

## Research Article

# On Existence of Stabilizing Switching Laws within a Class of Unstable Linear Systems

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The equivalence of two conditions, condition (3) and condition (4) stated in Problem Statement section, regarding the existence of stabilizing switching laws between two unstable linear systems first appeared in (Feron 1996). Although Feron never published this result, it has been referenced in almost every survey on switched systems; see, for example, (Liberzon and Morse 1999). This paper proposes another way to prove the equivalence of two conditions regarding the existence of stabilizing switching laws between two unstable linear systems. One is effective for theoretical derivation, while the other is implementable, and a class of stabilizing switching laws have been explicitly constructed by Wicks et al. (1994). With the help of the equivalent relation, a condition for the existence of controllers and stabilizing switching laws between two unstabilizable linear control systems is then proposed. Then, the study is further extended to the issue concerning the construction of quadratically stabilizing switching laws among  $N$  unstable linear systems and  $N$  unstabilizable linear control systems. The obtained results are employed to study the existence of control laws and quadratically stabilizing switching laws within a class of unstabilizable linear control systems. The numerical examples are illustrated and simulated to show the feasibility and effectiveness of the proposed methods.

## 1. Introduction

The study of switched systems and switching strategies has recently attracted considerable attention (see, e.g., [1–40]). A switched system is known as a hybrid dynamical system comprising a family of continuous-time subsystems and a rule that conducts the switching among them. This kind of system arises from many practical applications such as adaptive control, intelligent control, and the control of many mechanical systems (see, e.g., [4, 5]). The key motivations for having “multiple modalities” or “variable structure” during the control period include the improvement of the transient response of an adaptive system [2] and the existence of systems (such as nonholonomic systems) that cannot be asymptotically stabilizable by a single continuous feedback control law [8, 41], which makes switch control technique especially suitable. Recently, the fault-tolerant control (FTC) issues with fault detection and diagnosis (FDD) schemes have been

largely discussed [38, 42–46]. Herein, the switching strategies will also be applied in the active, passive, or data-driven framework in the current and future works. Another example in the real control application of an air-breathing hypersonic aircraft with high complexity of the motion equations can be achieved by applying a switched linear-parameter-varying systems approach [32]. In mathematical modeling of physical systems, Wu and Zheng explored the topics concerning weighted  $H^\infty$  model reduction for linear switched systems with time-varying delay [18] and dissipativity-based sliding mode control of switched stochastic systems [39]. Lian et al. studied the dwell time method and exponential stability for uncertain switched stochastic time-delay systems [37].

In general, the issue regarding stability and design of switched systems is characterized into three categories [9]. The first is to find conditions that guarantee that the switched system is asymptotically stable for any switching signal. An intuitive approach for this category is to investigate

the existence of common Lyapunov function for all subsystems [2, 6, 12]. Clearly, all subsystems in this case are necessarily required to be stable. The second is to identify those classes of switching signals for which the switched system, with all subsystems being stable, is asymptotically stable. A strategy to carry out the analysis is through the use of multiple Lyapunov function technique [7]. On the other hand, instead of assuming that all subsystems are stable, the final category allowed them to be unstable. The objectives of the final category are then to explore the existence conditions for stabilizing switching signals and to explicitly construct the signals, which makes the switched system asymptotically stable [1]. In this paper, we only consider the issue of the final category.

It is known that the switching among stable systems might result in the system state being unbounded [9]. On the other hand, with suitably selecting the switching signal, the switching among unstable systems might have the property that the system state converges to zero [1]. One of the sufficient conditions regarding the existence of stabilizing switching laws between two unstable linear systems is given by (2) of Section 2 below. Though this condition is effective for theoretical derivation, it does not provide any guidelines for the design of switching laws. In 1994, Wicks et al. [1] showed that a matrix pencil condition, as given by (3) of Section 2, is also a sufficient condition for the existence of stabilizing switching laws. Although they pointed out that the latter condition is more restricted than the former one, they had presented an algorithm to facilitate the checking of the latter condition and a guideline to explicitly construct the stabilizing switching laws [1]. Recently, there are so many new research results about the switched systems that have been proposed (e.g., see [11–14, 16]). However, as far as the construction of quadratically stabilizing switching laws among  $N$  systems is concerned, there are so many detailed and extendable parts which still need further discussion and exploration.

