Research Article

Convergence Behavior for Newton-Steffensen's Method under γ-Condition of Second Derivative

Yonghui Ling,[1] Xiubin Xu,[2] and Shaohua Yu[2]

[1] *Department of Mathematics, Zhejiang University, Hangzhou 310027, China*
[2] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China*

Correspondence should be addressed to Xiubin Xu; xxu@zjnu.edu.cn

The present paper is concerned with the semilocal as well as the local convergence problems of Newton-Steffensen's method to solve nonlinear operator equations in Banach spaces. Under the assumption that the second derivative of the operator satisfies γ-condition, the convergence criterion and convergence ball for Newton-Steffensen's method are established.

1. Introduction

Let X and Y be real or complex Banach spaces, let $D \subset X$ be an open subset, and let $F : D \subset X \to Y$ be the Fréchet differentiable nonlinear operator. Approximating a solution of a nonlinear equation

$$F(x) = 0 \tag{1}$$

is widely studied in both theoretical and applied areas of mathematics.

One of the most famous methods to solve this problem is Newton's method defined by

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n), \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $x_0 \in D$ is an initial point. Usually, the study about convergence issue of Newton's method includes local and semilocal convergence analyses. The local convergence issue is, based on the information around a solution, to seek estimates of the radii of convergence balls, while the semilocal one is, based on the information around an initial point, to give criteria ensuring the convergence. Among the semilocal convergence results on Newton's method, one of the famous results is Smale's point estimate theory which gives a convergence criterion of Newton's method only based on the information at the initial point for analytic functions; see for example, [1–6]. To extend and improve Smale's theory,

Wang and Han proposed in [7, 8] the notion of γ-condition, which is weaker than Smale's assumption in [5] for analytic operators.

There are several kinds of cubic generalizations for Newton's method. The most important family is the Euler-Halley family and its variations which include Chebyshev's method and Halley's method as special cases; see for example, [9–16] and references therein. However, the disadvantage of this family is that the evaluation of the second derivative of the operator F is required at every step, the operation cost of which may be very large in fact. To reduce the operation cost but also retain the cubic convergence, Sharma in [17] proposed the following Newton-Steffensen's method which avoids the computation of the second Fréchet derivative. Let $f : \mathbb{R} \to \mathbb{R}$. The method is defined as follows:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} &= x_n - \frac{f(x_n)}{g(x_n)}, \quad n = 0, 1, \ldots,
\end{aligned}
\tag{3}
$$

where $g(x_n) = ((f(y_n) - f(x_n))/(y_n - x_n))$. The author obtained cubic convergence for (3) under the assumption that f is sufficiently smooth in the neighborhood of the solution.

Motivated by the work mentioned above, we extend this method to Banach spaces and present its semilocal and local convergence. The extension is described as follows:

$$y_n = x_n - F'(x_n)^{-1} F(x_n),$$

$$x_{n+1} = x_n - [y_n, x_n; F]^{-1} F(x_n), \quad n = 0, 1, \ldots, \tag{4}$$

where the divided difference operator is defined by

$$[y_n, x_n; F] = \int_0^1 F'(x_n + t(y_n - x_n)) \, dt. \tag{5}$$

In Section 2, we introduce some preliminary notions and important majorizing functions with properties. In Sections 3 and 4, we study the semilocal convergence and local convergence results of Newton-Steffensen's method under γ-condition, respectively. We obtain the uniqueness ball and the convergence ball.

2. Notations and Preliminary Results

Throughout this paper, we assume that X and Y are two Banach spaces. Let $D \subset X$ be an open subset and let $F : D \subset X \to Y$ be a nonlinear operator with the continuous twice Fréchet derivative. For $x \in X$ and $r > 0$, we use $\mathbf{B}(x, r)$ and $\overline{\mathbf{B}}(x, r)$ to denote the open ball with radius r and center x and its closure, respectively. Let $\overline{x} \in D$ be such that $F'(\overline{x})^{-1}$ exists and $\overline{\mathbf{B}}(\overline{x}, r) \subset D$.

Let γ be some positive constant and $0 < r \leqslant (1/\gamma)$. We say that F satisfies γ-condition on $\mathbf{B}(\overline{x}, r)$ if the following relation holds:

$$\left\| F'(\overline{x})^{-1} F''(x) \right\| \leqslant \frac{2\gamma}{(1 - \gamma \|x - \overline{x}\|)^3}, \quad x \in \mathbf{B}(\overline{x}, r). \tag{6}$$

For simplicity, we write

$$r_0 = \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{\gamma}. \tag{7}$$

The lemma below is useful in the next two sections.

Lemma 1. *Suppose that $r \leqslant r_0$ and that F satisfies γ-condition (6) on $\mathbf{B}(\overline{x}, r)$. Then for any $x \in \mathbf{B}(\overline{x}, r)$, $F'(x)^{-1}$ exists and the following inequality holds:*

$$\left\| F'(x)^{-1} F'(\overline{x}) \right\| \leqslant \frac{(1 - \gamma \|x - \overline{x}\|)^2}{2(1 - \gamma \|x - \overline{x}\|)^2 - 1}. \tag{8}$$

Proof. We can derive the following relation:

$$I - F'(\overline{x})^{-1} F'(x)$$

$$= -\int_0^1 F'(\overline{x})^{-1} F''(\overline{x} + t(x - \overline{x}))(x - \overline{x}) \, dt. \tag{9}$$

For any $x \in B(\overline{x}, r)$, it follows from γ-condition and $r \leqslant r_0$ that

$$\left\| I - F'(\overline{x})^{-1} F'(x) \right\| \leqslant \int_0^1 \frac{2\gamma \|x - \overline{x}\|}{(1 - \gamma \|x - \overline{x}\| t)^3} \, dt$$

$$= \frac{1}{(1 - \gamma \|x - \overline{x}\|)^2} - 1 < 1. \tag{10}$$

Then, by Banach lemma, one has that $F'(x)^{-1}$ exists and the following inequality holds:

$$\left\| F'(x)^{-1} F'(\overline{x}) \right\| \leqslant \frac{1}{1 - \left[\left(1/(1 - \gamma \|x - \overline{x}\|)^2\right) - 1\right]}$$

$$= \frac{(1 - \gamma \|x - \overline{x}\|)^2}{2(1 - \gamma \|x - \overline{x}\|)^2 - 1}. \tag{11}$$

□

Let $\beta > 0$ be some positive constant. The following majorizing function h introduced by Wang and Han in [18] will be used to obtain a Smale-type semilocal convergence criterion:

$$h(t) = \beta - t + \frac{\gamma t^2}{1 - \gamma t}, \quad t \in \left[0, \frac{1}{\gamma}\right). \tag{12}$$

Let $\{s_n\}$ and $\{t_n\}$ denote the corresponding sequences generated by Newton-Steffensen's method for the majorizing function h with the initial point $t_0 = 0$; that is,

$$s_n = t_n - \frac{h(t_n)}{h'(t_n)},$$

$$t_{n+1} = t_n - \left(\frac{h(s_n) - h(t_n)}{s_n - t_n}\right)^{-1} h(t_n), \quad n = 0, 1, \ldots. \tag{13}$$

The following lemma taken from [19] describes some useful properties about h.

