Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 704160, 7 pages
http://dx.doi.org/10.1155/2013/704160

Research Article

Existence of Solutions for a Periodic Boundary Value Problem via Generalized Weakly Contractions

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Received 21 December 2012; Accepted 19 February 2013

Academic Editor: Abdul Latif

We discuss the existence of solutions for a periodic boundary value problem and for some polynomials. For this purpose, we present some fixed point theorems for weakly and generalized weakly contractive mappings in the setting of partially ordered complete metric spaces.

1. Introduction

Existence of solutions for a periodic boundary value problem by using upper and lower solution methods has attracted the attention of many authors (see, e.g., [1–5]).

We consider a special case of the following boundary value problem:

$$\begin{gathered} u'(t) = f(t, u(t)) \quad \text{if } t \in [0, T], \\ u(0) = u(T) + \zeta_0, \end{gathered} \tag{1}$$

where $T > 0$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous map and ζ_0 is constant.

Obviously, if $\zeta_0 = 0$, then the problem (1) becomes the following periodic boundary value problem:

$$\begin{gathered} u'(t) = f(t, u(t)) \quad \text{if } t \in [0, T], \\ u(0) = u(T). \end{gathered} \tag{2}$$

Definition 1. A lower solution for (1) is a function $\alpha \in C^1([0, T])$ such that

$$\begin{gathered} \alpha'(t) \leq f(t, \alpha(t)) \quad \text{if } t \in [0, T], \\ \alpha(0) \leq \alpha(T) + \zeta_0. \end{gathered} \tag{3}$$

Let $\mathscr{A}$ stand for the class of functions $\phi : [0, +\infty) \to [0, +\infty)$, which satisfy the following conditions:

(i) ϕ is nondecreasing,

(ii) $\phi(x) < x$, for each $x > 0$,

(iii) $\beta(x) = \phi(x)/x \in \mathcal{S}$.

Very recently, Amini-Harandi and Emami [1] proved the following existence theorem, which extended the main theorem of Harjani and Sadarangani [2].

Theorem 2. *Consider problem* (2), *with f being continuous. Suppose that there exists $\lambda > 0$ such that for $x, y \in \mathbb{R}$ with $y \geq x$,*

$$0 \leq f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda \phi(y - x), \tag{4}$$

where $\phi \in \mathscr{A}$. Then, the existence of a lower solution for (2) *provides the existence of a unique solution of* (2).

In this paper, we solve (2) by extending a fixed point theorem in the context of partially ordered metric space. Our results improve/extend/generalize some results in the literature, in particular, the results of Amini-Harandi and

Emami [1] and Harjani and Sadarangani [2]. Finally, in the last section, we prove the existence of a solution for some polynomials, as applications.

2. Preliminaries

In this section, we state a necessary background on the topic of fixed point theory, one of the core subjects of nonlinear analysis, for the sake of completeness of the paper. Fixed point theory has a wide potential application not only in the branches of mathematics, but also in several disciplines such as economics, computer science, and biology (see, e.g., [6, 7]). The most beautiful and elementary result in this direction is the Banach contraction mapping principle [8]. After this substantial result of Banach, several authors have extended this principle in many different ways (see, e.g., [1–7, 9–31]). In particular, the authors have introduced new type of contractions and researched the existence and uniqueness of the fixed point in various spaces. One of the important contraction types, a ϕ-contraction, was introduced by Boyd and Wong [14]. In 1997, Alber and Duerre-Delabriere [10] defined the concept of a weak-φ-contraction which is a generalization of the ϕ-contraction. A self-mapping f on a metric space (X, d) is said to be weak-φ-contractive if there exists a map $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(f(x), f(y)) \le d(x, y) - \varphi(d(x, y)), \tag{5}$$

for all $x, y \in X$.

Later, Zhang and Song [31] introduced the notion of a generalized weak-φ-contraction which is a natural extension of the weak-φ-contraction. A self-mapping f on a metric space (X, d) is said to be generalized weak-φ-contractive if there exists a map $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(f(x), f(y)) \le N(x, y) - \varphi(N(x, y)), \tag{6}$$

for all $x, y \in X$, where

$$N(x, y) = \max\left\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2}\right\}. \tag{7}$$

For more details on weak φ-contractions, we refer to, for example, [20, 21, 28].

On the other hand, the existence and uniqueness of a fixed point in the context of partially ordered metric spaces were first investigated in 1986 by Turinici [30]. After this pivotal paper, a number of results were reported in this direction with applications to matrix equations, ordinary differential equations, and integral equations (see, e.g., [1, 2, 4, 5, 7, 9, 11–13, 15–19, 22, 25–27]).