The issue, concerning the existence of stabilizing switching laws between two unstable linear systems, has been studied by Feron [3] and Liberzon and Morse [9]. The equivalence of two conditions (condition (3) and condition (4) stated in Section 2 below) regarding the existence of stabilizing switching laws between two unstable linear systems first appeared in Feron (1996) [3]. Although Feron never published this result, it has been referenced in almost every survey on switched systems; see, for example, [9]. In this paper, we will propose another way to show that the two conditions stated above are indeed equivalent and then apply the equivalent relation to explore the existence of controllers and stabilizing switching laws between two unstabilizable linear control systems in an implementable way. In addition, in order to further understand the geometrical interpretations of the two sufficient conditions, we also specialize the two existence conditions to a planar case. Conditions involving eigenvalues and eigenvectors for a planar system are then presented.

This paper is organized as follows. Section 2 introduces the two sufficient conditions regarding the existence of stabilizing switching laws between two unstable linear systems, stabilizing switching laws among  $N$  unstable linear

systems, and the objectives of the paper. Section 3 discusses another way to prove the equivalence of the two sufficient conditions for two linear systems. Section 4 specializes the existence condition to planar systems and gives a numerical example. Section 5 extends the study to the application of the equivalent relation to the design of controllers and stabilizing switching laws among  $N$  unstabilizable linear systems and linear control systems and gives two numerical examples. Finally, Section 6 summarizes the main results.

## 2. Problem Statement

Consider the two linear systems

$$\dot{\mathbf{x}} = A_i \mathbf{x}, \quad i = 1, 2, \quad (1)$$

where we assume that neither of the two systems is stable; that is, neither  $A_1$  nor  $A_2$  is a Hurwitz matrix. It is known that a sufficient condition for the existence of stabilizing switching laws between the two unstable systems can be stated as in condition (2) as follows:

$$\begin{aligned} &\exists P > 0 \\ &\text{such that } \bigcup_{i=1}^2 \{ \mathbf{x} \mid \mathbf{x}^T (A_i^T P + P A_i) \mathbf{x} < 0 \} = \mathbb{R}^n \setminus \{0\}. \end{aligned} \quad (2)$$

Clearly, under this condition and by defining  $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$ , one can find a proper switching law between the two systems such that  $\dot{V} < 0$  for all the time. It then results in  $\mathbf{x}(t) \rightarrow 0$  as  $t \rightarrow \infty$ . However, condition (2) does not provide a means to obtain such a matrix  $P$ . In 1994, Wicks et al. [1] verified that the matrix pencil condition as given by (3) is also a sufficient condition for the existence of stabilizing switching laws:

$$\exists \beta > 0 \text{ such that } A_1 + \beta A_2 \text{ is a Hurwitz matrix.} \quad (3)$$

They also proposed an algorithm to determine the existence of  $\beta$  and to explicitly construct a class of stabilizing switching laws if condition (3) holds. It was pointed out by [1] that condition (2) is more general than condition (3).

In 1996, Feron [3] proved that the sufficient condition (3) is also a necessary one for the existence of stabilizing switching laws. At the same time, the study concerning quadratic stabilizability was also extended to dynamic output feedback with a robust detectability condition. In 2000, Decarlo et al. [11] extended the sufficient condition (3) for two systems [1] to the sufficient condition (4) for more than two systems as follows:

$$\begin{aligned} &\exists \alpha_i > 0 \text{ and } \sum_{i=1}^N \alpha_i = 1 \\ &\text{such that } A_{eq} := \sum_{i=1}^N \alpha_i A_i \text{ is a Hurwitz matrix.} \end{aligned} \quad (4)$$

However, the literature search indicates that the implementation of stabilizing  $N$  unstable linear systems needs to be further explored.