Lemma 2. *Suppose that*

$$\alpha := \beta\gamma \leqslant 3 - 2\sqrt{2}. \tag{14}$$

Then h has two zeros in $[0, 1/\gamma)$ denoted by t^* and t^{**}. They satisfy the following relations:

$$\beta \leqslant t^* \leqslant r_0 \leqslant t^{**} \leqslant \frac{1}{2\gamma}. \tag{15}$$

Moreover, h is decreasing monotonically in interval $[0, r_0]$, while it is increasing monotonically in interval $[r_0, 1/(2\gamma)]$.

The lemma below describes the convergence property of the sequences $\{s_n\}$ and $\{t_n\}$, which is crucial for the semilocal convergence analysis of Newton-Steffensen's method (4) under γ-condition.

Lemma 3. *Suppose that (14) holds. Let $\{s_n\}$ and $\{t_n\}$ be the sequences generated by (13). Then*

$$0 \leqslant t_n < s_n < t_{n+1} < t^* \quad \forall n \geqslant 0. \tag{16}$$

Moreover, $\{s_n\}$ and $\{t_n\}$ converge increasingly to the same point t^*.

Proof. To show that (16) holds for $n = 0$, we note that $0 = t_0 < s_0 = \beta$ and that $t_1 = (\beta(1 - \gamma\beta)/(1 - 2\gamma\beta))$. By (15), we have

$$\frac{1 - \gamma\beta}{1 - 2\gamma\beta} > 1. \tag{17}$$

This implies that $t_1 > \beta = s_0$. It remains to show that $t_1 < t^*$ for the case $n = 0$. To this end, we define a real function as

$$\Phi(t) = 1 - \frac{\gamma t}{1 - \gamma t}, \quad t \in [0, +\infty). \tag{18}$$

It is clear that $\Phi(t) = -(h(t) - \beta)/t$ and that $\Phi(t)$ is decreasing monotonically in $[0, +\infty)$. It follows from (15) that $\Phi(\beta) > \Phi(t^*)$. In view of the fact that t^* is the unique zero of h in $[0, r_0]$, we obtain $\beta/t^* = \Phi(t^*) < \Phi(\beta)$. This is equivalent to

$$t^* > \frac{\beta}{1 - (\gamma\beta/(1 - \gamma\beta))} = t_1. \tag{19}$$

Hence (16) holds for $n = 0$.

Now we assume that

$$0 \leqslant t_{n-1} < s_{n-1} < t_n < t^* \quad \text{for some } n \geqslant 1. \tag{20}$$

From Lemma 2, we have $h(t) \geqslant 0$, for each $t \in [0, t^*]$, and $h(t_n)/h'(t_n) < 0$. The later one implies that $s_n > t_n$. Define function

$$N(t) = t - \frac{h(t)}{h'(t)}, \quad t \in [0, t^*]. \tag{21}$$

Then, $N'(t) = h(t)h''(t)/h'(t)^2 > 0$, which implies that N is increasing monotonically in $[0, t^*]$. Hence, we have

$$s_n = t_n - \frac{h(t_n)}{h'(t_n)} < t^* - \frac{h(t^*)}{h'(t^*)} = t^*. \tag{22}$$

Since h is convex in $[0, t^*]$, we get $h'(t_n) < (h(s_n) - h(t_n))/(s_n - t_n)$ and so $s_n < t_{n+1}$.

Furthermore, we claim that

$$\begin{aligned} t' &- \left(\frac{h(t) - h(t')}{t - t'} \right)^{-1} h(t') \\ &< t'' - \left(\frac{h(t'') - h(t)}{t'' - t} \right)^{-1} h(t'') \end{aligned} \tag{23}$$

for all $t', t, t'' \in [0, t^*]$ and $t' < t < t''$. Indeed, it follows from the convexity of h that

$$\begin{aligned} -\frac{h(t'')}{h'(t)} &\leqslant -\left(\frac{h(t'') - h(t)}{t'' - t} \right)^{-1} h(t'') \\ &\leqslant -\frac{h(t'')}{h'(t'')}, \\ -\frac{h(t')}{h'(t')} &\leqslant -\left(\frac{h(t) - h(t')}{t - t'} \right)^{-1} h(t') \\ &\leqslant -\frac{h(t')}{h'(t)}, \end{aligned} \tag{24}$$

from which we have

$$\begin{aligned} (t'' - t') + \frac{h(t') - h(t'')}{h'(t)} &\leqslant (t'' - t') + (T'' - T') \\ &\leqslant (t'' - t') - \frac{h(t'')}{h'(t'')} + \frac{h(t')}{h'(t')}, \end{aligned} \tag{25}$$

where

$$T' = -\left(\frac{h(t) - h(t')}{t - t'} \right)^{-1} h(t'), \tag{26}$$

$$T'' = -\left(\frac{h(t'') - h(t)}{t'' - t} \right)^{-1} h(t'').$$

Noting that $-1 < h'(t) < 0$ for all $t \in [0, t^*]$, we obtain

$$(t'' - t') + \frac{h(t') - h(t'')}{h'(t)} > (t'' - t') + [h(t'') - h(t')]$$

$$\geqslant 0. \tag{27}$$

Then (23) follows. By (23), we conclude that

$$\begin{aligned} t_{n+1} &= t_n - \left(\frac{h(s_n) - h(t_n)}{s_n - t_n} \right)^{-1} h(t_n) \\ &< t^* - \left(\frac{h(t^*) - h(s_n)}{t^* - s_n} \right)^{-1} h(t^*) = t^*. \end{aligned} \tag{28}$$