Recently, the main theorem of Geraghty [16, Theorem 2.1] is reproved by Amini-Harandi and Emami [1] in the context of partially ordered metric space. On the other hand, the main theorem of Amini-Harandi and Emami [1, Theorem 2.1] extends the theorem of Harjani and Sadarangani [2]. The authors in [1, 2] also proved the existence and uniqueness of a solution for a periodic boundary value problem.

Before stating the main theorem in [1], we recall the following class of functions introduced by Geraghty [16]. Let $\mathcal{S}$ denote the set of all functions $\psi : [0, +\infty) \to [0, 1)$ such that

$$\psi(t_n) \longrightarrow 1 \quad \text{implies } t_n \longrightarrow 0. \tag{8}$$

Theorem 3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\beta \in \mathcal{S}$ such that*

$$d(f(x), f(y)) \le \beta(d(x, y))\, d(x, y) \quad \text{for each } x, y \in X \text{ with } x \succeq y. \tag{9}$$

Assume that either

(a) *f is continuous or*

(b) *for every nondecreasing sequence $\{x_n\}$ if $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then f has a unique fixed point.

Let $F(f)$ denote the set of fixed points of f.

We give the following classes of functions. Let Φ denote the set of all mappings $\varphi : [0, +\infty) \to [0, +\infty)$ verifying that

$$\varphi(t_n) \longrightarrow 0 \quad \text{implies } t_n \longrightarrow 0. \tag{10}$$

It is clear that if $\varphi \in \Phi$, we have that

$$\varphi(t) = 0 \quad \text{implies } t = 0. \tag{11}$$

3. Some Auxiliary Fixed Point Theorems

In the following theorem, we prove the existence and uniqueness of a fixed point for generalized weak-φ-contractive mappings in partially ordered complete metric spaces.

Theorem 4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\varphi \in \Phi$ such that*

$$d(f(x), f(y)) \le N(x, y) - \varphi(N(x, y)), \tag{12}$$

for each $x, y \in X$ with $x \preceq y$ (i.e., a generalized weak-φ-contraction).

Suppose also that either

(a) *f is continuous or*

(b) *for every nondecreasing sequence $\{x_n\}$ if $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then f has a fixed point. Moreover, if every $x, y \in F(f)$ is comparable, then the fixed point of f is unique.

Proof. First, we prove the existence of a fixed point of f. Since the self-mapping f is nondecreasing and $x_0 \preceq f(x_0)$, we get that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq \cdots. \tag{13}$$

Define $x_n = f^n(x_0)$, $n = 1, 2, 3, \ldots$. Then, expression (13) is equivalent to

$$x_n \preceq x_{n+1} \quad \forall n \in \mathbb{N}. \tag{14}$$

Assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. Otherwise, the proof is completed. From (12), we derive that

$$d(x_{n+1}, x_n) \leq N(x_n, x_{n-1}) - \varphi(N(x_n, x_{n-1})), \tag{15}$$

where

$$\begin{aligned} N(x_n, x_{n-1}) = \max &\left\{ d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \right. \\ &\left. \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2} \right\} \\ = \max &\{ d(x_n, x_{n-1}), d(x_n, x_{n+1}) \}. \end{aligned} \tag{16}$$

If $N(x_n, x_{n-1}) = d(x_n, x_{n+1})$ for some n, then from (15) and (16), we have

$$\begin{aligned} 0 < d(x_{n+1}, x_n) &\leq d(x_n, x_{n+1}) - \varphi(d(x_n, x_{n+1})) \\ &< d(x_n, x_{n+1}). \end{aligned} \tag{17}$$

This is a contradiction. Hence, $N(x_n, x_{n-1}) = d(x_n, x_{n-1})$ for all $n \geq 1$. So by (15) and (16), we have for all $n \geq 1$,

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(x_n, x_{n-1}) - \varphi(d(x_n, x_{n-1})) \\ &< d(x_n, x_{n-1}). \end{aligned} \tag{18}$$

Thus, we conclude that the nonnegative sequence $\{d(x_{n+1}, x_n)\}$ is decreasing. Therefore, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$. By using (18), we find that

$$\begin{aligned} 0 &\leq \varphi(d(x_n, x_{n-1})) \\ &\leq d(x_n, x_{n-1}) - d(x_{n+1}, x_n). \end{aligned} \tag{19}$$

Taking $n \to \infty$ in (19), we get $\lim_{n\to\infty} \varphi(d(x_n, x_{n-1})) = 0$. Since $\varphi \in \Phi$, we obtain that $\lim_{n\to\infty} d(x_n, x_{n-1}) = 0$; that is, $r = 0$.