Therefore, the first goal of this paper is to propose another way to show the equivalence of conditions (2) and (3) and then to apply the equivalent relation to study the existence of controllers and stabilizing switching laws between two unstabilizable linear control systems in an implementable way. Then, we propose a way to implement the abovementioned problem for  $N$  unstable linear systems. Finally, we employ the results to design control laws and quadratically stabilizing switching laws among  $N$  unstabilizable linear control systems.

### 3. The Proof of Equivalence

To achieve the objectives as described in Section 2, we introduce the following condition:

$$\exists P > 0 \text{ and } \beta > 0 \text{ such that } L_1 + \beta L_2 < 0, \quad (5)$$

where

$$L_i := A_i^T P + P A_i, \quad i = 1, 2. \quad (6)$$

In the following, we will show the equivalence of the three conditions (2)–(5). Firstly, we have the relation between conditions (3) and (5) as follows.

**Lemma 1.** *Conditions (3) and (5) are equivalent.*

*Proof.* Condition (3) holds

$$\begin{aligned} &\iff \exists \beta > 0 \text{ and } P > 0 \\ &\text{such that } (A_1 + \beta A_2)^T P + P (A_1 + \beta A_2) < 0 \\ &\iff \exists \beta > 0 \text{ and } P > 0 \\ &\text{such that } (A_1^T P + P A_1) + \beta (A_2^T P + P A_2) < 0. \end{aligned} \quad (7)$$

This proves the result.  $\square$

Next, we prove that “condition (3)  $\Rightarrow$  condition (2),” which can also be found in [1].

**Lemma 2.** *Condition (3)  $\Rightarrow$  condition (2).*

*Proof.*  $A_1 + \beta A_2$  is Hurwitz

$$\begin{aligned} &\implies \forall Q > 0, \exists P > 0 \\ &\text{such that } (A_1 + \beta A_2)^T P + P (A_1 + \beta A_2) = -Q < 0, \\ &\implies \forall \mathbf{x} \neq 0, \\ &\mathbf{x}^T [(A_1^T P + P A_1) + \beta (A_2^T P + P A_2)] \mathbf{x} = -\mathbf{x}^T Q \mathbf{x} < 0, \\ &\implies \forall \mathbf{x} \neq 0. \end{aligned}$$

At least one of  $\mathbf{x}^T (A_i^T P + P A_i) \mathbf{x}, i = 1, 2$ , is less than zero.  $(8)$

The result then follows.  $\square$

Finally, we will show that “condition (2)  $\Rightarrow$  condition (5).” Before the proof, we define two subsets of the unit sphere as follows:

$$\Omega_i := \{ \mathbf{x} \mid \mathbf{x}^T L_i \mathbf{x} \geq 0, \|\mathbf{x}\| = 1 \}, \quad i = 1, 2. \quad (9)$$

The two sets have the following properties.

**Lemma 3.** (a) *Both sets  $\Omega_1$  and  $\Omega_2$  are symmetric about the origin. That is,  $\Omega_i = -\Omega_i$  for  $i = 1, 2$ , where  $-\Omega_i := \{-\mathbf{x} \mid \mathbf{x} \in \Omega_i\}$ .*

(b) *Each unit eigenvector associated with unstable eigenvalues of  $L_i$  belongs to  $\Omega_i$  (by unstable eigenvalues one means those eigenvalues with nonnegative real parts).*

(c) *Either  $\Omega_i$  is a connected set or it consists of two disjoint connected sets  $\Omega_i = \Omega_i^+ \cup \Omega_i^-$  with  $\Omega_i^+ = -\Omega_i^-$ .*

(d) *If condition (2) holds, then  $\Omega_1 \cap \Omega_2 = \emptyset$  and  $S \setminus \{\Omega_1 \cup \Omega_2\} \neq \emptyset$ , where  $S = \{\mathbf{x} \mid \|\mathbf{x}\| = 1\}$  denotes the unit sphere.*

*Proof.* Parts (a), (b), and (d) are trivial, so we omit their proofs.