Therefore, (16) holds for all $n \geqslant 0$. The inequlities in (16) imply that $\{s_n\}$ and $\{t_n\}$ converge increasingly to some same points, say τ. Clearly $\tau \in [0, t^*]$ and τ is a zero of h in $[0, t^*]$. Noting that t^* is the unique zero of h in $[0, r_0]$, one has that $\tau = t^*$. The proof is complete. $\quad \square$

3. Convergence Criterion

Throughout this subsection, let $x_0 \in D$ be the initial point such that the inverse $F'(x_0)^{-1}$ exists and let $\mathbf{B}(x_0, r_0) \subset D$, where r_0 is defined by (7). Moreover, we assume that F satisfies γ-condition on $\mathbf{B}(x_0, r_0)$; that is, the following relation holds:

$$\left\| F'(x_0)^{-1} F''(x) \right\| \leqslant \frac{2\gamma}{\left(1 - \gamma \left\| x - x_0 \right\| \right)^3}, \tag{29}$$

$$x \in \mathbf{B}(x_0, r_0).$$

Then, for any $x \in \mathbf{B}(x_0, r_0)$, it follows from Lemma 1 that $F'(x)^{-1}$ exists and the following inequality holds:

$$\left\| F'(x)^{-1} F'(x_0) \right\| \leqslant \frac{\left(1 - \gamma \left\| x - x_0 \right\| \right)^2}{2 \left(1 - \gamma \left\| x - x_0 \right\| \right)^2 - 1}. \tag{30}$$

Below we list two useful lemmas.

Recall that the divided difference operator $[y, x; F]$ is defined by (5). The following lemma gives the expressions of some desired estimates in the proof of Lemma 5.

Lemma 4. Let $x \in \mathbf{B}(x_0, r_0)$. Define

$$y := x - F'(x)^{-1} F(x), \qquad \overline{x} := x - [y, x; F]^{-1} F(x). \tag{31}$$

Then the following formulas hold:

(i) $[y, x; F] - F'(x_0) = \int_0^1 \int_0^1 F''(x_0 + s[(x - x_0) + t(y - x)])[(x - x_0) + t(y - x)] \, ds \, dt$.

(ii) $\overline{x} - y = -F'(x)^{-1} \int_0^1 \int_0^1 F''(x + st(y - x))(y - x)(\overline{x} - x) t \, ds \, dt$.

(iii) $F(\overline{x}) = \int_0^1 \int_0^1 F''(x + t(y - x) + st(\overline{x} - y))(\overline{x} - x)(\overline{x} - y) t \, ds \, dt$.

Proof. For (i), we notice that

$$[y, x; F] - F'(x_0) = \int_0^1 \left[F'(x + t(y - x)) - F'(x_0) \right] dt$$

$$= \int_0^1 \int_0^1 F''(x_0 + s[(x - x_0) + t(y - x)])$$

$$\times [(x - x_0) + t(y - x)] \, ds \, dt. \tag{32}$$

As for (ii), one has

$$\overline{x} - y = F'(x)^{-1} F(x) - [y, x; F]^{-1} F(x)$$

$$= F'(x)^{-1} \left[F'(x) - \int_0^1 F'(x + t(y - x)) dt \right]$$

$$\times (\overline{x} - x)$$

$$= -F'(x)^{-1}$$

$$\times \int_0^1 \int_0^1 F''(x + st(y - x))(y - x)(\overline{x} - x) t \, ds \, dt. \tag{33}$$

Similarly, we obtain

$$F(\overline{x}) = F(\overline{x}) - F(x) - [y, x; F](\overline{x} - x)$$

$$= \int_0^1 F'(x + t(\overline{x} - x))(\overline{x} - x) dt$$

$$- \int_0^1 F'(x + t(y - x))(\overline{x} - x) dt \tag{34}$$

$$= \int_0^1 \int_0^1 F''(x + t(y - x) + st(\overline{x} - y))$$

$$\times (\overline{x} - x)(\overline{x} - y) t \, ds \, dt.$$

The proof is complete. $\square$

Lemma 5. Suppose that (14) holds. Then the sequence $\{x_n\}$ generated by (4) with the initial point x_0 is well defined and the following estimates hold for any natural number $n \geqslant 1$:

(i) $\| y_{n-1} - x_{n-1} \| \leqslant s_{n-1} - t_{n-1}$, $\| x_n - x_{n-1} \| \leqslant t_n - t_{n-1}$, $\| x_n - y_{n-1} \| \leqslant t_n - s_{n-1}$.

(ii) $\| [y_{n-1}, x_{n-1}; F]^{-1} F'(x_0) \| \leqslant -((s_{n-1} - t_{n-1})/(h(s_{n-1}) - h(t_{n-1})))$.

(iii) $\| F'(x_0)^{-1} F(x_n) \| \leqslant h(t_n)(\| x_n - x_{n-1} \|/(t_n - t_{n-1}))^2 (\| y_{n-1} - x_{n-1} \|/(s_{n-1} - t_{n-1}))$.

Proof. For the case $n = 1$ in (i), it is clear that $\| y_0 - x_0 \| \leqslant \beta = s_0 - t_0$. By Lemma 4 and (29), we have

$$\left\| F'(x_0)^{-1} \left([y_0, x_0; F] - F'(x_0) \right) \right\|$$

$$= \left\| F'(x_0)^{-1} \right.$$

$$\times \left. \int_0^1 \int_0^1 F''(x_0 + st(y_0 - x_0))(y_0 - x_0) t \, ds \, dt \right\|$$

$$\leqslant \int_0^1 \int_0^1 \frac{2\gamma(s_0 - t_0) t}{\left[1 - \gamma(s_0 - t_0) st \right]^3} \, ds \, dt$$

$$= \frac{h(s_0) - h(t_0)}{s_0 - t_0} + 1. \tag{35}$$

In view of the monotonicity of h, one has that $(h(s_0) - h(t_0))/(s_0 - t_0) < 0$. Hence, we get from Banach lemma that $[y_0, x_0; F]^{-1}$ exists and satisfies

$$\left\| [y_0, x_0; F]^{-1} F'(x_0) \right\|$$

$$\leqslant \frac{1}{1 - (((h(s_0) - h(t_0))/(s_0 - t_0)) + 1)} \tag{36}$$

$$= -\frac{s_0 - t_0}{h(s_0) - h(t_0)}.$$

Combining (36) inequality with the definitions of $\{s_n\}$ and $\{t_n\}$ given in (13), one has

$$\|x_1 - x_0\| \le \|[y_0, x_0; F]^{-1} F'(x_0)\| \|F'(x_0)^{-1} F(x_0)\|$$
$$\le -\frac{s_0 - t_0}{h(s_0) - h(t_0)} h(t_0) \tag{37}$$
$$= t_1 - t_0.$$