We prove that the iterative sequence $\{x_n\}$ is Cauchy. Take $m > n$, then $x_n \preceq x_m$. From (12), we obtain that

$$d(x_{m+1}, x_{n+1}) \leq N(x_m, x_n) - \varphi(N(x_m, x_n)), \tag{20}$$

and thus,

$$0 \leq \varphi(N(x_m, x_n)) \leq N(x_m, x_n) - d(x_{m+1}, x_{n+1}), \tag{21}$$

where

$$\begin{aligned} N(x_m, x_n) = \max &\left\{ d(x_m, x_n), d(x_m, x_{m+1}), d(x_n, x_{n+1}), \right. \\ &\left. \frac{d(x_m, x_{n+1}) + d(x_n, x_{m+1})}{2} \right\} \\ \leq\ & d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_n). \end{aligned} \tag{22}$$

Hence, by (21),

$$0 \leq \varphi(N(x_m, x_n)) \leq d(x_m, x_{m+1}) + d(x_{n+1}, x_n). \tag{23}$$

This shows that $\lim_{m,n\to\infty} \varphi(N(x_m, x_n)) = 0$; that is, $\{x_n\}$ is Cauchy. Since (X, d) is a complete metric space, then there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, we prove that x is a fixed point of f.

If (a) holds, that is, if f is continuous, then

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(x). \tag{24}$$

Suppose that (b) holds. By using (12), we derive that

$$0 \leq \varphi(N(x_n, x)) \leq N(x_n, x) - d(x_{n+1}, f(x)), \tag{25}$$

where

$$\begin{aligned} N(x_n, x) = \max &\left\{ d(x_n, x), d(x_n, x_{n+1}), d(x, f(x)), \right. \\ &\left. \frac{d(x_n, f(x)) + d(x, x_{n+1})}{2} \right\}. \end{aligned} \tag{26}$$

So $\lim_{n\to\infty} N(x_n, x) = d(x, f(x))$. Taking $n \to \infty$ in (25), we get $\lim_{n\to\infty} \varphi(N(x_n, x)) = 0$. Since $\varphi \in \Phi$, we conclude that $\lim_{n\to\infty} N(x_n, x) = 0$. So $d(x, f(x)) = 0$ and hence $x = f(x)$.

Now, we show that this fixed point x of the self-mapping f is unique. If for each $x, y \in F(f)$, x and y are comparable, then the fixed point is unique. Let x, y be two fixed points of f. Then $N(x, y) = d(x, y)$ and from (12), we conclude that $\varphi(d(x, y)) = 0$. Thus, $d(x, y) = 0$ and hence, $x = y$. This completes the proof. □

The following consequence of Theorem 4 plays a crucial role in the proof of our main result, Theorem 9.

Theorem 5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\varphi \in \Phi$ such that*

$$d(f(x), f(y)) \leq d(x, y) - \varphi(d(x, y)), \tag{27}$$

for each $x, y \in X$ with $x \preceq y$ (i.e., weak-φ-contraction). Suppose also that either

(a) *f is continuous or*

(b) *for every nondecreasing sequence $\{x_n\}$ if $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then f has a fixed point. Moreover, if for each $x, y \in F(f)$ there exists $z \in X$ which is comparable to x and y, then the fixed point of f is unique.

Remark 6. In Theorem 4, if the condition "every $x, y \in F(f)$ is comparable" is replaced by the condition "for each $x, y \in F(f)$ there exists $z \in X$ which is comparable to x and y," then we cannot conclude that the fixed point is unique. The following example illustrates our claim.

Example 7. Let $X = \{x, y, z, w\}$ be endowed with the relation $\preceq$ given as follows:

$$x \preceq z, \qquad x \preceq w, \qquad y \preceq z, \qquad y \preceq w, \tag{28}$$

and $a \preceq a$ for each $a \in X$. Obviously, $(X, \preceq)$ is a partially ordered set. Also, we may endow X with the following metric:

$$\begin{gathered} d(x,z) = d(x,w) = d(y,z) = d(y,w) = d(x,y) = 1, \\ d(z,w) = 2, \end{gathered} \tag{29}$$

and $d(a,a) = 0$ for each $a \in X$. Define $f : X \to X$ by $f(x) = x$, $f(y) = y$, $f(z) = w$, and $f(w) = z$. Obviously, the mapping f is nondecreasing and

$$d(f(a), f(b)) \le d(a,b) - \varphi(d(a,b)), \tag{30}$$

for all $a, b \in X$ with $a \preceq b$, where $\varphi(t) = (1/3)t$. Also $F(f) = \{x, y\}$, but $x \preceq z$ and $y \preceq z$.