We now prove part (c). Firstly, we assume that  $L_i$  is a diagonal matrix having the following form:

$$L_i = \text{diag}(\lambda_1^+, \dots, \lambda_{m_i}^+, \lambda_{m_i+1}^-, \dots, \lambda_n^-), \quad (10)$$

where  $\lambda_j^+ \geq 0$  for  $j = 1, \dots, m_i$  and  $\lambda_j^- < 0$  for  $j = m_i+1, \dots, n$ . Clearly, the set

$$\Gamma_i := \{ \mathbf{x} = (x_1, \dots, x_{m_i}, 0, \dots, 0)^T \mid \|\mathbf{x}\| = 1 \} \subset \Omega_i, \quad (11)$$

and  $\Gamma_i$ , a subset of the unit sphere, is either a connected set or a union of two connected sets which are symmetric about the origin. Now,  $\mathbf{x} \in \Omega_i$  means that

$$\mathbf{x}^T L_i \mathbf{x} = \sum_{j=1}^{m_i} \lambda_j^+ x_j^2 + \sum_{j=m_i+1}^n \lambda_j^- x_j^2 \geq 0. \quad (12)$$

This implies that

$$\sum_{j=1}^{m_i} \lambda_j^+ x_j^2 + \sum_{j=m_i+1}^n \lambda_j^- (t x_j)^2 \geq 0, \quad \forall 0 \leq t \leq 1, \quad (13)$$

or

$$\mathbf{x}_t^T L_i \mathbf{x}_t \geq 0, \quad \forall 0 \leq t \leq 1, \quad (14)$$

where  $\mathbf{x}_t := (x_1, \dots, x_{m_i}, t x_{m_i+1}, \dots, t x_n)^T$ . It follows that the set

$$\left\{ \frac{\mathbf{x}_t}{\|\mathbf{x}_t\|} \mid 0 \leq t \leq 1 \right\} \subseteq \Omega_i. \quad (15)$$

Therefore, for any point  $\mathbf{x}$  in  $\Omega_i$ , there exists a smooth path lying entirely inside  $\Omega_i$  with an end point  $\mathbf{x}$  and the other end point in  $\Gamma_i$ . This means that  $\Omega_i$  is also a connected set or a union of two connected sets as that of  $\Gamma_i$ .

Next, if the symmetric matrix  $L_i$  is not a diagonal matrix, then there exists an orthogonal matrix  $U$  such that  $U^T L_i U$

is a diagonal matrix; say that  $D_i = U^T L_i U$ , and the set  $\Omega_i$  is the image of  $\Omega_i^{\text{new}} := \{\mathbf{y} \mid \mathbf{y}^T D_i \mathbf{y} \geq 0, \|\mathbf{y}\| = 1\}$  under the continuous transformation  $\mathbf{x} = U\mathbf{y}$ , since the image of a connected set under a continuous transformation is also a connected set. The result then follows.  $\square$

Now, we are in the position to show that “condition (2)  $\Rightarrow$  condition (5).”

**Lemma 4.** *Condition (2)  $\Rightarrow$  condition (5).*

*Proof.* Since  $\mathbf{x}^T L_1 \mathbf{x} \geq 0$  and  $\mathbf{x}^T L_2 \mathbf{x} < 0$  for all  $\mathbf{x} \in \Omega_1$ , this implies that

$$g_1(\beta) := \sup_{\mathbf{x} \in \Omega_1} \mathbf{x}^T (L_1 + \beta L_2) \mathbf{x} \quad (16)$$

is a decreasing function of  $\beta$  and  $g_1(0) \geq 0$ . Since  $\Omega_1$  is a compact set, it follows that  $\sup_{\mathbf{x} \in \Omega_1} \mathbf{x}^T L_1 \mathbf{x} = l_1$  and  $\sup_{\mathbf{x} \in \Omega_1} \mathbf{x}^T L_2 \mathbf{x} = -l_2$  for some  $l_1 > 0$  and  $l_2 > 0$ . Thus,  $g_1(\beta) < 0$  for all  $\beta > l_1/l_2$ . By the Intermediate Value Theorem, there exists a real number  $\beta_1^* \geq 0$  such that  $g_1(\beta_1^*) = 0$ . Similarly,  $\mathbf{x}^T L_1 \mathbf{x} < 0$  and  $\mathbf{x}^T L_2 \mathbf{x} \geq 0$  for all  $\mathbf{x} \in \Omega_2$  yield that