As for the estimate $\|x_1 - y_0\|$, by Lemma 4, we have

$$x_1 - y_0 = -F'(x_0)^{-1}$$
$$\times \int_0^1 \int_0^1 F''(x_0 + st(y_0 - x_0)) \tag{38}$$
$$\times (y_0 - x_0)(x_1 - x_0) t \, ds \, dt.$$

This together with the obtained bounds $\|y_0 - x_0\|$, $\|x_1 - x_0\|$ and (29) yields that

$$\|x_1 - y_0\| \le \int_0^1 \int_0^1 \frac{2\gamma(s_0 - t_0)(t_1 - t_0)}{(1 - \gamma(s_0 - t_0)st)^3} t \, ds \, dt$$
$$\times \left(\frac{\|y_0 - x_0\|}{s_0 - t_0}\right)\left(\frac{\|x_1 - x_0\|}{t_1 - t_0}\right) \tag{39}$$
$$= (t_1 - s_0)\left(\frac{\|y_0 - x_0\|}{s_0 - t_0}\right)\left(\frac{\|x_1 - x_0\|}{t_1 - t_0}\right).$$

This implies that statement (i) holds for $n = 1$.

Statement (ii) for the case $n = 1$ is justified by (36). Below, we consider the case $n = 1$ for (iii). First we have the following expression of $F(x_1)$ due to Lemma 4:

$$F(x_1) = \int_0^1 \int_0^1 F''(x_0 + t(y_0 - x_0) + st(x_1 - y_0))$$
$$\times (x_1 - x_0)(x_1 - y_0) t \, ds \, dt, \tag{40}$$

from which we obtain that

$$\|F'(x_0)^{-1} F(x_1)\|$$
$$\le \int_0^1 \int_0^1 \frac{2\gamma(t_1 - t_0)(t_1 - s_0) t}{[1 - \gamma(t(s_0 - t_0) + st(t_1 - s_0))]^3} ds \, dt$$
$$\times \left(\frac{\|x_1 - x_0\|}{t_1 - t_0}\right)^2 \left(\frac{\|y_0 - x_0\|}{s_0 - t_0}\right) \tag{41}$$
$$= h(t_1)\left(\frac{\|x_1 - x_0\|}{t_1 - t_0}\right)^2 \left(\frac{\|y_0 - x_0\|}{s_0 - t_0}\right).$$

Therefore statement (iii) holds for $n = 1$.

Assume that statements (i)–(iii) are true for $n = k (\ge 1)$. Below, we will show that they also hold for $n = k + 1$. First, by statement (i), we have

$$\|x_k - x_0\| \le \sum_{i=0}^{k-1} \|x_{i+1} - x_i\| \le \sum_{i=0}^{k-1} (t_{i+1} - t_i) \tag{42}$$
$$= t_k < t^* < r_0.$$

Hence, $F'(x_k)^{-1}$ exists by Lemma 1.

Note that

$$\|F'(x_0)^{-1} F(x_k)\| \le h(t_k) \tag{43}$$

by the inductive hypotheses of (i) and (iii). Then it follows from (30) and (13) that

$$\|y_k - x_k\| \le \|F'(x_k)^{-1} F'(x_0)\| \|F'(x_0)^{-1} F(x_k)\|$$
$$\le -\frac{h(t_k)}{h'(t_k)} = s_k - t_k. \tag{44}$$

Hence by (29), (44), Lemma 4, and the inductive hypothesis of (i), we have

$$\|F'(x_0)^{-1}([y_k, x_k; F] - F'(x_0))\|$$
$$= \left\|F'(x_0)^{-1} \int_0^1 \int_0^1 F''(x_0 + s[(x_k - x_0) + t(y_k - x_k)])\right.$$
$$\left. \times [(x_k - x_0) + t(y_k - x_k)] \, ds \, dt\right\|$$
$$\le \int_0^1 \int_0^1 \frac{2\gamma}{[1 - \gamma s(t_k + t(s_k - t_k))]^3}$$
$$\times [t_k + t(s_k - t_k)] \, ds \, dt$$
$$= \int_0^1 \left(\frac{1}{[1 - \gamma(t_k + t(s_k - t_k))]^2} - 1\right) dt$$
$$= \frac{h(s_k) - h(t_k)}{s_k - t_k} + 1 < 1. \tag{45}$$

It follows from Banach lemma that $[y_k, x_k; F]^{-1} F'(x_0)$ exists and satisfies

$$\|[y_k, x_k; F]^{-1} F'(x_0)\|$$
$$\le \frac{1}{1 - (((h(s_k) - h(t_k))/(s_k - t_k)) + 1)} \tag{46}$$
$$= -\frac{s_k - t_k}{h(s_k) - h(t_k)}.$$

Hence, (ii) holds for $n = k + 1$.

Combining (46) with the inductive hypothesis (iii), one has

$$\|x_{k+1} - x_k\| \leqslant \left\|[y_k, x_k; F]^{-1} F'(x_0)\right\| \left\|F'(x_0)^{-1} F(x_k)\right\|$$

$$\leqslant -\left(\frac{h(s_k) - h(t_k)}{s_k - t_k}\right)^{-1} h(t_k)$$

$$\times \left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}}\right)^2 \left(\frac{\|y_{k-1} - x_{k-1}\|}{s_{k-1} - t_{k-1}}\right) \quad (47)$$

$$= (t_{k+1} - t_k)$$

$$\times \left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}}\right)^2 \left(\frac{\|y_{k-1} - x_{k-1}\|}{s_{k-1} - t_{k-1}}\right),$$

which implies that $\|x_{k+1} - x_k\| \leqslant t_{k+1} - t_k$.

