptions (A.3), (A.7), (A.8), and (A.9), the problem (32) has a unique solution, provided that*

$$\gamma_3 = ml_2 + \frac{c(\|L_3\|_{1/p} + \|L_4\|_{1/p}) T^{\alpha-p}}{\Gamma(\alpha)} < 1. \tag{34}$$

Proof. The proof is similar to the proof of Theorem 13. \square

Theorem 19. *Under the assumptions (A.3'), and (A.8'), the problem (32) has at least one solution in $PC([-r, T], X)$.*

Proof. The proof is similar to the proof of Theorem 14. \square

Theorem 20. *If f is bounded and Lipschitz, in particular, $\|f(t, x, \phi) - f(t, y, \psi)\|_X \leq L_3^*(t)\|x - y\|_X + L_4^*(t)\|\phi - \psi\|_r$, for all $(t, x, \phi), (t, y, \psi) \in I \times X \times \mathcal{C}$ and $L_3^*, L_4^* \in L_{1/p}([0, T], \mathbb{R}^+)$, then the problem (32) has a unique solution in $PC([-r, T], X)$, provided that*

$$\gamma_3^* = ml_2 + \frac{c(\|L_3^*\|_{1/p} + \|L_4^*\|_{1/p}) T^{\alpha-p}}{\Gamma(\alpha + 1)} < 1, \tag{35}$$

where $c = ((1 - p)/(\alpha - p))^{1-p}$.

4. Examples

Example 21 (fractional impulsive logistic equation). Consider the following class of fractional logistic equations in Banach space X with norm $\|\cdot\|_X$:

$$\begin{aligned} {}^C D^\alpha x(t) &= x(t)(a(t) - b(t)x(t)), \quad t \in [0, T], \quad t \neq t_k, \\ \Delta x(t)|_{t=t_k} &= I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ x(0) &= x_0, \end{aligned} \tag{36}$$

where $a(t) \in [a_*, a^*]$ and $b(t) \in [b_*, b^*]$ with $a_*, b_* > 0$.

Lemma 22 (Fečken et al. [24], Lemma 2). *The initial value problem (36) is equivalent to the nonlinear integral equation*

$$\begin{aligned} x(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (x(s)(a(s) - b(s)x(s))) ds, \\ & \quad t \in [0, t_1] \\ &= x_0 + I_1(x(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t-s)^{\alpha-1} x(s)a(s) ds \\ & \quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t-s)^{\alpha-1} b(s)x^2(s) ds, \quad t \in (t_1, t_2] \\ &= x_0 + \sum_{k=1}^2 I_k(x(t_k^-)) + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t-s)^{\alpha-1} x(s)a(s) ds \\ & \quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t-s)^{\alpha-1} b(s)x^2(s) ds, \\ & \quad t \in (t_2, t_3] \\ &= x_0 + \sum_{k=1}^m I_k(x(t_k^-)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)a(s) ds \\ & \quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b(s)x^2(s) ds, \quad t \in [t_m, T]. \end{aligned} \tag{37}$$

In other words, every solution of the integral equation (37) is also solution of our original initial value problem (36) and conversely.

We can easily see that for the problem (36), our functions are $f_1(t, x) = a(t)x$ and $f_2(t, x) = -b(t)x^2$. It is not difficult to deduce that

$$\begin{aligned} \|f_1(t, x)\|_X &\leq a^* \|x\|_X + ml_1, \\ \|f_1(t, x) - f_1(t, y)\|_X &\leq a^* \|x - y\|_X. \end{aligned} \tag{38}$$

Also $\|f_2(t, x)\|_X \leq b^* \|x\|^2$. From the integral representation of problem (36), we get

$$\|x(t)\|_X \leq \|x_0\|_X + ml_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a^* \|x(s)\|_X ds. \tag{39}$$

Using Gronwall's inequality (Diethelm, Lemma 6.19 [10, 31]), we get

$$\begin{aligned} \|x(t)\|_X &\leq (\|x_0\|_X + ml_1) \exp\left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a^* ds\right) \\ &\leq (\|x_0\|_X + ml_1) \exp\left(\frac{a^*}{\Gamma(\alpha + 1)}\right). \end{aligned} \tag{40}$$

Thus x is bounded which implies that all the assumptions of Theorem 8 are satisfied, and hence there exists a solution of the problem (36).