$$g_2(\beta) := \sup_{\mathbf{x} \in \Omega_2} \mathbf{x}^T (L_1 + \beta L_2) \mathbf{x} \quad (17)$$

is an increasing function of  $\beta$  and  $g_2(0) < 0$ . If  $\sup_{\mathbf{x} \in \Omega_2} \mathbf{x}^T L_2 \mathbf{x} > 0$ , then there exists a real number  $\beta_2^* > 0$  such that  $g_2(\beta_2^*) = 0$ . On the other hand, if  $\sup_{\mathbf{x} \in \Omega_2} \mathbf{x}^T L_2 \mathbf{x} = 0$  (this corresponds to the case that all of the unstable eigenvalues of  $L_2$  are zero), then we define  $\beta_2^* = \infty$ . From the definitions of  $\beta_1^*$  and  $\beta_2^*$ , we have that

$$\begin{aligned} \beta > \beta_1^* &\iff \mathbf{x}^T (L_1 + \beta L_2) \mathbf{x} < 0, \quad \forall \mathbf{x} \in \Omega_1, \\ \beta < \beta_2^* &\iff \mathbf{x}^T (L_1 + \beta L_2) \mathbf{x} < 0, \quad \forall \mathbf{x} \in \Omega_2. \end{aligned} \quad (18)$$

Next, we show that  $\beta_1^* < \beta_2^*$ . Suppose, on the contrary, that  $\beta_2^* \leq \beta_1^*$ . Then, there exists a nonnegative  $\beta^*$  satisfying  $\beta_2^* \leq \beta^* \leq \beta_1^*$ . From (18), there exist an  $\mathbf{x}_1 \in \Omega_1$  and an  $\mathbf{x}_2 \in \Omega_2$  such that  $\mathbf{x}_1^T (L_1 + \beta^* L_2) \mathbf{x}_1 \geq 0$  and  $\mathbf{x}_2^T (L_1 + \beta^* L_2) \mathbf{x}_2 \geq 0$ . It follows that

$$\Omega^* \cap \Omega_1 \neq \emptyset, \quad \Omega^* \cap \Omega_2 \neq \emptyset, \quad (19)$$

where  $\Omega^* = \{\mathbf{x} \mid \mathbf{x}^T (L_1 + \beta^* L_2) \mathbf{x} \geq 0, \|\mathbf{x}\| = 1\}$ . Moreover, since  $\mathbf{x}^T L_i \mathbf{x} < 0$  for all  $i = 1, 2$  and  $\mathbf{x} \in S \setminus (\Omega_1 \cup \Omega_2)$ , we have that  $\mathbf{x}^T (L_1 + \beta^* L_2) \mathbf{x} < 0$  for all  $\mathbf{x} \in S \setminus (\Omega_1 \cup \Omega_2)$ . This means that

$$\Omega^* \subseteq \Omega_1 \cup \Omega_2. \quad (20)$$

The relations (19)-(20) then yield

$$\Omega^* = (\Omega^* \cap \Omega_1) \cup (\Omega^* \cap \Omega_2). \quad (21)$$

Since, by (d) of Lemma 3,  $\Omega_1 \cap \Omega_2 = \emptyset$ , then (21) reveals that  $\Omega^*$  is the union of two disjoint sets which are not symmetric about the origin. But this is impossible since, by Lemma 3,  $\Omega^*$

should also have the property that it is a connected set or it consists of two disjoint connected sets which are symmetric about the origin. This contradiction then implies that  $\beta_1^* < \beta_2^*$ .