On the other hand, by (29), (30), (44), and Lemma 4, we conclude that

$$\|x_{k+1} - y_k\|$$

$$\leqslant \int_0^1 \int_0^1 \left\|F'(x_k)^{-1} F''(x_k + st(y_k - x_k))\right\|$$

$$\times \|y_k - x_k\| \|x_{k+1} - x_k\| t \, ds \, dt$$

$$\leqslant \left\|F'(x_k)^{-1} F'(x_0)\right\|$$

$$\times \int_0^1 \int_0^1 \frac{2\gamma \|y_k - x_k\| \|x_{k+1} - x_k\| t}{\left[1 - \gamma(\|x_k - x_0\| + st \|y_k - x_k\|)\right]^3} \, ds \, dt$$

$$\leqslant \frac{(1 - \gamma t_k)^2}{2(1 - \gamma t_k)^2 - 1} \quad (48)$$

$$\times \int_0^1 \int_0^1 \frac{2\gamma(s_k - t_k)(t_{k+1} - t_k) t}{\left[1 - \gamma(t_k + st(s_k - t_k))\right]^3} \, ds \, dt$$

$$\times \left(\frac{\|y_k - x_k\| \|x_{k+1} - x_k\|}{(s_k - t_k)(t_{k+1} - t_k)}\right)$$

$$= -\frac{1}{h'(t_k)} (t_{k+1} - t_k) \left(\frac{h(s_k) - h(t_k)}{s_k - t_k} - h'(t_k)\right)$$

$$\times \left(\frac{\|y_k - x_k\| \|x_{k+1} - x_k\|}{(s_k - t_k)(t_{k+1} - t_k)}\right)$$

$$= (t_{k+1} - s_k) \left(\frac{\|y_k - x_k\| \|x_{k+1} - x_k\|}{(s_k - t_k)(t_{k+1} - t_k)}\right),$$

which leads to $\|x_{k+1} - y_k\| \leqslant t_{k+1} - s_k$. Thus, (i) holds for $n = k + 1$.

Next, we will show that (iii) also holds for $n = k + 1$. In fact, by using Lemma 4, (29), (44), and (48), we obtain

$$\left\|F'(x_0)^{-1} F(x_{k+1})\right\|$$

$$\leqslant \int_0^1 \int_0^1 \frac{2\gamma(t_{k+1} - t_k)(t_{k+1} - s_k)}{\left[1 - \gamma(t_k + t(s_k - t_k) + st(t_{k+1} - s_k))\right]^3} t \, ds \, dt$$

$$\cdot \left(\frac{\|x_{k+1} - x_k\|}{t_{k+1} - t_k}\right)^2 \left(\frac{\|y_k - x_k\|}{s_k - t_k}\right)$$

$$= \left(h(t_{k+1}) - h(t_k) - \frac{t_{k+1} - t_k}{s_k - t_k}(h(s_k) - h(t_k))\right)$$

$$\cdot \left(\frac{\|x_{k+1} - x_k\|}{t_{k+1} - t_k}\right)^2 \left(\frac{\|y_k - x_k\|}{s_k - t_k}\right)$$

$$\leqslant h(t_{k+1}) \left(\frac{\|x_{k+1} - x_k\|}{t_{k+1} - t_k}\right)^2 \left(\frac{\|y_k - x_k\|}{s_k - t_k}\right).$$

$$(49)$$

Therefore statement (iii) holds for $n = k + 1$. Hence (i)–(iii) hold for all $n \geqslant 0$. Furthermore, by statement (i), one has, for any $n \geqslant 0$, $\|x_n - x_0\| \leqslant t_n < t^* < r_0$. Thus by Lemma 1 we know that $F'(x_n)^{-1}$ exists for each $n \geqslant 1$; that is, $\{x_n\}$ is well defined. The proof is complete. □

Recall that $\{s_n\}$ and $\{t_n\}$ are defined in (13). Based on the preceding useful lemmas, we are now ready to prove a Smale-type semilocal convergence theorem for Newton-Steffensen's method (4) under γ-condition.

Theorem 6. *Suppose that (14) holds. Then the sequence $\{x_n\}$ generated by (4) with the initial point x_0 is well defined and converges to a solution $x^* \in \overline{\mathbf{B}(x_0, t^*)}$ of (1) with Q-cubic rate, and this solution x^* is unique in $\overline{\mathbf{B}(x_0, r)}$, where $t^* \leqslant r < t^{**}$. Moreover, the following error bounds*

$$\|x^* - x_n\| \leqslant (t^* - t_n) \left(\frac{\|x^* - x_m\|}{t^* - t_m}\right)^{3^{n-m}} \quad \forall n \geqslant m \geqslant 0 \quad (50)$$

are valid, where t^* and t^{**} are defined in Lemma 2.

Proof. The uniqueness ball can be obtained by Theorem 5.2 in [19]. It follows from Lemma 1 that $\{x_n\}$ is well defined. In addition, from Lemmas 3 and 5 (i), we can see that $\{x_n\}$ is convergent to a limit, say x^*. Below, we show that x^* is a solution of (1). It follows from Lemma 5 (iii) that

$$\left\|F'(x_0)^{-1} F(x_n)\right\| \leqslant h(t_n) \quad \forall n \geqslant 0. \quad (51)$$

Letting $n \to \infty$ in the preceding relation gives that the limit x^* is a solution of (1). Moreover, we have

$$\|x^* - x_n\| \leqslant t^* - t_n. \quad (52)$$

Next, we verify that estimate (62) is true. By (29) and Lemma 5, one has

$$
\begin{aligned}
&\| x^* - y_n \| \\
&= \Big\| F'(x_n)^{-1} \\
&\quad \times \Big[\int_0^1 F'(x_n)(x^* - x_n)\, dt + F(x_n) - F(x^*) \Big] \Big\| \\
&= \Big\| F'(x_n)^{-1} \\
&\quad \times \int_0^1 \int_0^1 F''(x_n + st(x^* - x_n))(x^* - x_n)^2 t\, ds\, dt \Big\| \\
&\leqslant \| F'(x_n)^{-1} F'(x_0) \| \\
&\quad \times \Big\| \int_0^1 \int_0^1 F'(x_0)^{-1} F''(x_n + st(x^* - x_n)) \\
&\qquad \times (x^* - x_n)^2 t\, ds\, dt \Big\| \\
&\leqslant \frac{(1 - \gamma \| x_n - x_0 \|)^2}{2(1 - \gamma \| x_n - x_0 \|)^2 - 1} \\
&\quad \times \int_0^1 \int_0^1 \frac{2\gamma (t^* - t_n)^2 t}{[1 - \gamma(\| x_n - x_0 + st(x^* - x_n) \|)]^3}\, ds\, dt \\
&\leqslant (t^* - s_n)\left(\frac{\| x^* - x_n \|}{t^* - t_n} \right)^2 .
\end{aligned}
\tag{53}
$$