Remark 8. If $\beta \in \mathcal{S}$, then $\varphi(t) = t - \beta(t)t \in \Phi$. But if $\varphi \in \Phi$, then we can not conclude that the function

$$\beta(t) = \begin{cases} 1 - \dfrac{\varphi(t)}{t}, & t > 0 \\ 0, & t = 0 \end{cases} \tag{31}$$

belongs to $\mathcal{S}$. Consider, for example,

$$\varphi(t) = \begin{cases} \dfrac{1}{2}t, & 0 \le t < 1 \\ \dfrac{1}{2}, & 1 \le t. \end{cases} \tag{32}$$

which illustrates our claim. As a result, Theorem 5 is a proper extension of Theorem 3.

4. Applications

4.1. Solving a Boundary Value Problem. In this paragraph, we prove the existence of a solution of the problem (1).

Theorem 9. *Consider problem (1) with f being continuous. Suppose that there exists $\lambda > 0$ such that for $x, y \in \mathbb{R}$ with $y \ge x$*

$$\begin{aligned} 0 &\le f(t,y) + \lambda y - [f(t,x) + \lambda x] \\ &\le \lambda[(y-x) - \varphi(y-x)], \end{aligned} \tag{33}$$

where $\varphi \in \Phi$ and $t \mapsto t - \varphi(t)$ is nondecreasing. Then the existence of a lower solution for (1) provides the existence of a unique solution for (1).

Proof. Define $\zeta = \zeta_0/T$. Then, problem (1) becomes as follows

$$\begin{gathered} u'(t) = f(t, u(t)) \quad \text{if } t \in [0,T], \\ u(0) = u(T) + \zeta T. \end{gathered} \tag{34}$$

Suppose $y(t) = u(t) + \zeta t$. So $y'(t) = u'(t) + \zeta$ and hence problem (34) can be rewritten as

$$\begin{gathered} y'(t) = h(t, y(t)) \quad \text{if } t \in [0,T], \\ y(0) = y(T), \end{gathered} \tag{35}$$

where $h : I \times \mathbb{R} \to \mathbb{R}$ is defined by $h(t,z) = f(t, z - \zeta t) + \zeta$ and $I = [0,T]$. Obviously, h is continuous. Also the lower solution of (34) is replaced by the lower solution of (35). Now we prove that the problem (35) has a unique solution. Obviously, if $x, y \in \mathbb{R}$ and $y \ge x$, then for every $t \in I$, $y - \zeta t \ge x - \zeta t$ and hence from (33),

$$\begin{aligned} 0 &\le f(t, y-\zeta t) + \lambda(y - \zeta t) - [f(t, x - \zeta t) + \lambda(x - \zeta t)] \\ &\le \lambda[((y-\zeta t) - (x - \zeta t)) - \varphi((y - \zeta t) - (x - \zeta t))]. \end{aligned} \tag{36}$$

Inequality (36) implies that if $x, y \in \mathbb{R}$,

$$0 \le h(t,y) + \lambda y - [h(t,x) + \lambda x] \le \lambda[(y-x) - \varphi(y-x)]. \tag{37}$$

Problem (35) is equivalent to the following integral equation:

$$y(t) = \int_0^T G(t,s)[h(s, y(s)) + \lambda y(s)]\, ds, \tag{38}$$

where

$$G(t,s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda t} - 1}, & 0 \le s < t \le T, \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda t} - 1}, & 0 \le t < s \le T. \end{cases} \tag{39}$$

Let $C(I, \mathbb{R})$ be the set of continuous functions defined on $I = [0,T]$. Consider $F : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ given by

$$(Fy)(t) = \int_0^T G(t,s)[h(s,y(s)) + \lambda y(s)]\, ds. \tag{40}$$

Note that if $y \in C(I,\mathbb{R})$ is a fixed point of F, then $y \in C^1(I,\mathbb{R})$ is a solution of (35). Now, we check that hypotheses of Theorem 5 are satisfied.