Finally, from (18), any  $\beta$  with  $\beta_1^* < \beta < \beta_2^*$  will result in  $\mathbf{x}^T (L_1 + \beta L_2) \mathbf{x} < 0$  for all  $\mathbf{x} \in \Omega_1 \cup \Omega_2$ . In addition, since, for all  $\mathbf{x} \in S \setminus (\Omega_1 \cup \Omega_2)$ , both  $\mathbf{x}^T L_1 \mathbf{x} < 0$  and  $\mathbf{x}^T L_2 \mathbf{x} < 0$ , it follows that  $\mathbf{x}^T (L_1 + \beta L_2) \mathbf{x} < 0$  for any  $\beta \in (\beta_1^*, \beta_2^*)$  and for all  $\mathbf{x} \in S$ . Thus, the matrix  $L_1 + \beta L_2 < 0$  for any  $\beta \in (\beta_1^*, \beta_2^*)$ . This completes the proof.  $\square$

Theorem 5 below summarizes the conclusions.

**Theorem 5.** *Condition (2)  $\Leftrightarrow$  condition (3)  $\Leftrightarrow$  condition (5).*

We have shown that conditions (2)–(5) are equivalent. We now investigate the relation of the unstable eigenpairs of  $L_1$  and  $L_2$  under these conditions. Let  $\{\lambda_1^+, \dots, \lambda_{m_1}^+\}$  and  $\{\mu_1^+, \dots, \mu_{m_2}^+\}$  be the unstable eigenvalues of  $L_1$  and  $L_2$ , respectively, let  $\mathbf{x}_{1i}^+$  be a unit eigenvector of  $L_1$  associated with  $\lambda_i^+$ , and let  $\mathbf{x}_{2j}^+$  be a unit eigenvector of  $L_2$  associated with  $\mu_j^+$ . Define

$$\begin{aligned} \delta_{1j} &:= \mathbf{x}_{2j}^{+T} L_1 \mathbf{x}_{2j}^+, \quad j = 1, \dots, m_2, \\ \delta_{2i} &:= \mathbf{x}_{1i}^{+T} L_2 \mathbf{x}_{1i}^+, \quad i = 1, \dots, m_1. \end{aligned} \quad (22)$$

Clearly, if condition (2) holds, then

$$\delta_{1j} < 0, \quad \delta_{2i} < 0 \quad (23)$$

for all  $1 \leq i \leq m_1$  and  $1 \leq j \leq m_2$ . The following result is a direct consequence of Theorem 5.

**Corollary 6.** *Suppose that condition (2) holds. Then,  $\lambda_i^+ \mu_j^+ < \delta_{1j} \delta_{2i}$  for all  $1 \leq i \leq m_1$  and  $1 \leq j \leq m_2$ .*

*Proof.* If condition (2) holds, then by Theorem 5 there exists a  $\beta > 0$  such that  $L_1 + \beta L_2 < 0$ . It follows that

$$\begin{aligned} \mathbf{x}_{1i}^{+T} (L_1 + \beta L_2) \mathbf{x}_{1i}^+ &= \lambda_i^+ + \beta \delta_{2i} < 0, \\ \mathbf{x}_{2j}^{+T} (L_1 + \beta L_2) \mathbf{x}_{2j}^+ &= \delta_{1j} + \beta \mu_j^+ < 0 \end{aligned} \quad (24)$$

for all  $1 \leq i \leq m_1$  and  $1 \leq j \leq m_2$ . This then implies that

$$\lambda_i^+ < -\beta \delta_{2i}, \quad \mu_j^+ < -\frac{\delta_{1j}}{\beta}, \quad (25)$$

or  $\lambda_i^+ \mu_j^+ < \delta_{1j} \delta_{2i}$  for all  $1 \leq i \leq m_1$  and  $1 \leq j \leq m_2$ . This completes the proof.  $\square$

*Remark 7.* From Corollary 6 and (23), we have that

$$0 \leq \max_{1 \leq i \leq m_1} \left\{ \frac{-\lambda_i^+}{\delta_{2i}} \right\} < \min_{1 \leq j \leq m_2} \left\{ \frac{-\delta_{1j}}{\mu_j^+} \right\}. \quad (26)$$