In order to estimate $\| x_{n+1} - x^* \|$, we first notice that

$$
\begin{aligned}
&[y_n, x_n; F](x_{n+1} - x^*) \\
&= \int_0^1 \int_0^1 F''(x_n + z_n^{s,t})(x^* - y_n)(x^* - x_n) t\, ds\, dt,
\end{aligned}
\tag{54}
$$

where $z_n^{s,t} = t(y_n - x_n) + st(x^* - y_n)$. This together with Lemma 5(i), (29), (52), and (53) gives that

$$
\begin{aligned}
&\Big\| F'(x_0)^{-1} [y_n, x_n; F](x_{n+1} - x^*) \Big\| \\
&\leqslant \int_0^1 \int_0^1 \frac{2\gamma \| x^* - y_n \| \| x^* - x_n \| t}{[1 - \gamma(\| x_n - x_0 \| + t\| y_n - x_n \| + st(x^* - y_n))]^3}\, ds\, dt \\
&\leqslant \int_0^1 \int_0^1 \frac{2\gamma (t^* - s_n)(t^* - t_n) t}{[1 - \gamma(t_n + t(s_n - t_n) + st(t^* - s_n))]^3}\, ds\, dt \\
&\quad \times \left(\frac{\| x^* - x_n \|}{t^* - t_n} \right)^3
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-h(t_n) - \frac{h(s_n) - h(t_n)}{s_n - t_n}(t^* - t_n) \right) \left(\frac{\| x^* - x_n \|}{t^* - t_n} \right)^3 \\
&= (t_{n+1} - t^*)\left(\frac{h(s_n) - h(t_n)}{s_n - t_n} \right)\left(\frac{\| x^* - x_n \|}{t^* - t_n} \right)^3 .
\end{aligned}
\tag{55}
$$

Combining the above inequality with (46), we have

$$
\begin{aligned}
\| x_{n+1} - x^* \| &\leqslant \| [y_n, x_n; F]^{-1} F'(x_0) \| \\
&\quad \times \| F'(x_0)^{-1} [y_n, x_n; F](x_{n+1} - x^*) \| \\
&\leqslant -\left(\frac{h(s_n) - h(t_n)}{s_n - t_n} \right)^{-1}(t_{n+1} - t^*) \\
&\quad \times \left(\frac{h(s_n) - h(t_n)}{s_n - t_n} \right)\left(\frac{\| x^* - x_n \|}{t^* - t_n} \right)^3 \\
&\leqslant (t^* - t_{n+1})\left(\frac{\| x^* - x_n \|}{t^* - t_n} \right)^3 .
\end{aligned}
\tag{56}
$$

Therefore, the error estimate (62) follows. Also, from the previous inequality, we know that the convergence rate of $\{ x_n \}$ to x^* is Q-cubic. This completes the proof. □

One typical and important class of examples satisfying γ-condition is the one of analytic functions. Smale [5] studied the convergence and error estimation of Newton's method (2) under the hypotheses that F is analytic and satisfies

$$
\| F'(\overline{x})^{-1} F^{(n)}(\overline{x}) \| \leqslant n! \gamma^{n-1}, \quad n \geqslant 2,
\tag{57}
$$

where $\overline{x}$ is a fixed point in D and γ is defined by

$$
\gamma := \sup_{n>1} \left\| \frac{F'(\overline{x})^{-1} F^{(n)}(\overline{x})}{n!} \right\|^{1/(n-1)}.
\tag{58}
$$

The following lemma taken from [20] shows that an analytic operator satisfies γ-condition.

Lemma 7. *Let r_0 and γ be defined by (7) and (86), respectively. Suppose that F is analytic and satisfies (85). Then F satisfies γ-condition*

$$
\| F'(\overline{x})^{-1} F''(x) \| \leqslant \frac{2\gamma}{(1 - \gamma \| x - \overline{x} \|)^3}
\tag{59}
$$

on $B(\overline{x}, r_0)$.

According to this lemma, we can conclude that the semilocal results obtained in Theorem 6 also hold when F is an analytic operator.

Corollary 8. *Suppose that* (14) *holds. Suppose that F is analytic and satisfies*

$$\left\| F'(x_0)^{-1} F^{(n)}(x_0) \right\| \leqslant n! \gamma^{n-1}, \quad n \geqslant 2, \tag{60}$$

where γ is defined by

$$\gamma := \sup_{n>1} \left\| \frac{F'(x_0)^{-1} F^{(n)}(x_0)}{n!} \right\|^{1/(n-1)}. \tag{61}$$

Then the sequence $\{x_n\}$ generated by (4) *with the initial point x_0 is well defined and converges to a solution $x^* \in \overline{\mathbf{B}(x_0, t^*)}$ of* (1) *with Q-cubic rate and this solution x^* is unique in $\overline{\mathbf{B}(x_0, r)}$, where $t^* \leqslant r < t^{**}$. Moreover, the following error bounds*

$$\|x^* - x_n\| \leqslant (t^* - t_n) \left(\frac{\|x^* - x_m\|}{t^* - t_m} \right)^{3^{n-m}} \quad \forall n \geqslant m \geqslant 0 \tag{62}$$

are valid, where t^* and t^{**} are defined in Lemma 2.

4. Convergence Ball

Now we begin to study the local convergence properties for Newton-Steffensen's method (4) under γ-condition. Recall that r_0 is defined by (7). Throughout this section, suppose that $x^* \in D$ such that $F(x^*) = 0$, $\mathbf{B}(x^*, r_0) \subset D$, and the inverse $F'(x^*)^{-1}$ exists. Moreover, we assume that F satisfies the γ-condition on $\mathbf{B}(x^*, r_0)$; that is, the following relation holds:

$$\left\| F'(x^*)^{-1} F''(x) \right\| \leqslant \frac{2\gamma}{(1 - \gamma \|x - x^*\|)^3}, \quad x \in \mathbf{B}(x^*, r_0). \tag{63}$$

Then, for any $x \in \mathbf{B}(x^*, r_0)$, it follows from Lemma 1 that

$$\left\| F'(x)^{-1} F'(x^*) \right\| \leqslant \frac{(1 - \gamma \|x - x^*\|)^2}{2(1 - \gamma \|x - x^*\|)^2 - 1}. \tag{64}$$

Let

$$r_1 := \frac{(5 - \sqrt{17})}{(4\gamma)}. \tag{65}$$

Define function G as follows:

$$G(t) = \frac{\gamma t}{1 - 4\gamma t + 2\gamma^2 t^2}, \quad t \in (0, r_1). \tag{66}$$

It is clear that $r_1 \in (0, r_0)$ and that $G(r_1) = 1$. Moreover, G increases monotonically in $(0, r_1)$.