We give some more examples which are inspired by [24].

Example 23. Consider the following Caputo impulsive delay fractional differential equations

$$\begin{aligned} {}^C D^\alpha x(t) &= \frac{e^{-\nu t} \|x_t\|}{(1 + e^t)(1 + \|x_t\|)}, \quad t \in [0, 1], \quad t \neq t_1, \quad \nu > 0, \\ \Delta x(t)|_{t=t_1} &= \frac{1}{2}, \\ x(t) &= \phi(t), \quad t \in [-r, 0]. \end{aligned} \tag{41}$$

Set $C_1 = C([-r, 0], \mathbb{R}^+)$, $f(t, \phi) = e^{-\nu t} \phi / ((1 + e^t)(1 + \phi))$, $(t, \phi) \in [0, 1] \times C_1$.

Let $\phi_1, \phi_2 \in C_1$ and let $t \in [0, 1]$. Then, we have

$$\begin{aligned} |f(t, \phi_1) - f(t, \phi_2)| &= \frac{e^{-\nu t}}{1 + e^t} \left| \frac{\phi_1}{1 + \phi_1} - \frac{\phi_2}{1 + \phi_2} \right| \\ &= \frac{e^{-\nu t} \|\phi_1 - \phi_2\|_r}{(1 + e^t)(1 + \phi_1)(1 + \phi_2)} \\ &\leq \frac{e^{-\nu t} \|\phi_1 - \phi_2\|_r}{(1 + e^t)} \leq L^*(t) \|\phi_1 - \phi_2\|_r, \end{aligned} \tag{42}$$

where $L^*(t) = e^{-\nu t}/2$.

Again, for all $\phi \in C_1$ and each $t \in [0, T]$,

$$\begin{aligned} |f(t, \phi)| &= \frac{e^{-\nu t}}{1 + e^t} \left| \frac{\phi}{1 + \phi} \right| \leq \frac{e^{-\nu t}}{(1 + e^t)} \\ &< m_1(t), \end{aligned} \tag{43}$$

where $m_1(t) = e^{-\nu t}/2$.

For $t \in [0, 1]$ and some $p \in (0, \alpha)$, $L^*(t) = m_1(t) = e^{-\nu t}/2 \in L_{1/p}([0, 1], \mathbb{R}^+)$ with $M_1^* = \|e^{-\nu t}/10\|_{1/p}$, we can arrive at the inequality $1/4 + cM_1^*/\Gamma(\alpha) < 1$. We can see that all the assumptions of Theorem 13 are satisfied, and hence the problem (41) has a unique solution in $[0, 1]$.

Example 24. Consider the following Caputo impulsive delay fractional differential equation

$${}^C D^\alpha x(t) = \frac{\|x_t\|}{(1+e^t)(1+\|x_t\|)}, \quad t \in [0, 1], \quad t \neq t_1, \quad \nu > 0,$$

$$\Delta x(t)|_{t=t_1} = \frac{1}{2},$$

$$x(t) = \phi(t), \quad t \in [-r, 0]. \quad (44)$$

Set $C_2 = C([0, 1], \mathbb{R}^+)$, $f(t, \phi) = \phi / ((1 + e^t)(1 + \phi))$, $(t, \phi) \in [-r, 0] \times C_2$.

Again, for all $\phi \in C_2$ and each $t \in [0, T]$,

$$|f(t, \phi)| = \frac{e^{-t}}{1+e^t} \left| \frac{\phi}{1+\phi} \right|$$

$$\leq \frac{e^{-t}}{(1+e^t)} < m_2(t)(1+\|\phi\|), \quad (45)$$

where $m_2(t) = e^{-t}/4$.

For $t \in [0, 1]$ and some $p \in (0, \alpha)$, $m_2(t) = e^{-t}/4 \in L_{1/p}([0, 1], \mathbb{R}^+)$ with $M_2^* = \|e^{-t}/4\|_{1/p}$, we can arrive at the inequality $1/4 + cM/\Gamma(\alpha) < 1$. We can see that all the assumptions of Theorem 14 are satisfied, and hence the problem (44) has a solution in $[0, 1]$.

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