Take $X = C(I, \mathbb{R})$. The space X can be equipped with a partial order $\leq$ given by

$$x, y \in C(I, \mathbb{R}), \qquad x \leq y \Longleftrightarrow x(t) \leq y(t), \quad \forall t \in I. \tag{41}$$

Also, X can be equipped with the following metric:

$$x, y \in C(I, \mathbb{R}), \qquad d(x, y) = \sup_{t \in I} |x(t) - y(t)|. \tag{42}$$

We have that (X, d) is complete. For every $y \geq x$ and for every $t \in I$, we have $y - t\zeta \geq x - t\zeta$ and by hypothesis,

$$f(t, y - t\zeta) + \lambda(y - t\zeta) \geq f(t, x - t\zeta) + \lambda(x - t\zeta). \tag{43}$$

Therefore,

$$h(t, y) + \lambda y \geq h(t, x) + \lambda x, \tag{44}$$

and since $G(t, s) > 0$ for $(t, s) \in I \times I$, hence

$$(Fy)(t) \geq (Fx)(t), \tag{45}$$

for all $x, y \in C(I, \mathbb{R})$ with $y \geq x$.

Also, for all $x, y \in C(I, \mathbb{R})$ with $y \geq x$, we find (using the fact that $t \mapsto t - \varphi(t)$ is nondecreasing)

$$\begin{aligned}
&d(Fy, Fx) \\
&\quad = \sup_{t \in I} |(Fy)(t) - (Fx)(t)| \\
&\quad \leq \sup_{t \in I} \int_0^T G(t, s) \\
&\qquad\qquad \times |h(s, y(s)) + \lambda y(s) - h(s, x(s)) - \lambda x(s)| \, ds \\
&\quad \leq \sup_{t \in I} \int_0^T G(t, s) \lambda |(y(s) - x(s)) - \varphi(y(s) - x(s))| \, ds \\
&\quad \leq \lambda [d(y, x) - \varphi(d(y, x))] \sup_{t \in I} \int_0^T G(t, s) \, ds \\
&\quad = \lambda [d(y, x) - \varphi(d(y, x))] \\
&\qquad \times \sup_{t \in I} \frac{1}{e^{\lambda T} - 1} \left(\frac{1}{\lambda} e^{\lambda(T+s-t)} \Big]_0^t + \frac{1}{\lambda} e^{\lambda(s-t)} \Big]_t^T \right) \\
&\quad = \lambda [d(y, x) - \varphi(d(y, x))] \frac{1}{\lambda(e^{\lambda T} - 1)} \left(e^{\lambda T} - 1\right) \\
&\quad = d(y, x) - \varphi(d(y, x)).
\end{aligned} \tag{46}$$

Finally, let $\alpha(t)$ be a lower solution for (35). We can show that $\alpha \leq F\alpha$ by a method similar to that in [1, 2]. Also, X is totally ordered. Hence, due to Theorem 5, F has a unique fixed point. Therefore, problem (35) has a unique solution $y \in C^1(I, \mathbb{R})$. Thus, $x(t) = y(t) - \zeta t$ is the unique solution of (34) and this completes the proof. □

Remark 10. If the mapping $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies the condition (33), then for $x, y \in \mathbb{R}$ with $y \geq x$ and for $t \in [0, T]$,

$$-\lambda(y - x) \leq f(t, y) - f(t, x) \leq -\lambda\varphi(y - x) \leq 0. \tag{47}$$

Hence, for all $x, y \in \mathbb{R}$ and all $t \in [0, T]$, we have

$$|f(t, y) - f(t, x)| \leq \lambda |y - x|. \tag{48}$$

Therefore, by using Banach contraction principle, for every $\eta \in \mathbb{R}$, the problem

$$\begin{gathered} u'(t) = f(t, u(t)) \quad \text{if } t \in [0, T] \\ u(0) = \eta \end{gathered} \tag{49}$$

has a unique solution $u_\eta \in C^1([0, T])$. So there exists a unique $\eta \in \mathbb{R}$ such that u_η is a solution of (1).

Now let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a mapping such that for all $x, y \in \mathbb{R}$ and all $t \in [0, T]$,

$$|f(t, y) - f(t, x)| \leq R |y - x|, \tag{50}$$

for some $R > 0$. We know that for every $\eta \in \mathbb{R}$, problem (49) has a unique solution $u_\eta \in C^1([0, T])$.

Question 1. It is natural to ask whether there is an $\eta \in \mathbb{R}$ where u_η is a solution of problem ((2), i.e., $(u_\eta(0) = u_\eta(T))$?