Theorem 9. *Let r_1 be defined in* (65). *Then, for any $x_0 \in B(x^*, r_1)$, the sequence $\{x_n\}$ generated by* (4) *converges to x^* and satisfies that*

$$\|x_n - x^*\| \leqslant q^{3^n - 1} \|x_0 - x^*\|, \quad n = 0, 1, \ldots, \tag{67}$$

where

$$q = G(t_0) < 1, \quad t_0 = \|x_0 - x^*\|. \tag{68}$$

Proof. For $n = 0, 1, \ldots$, we write $t_n = \|x_n - x^*\|$. It is sufficient to show that

$$t_{n+1} \leqslant t_n, \qquad \|x_{n+1} - x^*\| \leqslant \left(\frac{G(t_n)}{t_n} \right)^2 \|x_n - x^*\|^3, \tag{69}$$

$$n = 0, 1, \ldots.$$

In fact, by noticing the monotonicity of G/t, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leqslant \left(\frac{G(t_n)}{t_n} \right)^2 \|x_n - x^*\|^3 \\
&\leqslant \left(\frac{G(t_0)}{t_0} \right)^2 \|x_n - x^*\|^3 \\
&= \left(\frac{q}{t_0} \right)^2 \|x_n - x^*\|^3, \quad n = 0, 1, \ldots.
\end{aligned} \tag{70}$$

From this we can easily establish (67) by mathematical induction.

We now prove (69). First we get the following expression of $x_{n+1} - x^*$:

$$\begin{aligned}
x_{n+1} &- x^* \\
&= [y_n, x_n; F]^{-1} \\
&\quad \times \left[\int_0^1 F'(x_n + t(y_n - x_n))(x_n - x^*) \, dt \right. \\
&\quad \left. - (F(x_n) - F(x^*)) \right] \\
&= [y_n, x_n; F]^{-1} \\
&\quad \times \int_0^1 \int_0^1 F''(x^* + (1-t)(x_n - x^*) + st(y_n - x^*)) \\
&\quad \times (y_n - x^*)(x_n - x^*) t \, ds \, dt.
\end{aligned} \tag{71}$$

Similarly, we also have

$$\begin{aligned}
y_n - x^* &= x_n - F'(x_n)^{-1} F'(x_n) - x^* \\
&= F'(x_n)^{-1} \\
&\quad \times \left[F(x^*) - F(x_n) + F'(x_n)(x_n - x^*) \right] \\
&= F'(x_n)^{-1} \\
&\quad \times \int_0^1 \left[F'(x_n + t(x^* - x_n)) - F'(x_n) \right] \\
&\quad \times (x^* - x_n) \, dt \\
&= F'(x_n)^{-1} \\
&\quad \times \int_0^1 \int_0^1 F''(x_n + st(x^* - x_n))(x^* - x_n)^2 t \, ds \, dt.
\end{aligned} \tag{72}$$

This together with (63) and (64) yields

$$\|y_n - x^*\| \leqslant \left\|F'(x_n)^{-1} F'(x^*)\right\|$$

$$\times \left\|\int_0^1 \int_0^1 F'(x^*)^{-1} F''(x_n + st(x^* - x_n)) \, t \, ds \, dt\right\|$$

$$\times \|x^* - x_n\|^2$$

$$\leqslant \left(\frac{1}{(1 - \gamma \|x^* - x_n\|)^2} - 2\right)$$

$$\times \int_0^1 \int_0^1 \frac{2\gamma}{[1 - \gamma \|x_n + st(x^* - x_n) - x^*\|]^3} \, ds \, dt$$

$$\times \|x^* - x_n\|^2$$

$$= \left(\frac{1}{(1 - \gamma t_n)^2} - 2\right)$$

$$\times \int_0^1 \int_0^1 \frac{2\gamma}{[1 - \gamma(1 - st) t_n]^3} \, ds \, dt$$

$$\times \|x^* - x_n\|^2$$

$$= \frac{\gamma}{1 - 4\gamma t_n + 2\gamma^2 t_n^2} \|x^* - x_n\|^2$$

$$= \left(\frac{G(t_n)}{t_n}\right) \|x^* - x_n\|^2. \tag{73}$$

On the other hand, we notice that

$$F'(x^*)^{-1} \left([y_n, x_n; F] - F'(x^*)\right)$$

$$= F'(x^*)^{-1}$$

$$\times \int_0^1 \int_0^1 F''(x^* + s(1 - t)(x_n - x^*)$$

$$+ st(y_n - x^*))$$

$$\times [(1 - t)(x_n - x^*) + t(y_n - x^*)] \, ds \, dt. \tag{74}$$

It follows from (63) that

$$\left\|F'(x^*)^{-1} \left([y_n, x_n; F] - F'(x^*)\right)\right\|$$

$$\leqslant \int_0^1 \int_0^1 \frac{2\gamma [(1 - t) \|x_n - x^*\| + t \|y_n - x^*\|]}{[1 - \gamma(s(1 - t) \|x_n - x^*\| + st \|y_n - x^*\|)]^3} \, ds \, dt. \tag{75}$$

For the case $n = 0$, by (88) and (73), we get

$$\|y_0 - x^*\| \leqslant G(t_0) \|x^* - x_0\| \leqslant \|x^* - x_0\|. \tag{76}$$

Combining Lemma 2 with (75) and (76), we obtain

$$\left\|F'(x^*)^{-1} \left([y_0, x_0; F] - F'(x^*)\right)\right\|$$

$$\leqslant \int_0^1 \int_0^1 \frac{2\gamma [(1 - t) \|x_0 - x^*\| + t \|y_0 - x^*\|]}{[1 - \gamma(s(1 - t) \|x_0 - x^*\| + st \|y_0 - x^*\|)]^3} \, ds \, dt$$

$$\leqslant \int_0^1 \int_0^1 \frac{2\gamma [(1 - t) t_0 + tt_0]}{[1 - \gamma(s(1 - t) t_0 + stt_0)]^3} \, ds \, dt$$

$$= \frac{1}{(1 - \gamma t_0)^2} - 1 < 1. \tag{77}$$

It follows from Banach lemma that

$$\left\|[y_0, x_0; F]^{-1} F'(x^*)\right\| \leqslant \frac{1}{1 - \left((1/(1 - \gamma t_0)^2) - 1\right)}$$

$$= \frac{(1 - \gamma t_0)^2}{2(1 - \gamma t_0)^2 - 1}. \tag{78}$$

This together with (63), (71) and (76) yields

$$\|x_1 - x^*\|$$

$$\leqslant \left\|[y_0, x_0; F]^{-1} F'(x^*)\right\|$$

$$\times \left\|F'(x^*)^{-1}\right.$$

$$\times \int_0^1 \int_0^1 F''(x^* + (1 - t)(x_0 - x^*)$$

$$+ st(y_0 - x^*)) \, t \, ds \, dt\Big\|$$

$$\times \|y_0 - x^*\| \|x_0 - x^*\|$$

$$\leqslant \frac{(1 - \gamma t_0)^2}{2(1 - \gamma t_0)^2 - 1} \tag{79}$$

$$\times \int_0^1 \int_0^1 \frac{2\gamma}{[1 - \gamma((1 - t) t_0 + stt_0)]^3} \, t \, ds \, dt$$

$$\times G(t_0) \|x^* - x_0\|^3$$

$$= \frac{(1 - \gamma t_0)^2}{2(1 - \gamma t_0)^2 - 1} \frac{1}{t_0} \left(\frac{1}{(1 - \gamma t_0)^2} - \frac{1}{(1 - \gamma t_0)}\right)$$

$$\times G(t_0) \|x^* - x_0\|^3$$

$$= \left(\frac{G(t_0)}{t_0}\right)^2 \|x^* - x_0\|^3.$$

Hence (69) holds for $n = 0$.

Now assume that the inequalities in (69) hold for up to some $n \geqslant 1$. Then by (73), one has

$$\|y_{n+1} - x^*\| = G(t_{n+1}) t_{n+1} \leqslant G(t_0) t_{n+1} \leqslant t_{n+1}. \tag{80}$$

Thus (75) can be further reduced to

$$\left\| F'(x^*)^{-1} \left([y_{n+1}, x_{n+1}; F] - F'(x^*) \right) \right\|$$

$$\leqslant \int_0^1 \int_0^1 \frac{2\gamma \left[(1-t)\, t_{n+1} + t t_{n+1} \right]}{\left[1 - \gamma \left(s(1-t)\, t_{n+1} + s t t_{n+1} \right) \right]^3}\, ds\, dt \qquad (81)$$

$$= \frac{1}{\left(1 - \gamma t_{n+1} \right)^2} - 1 < 1.$$

Using Banach lemma again, one has

$$\left\| [y_{n+1}, x_{n+1}; F]^{-1} F'(x^*) \right\| \leqslant \frac{\left(1 - \gamma t_{n+1} \right)^2}{2\left(1 - \gamma t_{n+1} \right)^2 - 1}. \qquad (82)$$

This together with (63), (71), and (73) yields

$$\| x_{n+2} - x^* \|$$

$$\leqslant \left\| [y_{n+1}, x_{n+1}; F]^{-1} F'(x^*) \right\|$$

$$\times \left\| F'(x^*)^{-1} \right.$$

$$\cdot \int_0^1 \int_0^1 F'' \left(x^* + (1-t)(x_{n+1} - x^*) \right.$$

$$\left. + st (y_{n+1} - x^*) \right) t\, ds\, dt \Big\|$$

$$\times \| y_{n+1} - x^* \| \, \| x_{n+1} - x^* \|$$

$$\leqslant \frac{\left(1 - \gamma t_{n+1} \right)^2}{2\left(1 - \gamma t_{n+1} \right)^2 - 1}$$

$$\times \int_0^1 \int_0^1 \frac{2\gamma}{\left[1 - \gamma \left((1-t)\, t_{n+1} + s t t_{n+1} \right) \right]^3}\, t\, ds\, dt \qquad (83)$$

$$\times G(t_{n+1}) \, \| x^* - x_0 \|^3$$

$$= \frac{\left(1 - \gamma t_{n+1} \right)^2}{2\left(1 - \gamma t_{n+1} \right)^2 - 1}$$

$$\times \frac{1}{t_{n+1}} \left(\frac{1}{\left(1 - \gamma t_{n+1} \right)^2} - \frac{1}{\left(1 - \gamma t_{n+1} \right)} \right)$$

$$\times G(t_{n+1}) \, \| x^* - x_0 \|^3$$

$$= \left(\frac{G(t_{n+1})}{t_{n+1}} \right)^2 \| x^* - x_{n+1} \|^3.$$

Thus

$$t_{n+2} \leq G(t_{n+1})^2 t_{n+1} \leq G(t_0)^2 t_{n+1} \leq t_{n+1}. \qquad (84)$$

This and (83) show that the inequalities in (69) hold for $n + 1$ and hence they hold for each n. The proof is complete. □

Applying Lemma 7 to the above theorem, we get the following corollary:

Corollary 10. *Suppose that F is analytic and satisfies*

$$\left\| F'(x^*)^{-1} F^{(n)}(x^*) \right\| \leqslant n! \gamma^{n-1}, \quad n \geqslant 2, \qquad (85)$$

where γ is defined by

$$\gamma := \sup_{n>1} \left\| \frac{F'(x^*)^{-1} F^{(n)}(x^*)}{n!} \right\|^{1/(n-1)}. \qquad (86)$$

Let r_1 and G be defined by (65) *and* (66), *respectively. Then, for any $x_0 \in B(x^*, r_1)$, the sequence $\{x_n\}$ generated by* (4) *converges to x^* and satisfies that*

$$\| x_n - x^* \| \leqslant q^{3^{n}-1} \| x_0 - x^* \|, \quad n = 0, 1, \ldots, \qquad (87)$$

where

$$q = G(t_0) < 1, \quad t_0 = \| x_0 - x^* \|. \qquad (88)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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