The following example shows that the above question is not true.

Example 11. Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be defined by $f(t, x) = t + |x|$. Obviously, (50) holds for $R = 1$. Let $u \in C^1([0, T])$ be a solution for problem (2). From $u'(t) = f(t, u(t)) = t + |u(t)|$, we conclude that $u'(t) > 0$ for all $t > 0$. Hence, u is monotone nondecreasing. Using $u(0) = u(T)$, we conclude that $u \equiv 0$. Since $u'(t) = f(t, u(t)) = t + |u(t)|$ and $u \equiv 0$, then $t = 0$ for all $t \in [0, T]$ and this is a contradiction. So, problem (2) has no solution.

Example 12. Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be defined by $f(t, x) = \exp(t) - (1/2)x$ and let $\varphi : [0, +\infty) \to [0, +\infty)$ be defined by $\varphi(t) = (1/3)t$. Take $\lambda = 1$. One can show that inequality (33) holds. Suppose that $\alpha : \mathbb{R} \to \mathbb{R}$ is defined by $\alpha(t) = 0$. Obviously, α is a lower solution of problem (2). Hence, problem (2) has a unique solution, which is

$$u(t) = \frac{2}{3} \exp(t) + C \exp\left(\frac{-1}{2} t\right), \tag{51}$$

where $C = 2(\exp(T) - 1)/3(1 - \exp((-1/2)t))$.

4.2. Solving Some Polynomials. In this paragraph, we prove the existence and uniqueness of a solution of some polynomials.

Theorem 13. *Let $a_0, a_1, \ldots, a_{k-1} \in [0, +\infty)$ be such that $a_1 + a_2 + \cdots + a_{k-1} < 1$ and $a_0 \geq 1$. Then,*

$$y^k = a_{k-1} y^{k-1} + a_{k-2} y^{k-2} + \cdots + a_1 y + a_0 \tag{52}$$

has a unique solution on $[\sqrt[k]{a_0}, +\infty)$.

Proof. Suppose that $f : [a_0, +\infty) \to [a_0, +\infty)$ is defined by

$$f(x) = a_{k-1}\sqrt[k]{x^{k-1}} + a_{k-2}\sqrt[k]{x^{k-2}} + \cdots + a_1\sqrt[k]{x} + a_0. \quad (53)$$

If $x \le y$, then $f(x) \le f(y)$. So f is nondecreasing. Also for $x, y \in [a_0, +\infty)$ with $x \le y$, we derive that

$$\begin{aligned} 0 &\le f(y) - f(x) \\ &= a_{k-1}\left(\sqrt[k]{y^{k-1}} - \sqrt[k]{x^{k-1}}\right) + a_{k-2}\left(\sqrt[k]{y^{k-2}} - \sqrt[k]{x^{k-2}}\right) \\ &\quad + \cdots + a_1\left(\sqrt[k]{y} - \sqrt[k]{x}\right). \end{aligned} \quad (54)$$

Suppose that $g_i : [1, +\infty) \to \mathbb{R}$ is defined by $g_i(t) = t - t^{1-i/k}$, for $i = 1, 2, \ldots, k-1$. Since $g_i'(t) = 1 - (1 - i/k)1/t^{i/k} \ge 0$, then g_i is monotone nondecreasing. Hence, if $1 \le x \le y$, then $g_i(x) \le g_i(y)$. So, $y^{1-i/k} - x^{1-i/k} \le y - x$. Therefore, from (54), we get

$$0 \le f(y) - f(x) \le (y - x) - \varphi(y - x), \quad (55)$$

where $\varphi(t) = [1 - (a_{k-1} + a_{k-2} + \cdots + a_1)]t$. Also $a_0 \le f(a_0)$. Thus, using Theorem 5, the mapping f has a unique fixed point $x \in [a_0, +\infty)$. Moreover, the sequence $\{f^n(a_0)\}$ converges to this fixed point. Note that here the space X is taken to be $[a_0, +\infty)$, which is equipped with the usual Euclidian metric and the usual partial order.

On the other hand, there exists a unique $y \in [\sqrt[k]{a_0}, +\infty)$ such that $y^k = x$. So, from $x = f(x)$, we have $y^k = f(y^k)$ and therefore we find

$$y^k = a_{k-1}y^{k-1} + a_{k-2}y^{k-2} + \cdots + a_1 y + a_0. \quad (56)$$

Also the sequence $\{\sqrt[k]{f^n(a_0)}\}$ converges to y and this completes the proof. □

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