If we define

$$\beta_1 := \max_{1 \leq i \leq m_1} \left\{ \frac{-\lambda_i^+}{\delta_{2i}} \right\}, \quad \beta_2 := \min_{1 \leq j \leq m_2} \left\{ \frac{-\delta_{1j}}{\mu_j^+} \right\}, \quad (27)$$

then  $\beta_1 < \beta_2$ . Moreover,  $\beta \in (\beta_1, \beta_2)$  yields

$$\begin{aligned} \mathbf{x}_{1i}^{+T} (L_1 + \beta L_2) \mathbf{x}_{1i}^+ &= \lambda_i^+ + \beta \delta_{2i} < \lambda_i^+ + \beta_1 \delta_{2i} < 0, \\ \mathbf{x}_{2j}^{+T} (L_1 + \beta L_2) \mathbf{x}_{2j}^+ &= \delta_{1j} + \beta \mu_j^+ < \delta_{1j} + \beta_2 \mu_j^+ < 0. \end{aligned} \tag{28}$$

This means that  $\mathbf{x}^T(L_1 + \beta L_2)\mathbf{x}$  is negative at all of the eigenvectors associated with unstable eigenvalues of  $L_1$  and  $L_2$ . However,  $L_1 + \beta L_2$  might not be a negative definite matrix, which will be discussed in more details in the following section. From this observation, we then have the relation  $\beta_1 \leq \beta_1^* < \beta_2^* \leq \beta_2$ .

### 4. Specialization to Planar Systems

In order to explore the geometrical interpretations of conditions (2)-(3), (5), in this section, we focus the study on planar systems.

Consider the two linear systems as given by (44) with  $n = 2$ . Since neither  $A_1$  nor  $A_2$  is Hurwitz, we have that, for any  $P > 0$ , none of  $L_i = A_i^T P + P A_i$ ,  $i = 1, 2$ , is a Hurwitz matrix. This means that both  $L_1$  and  $L_2$  contain exactly one unstable eigenvalue if condition (2) holds. Denote by  $(\lambda_1^+, \mathbf{x}_{11}^+)$  and  $(\mu_1^+, \mathbf{x}_{21}^+)$  the unstable eigenpairs of  $L_1$  and  $L_2$ , respectively, with  $\|\mathbf{x}_{11}^+\| = \|\mathbf{x}_{21}^+\| = 1$  and

$$\mathbf{x}_{11}^{+T} \mathbf{x}_{21}^+ \geq 0. \tag{29}$$

Then, by Corollary 6, a necessary condition for condition (2) is

$$\lambda_1^+ \mu_1^+ < \delta_{11} \delta_{21}, \tag{30}$$

where  $\delta_{11}$  and  $\delta_{21}$  are given by (22). Clearly, condition (30) implies that

$$\lambda_1^+ \mu_1^+ < \lambda_1^- \mu_1^-, \tag{31}$$

where  $\lambda_1^-$  and  $\mu_1^-$  denote stable eigenvalues of  $L_1$  and  $L_2$ , respectively. Condition (30) or (31) is only a necessary one for condition (2). However, if we add an extra condition as described in (32), we have a necessary and sufficient one for condition (2).

**Theorem 8.** Consider the two systems as given by (44) with  $n = 2$ . Then, condition (2) holds if and only if condition (30) or (31) holds and

$$\begin{aligned} \cos^2 \theta &< \frac{\lambda_1^+ \mu_1^+ + \lambda_1^- \mu_1^- - 2\sqrt{\lambda_1^+ \mu_1^+ \lambda_1^- \mu_1^-}}{(\lambda_1^+ - \lambda_1^-)(\mu_1^+ - \mu_1^-)} \\ &\text{or} \\ \theta &> \cos^{-1} \left( \frac{\sqrt{\lambda_1^- \mu_1^-} - \sqrt{\lambda_1^+ \mu_1^+}}{\sqrt{(\lambda_1^+ - \lambda_1^-)(\mu_1^+ - \mu_1^-)}} \right), \end{aligned} \tag{32}$$

where  $\theta$  is the angle between  $\mathbf{x}_{11}^+$  and  $\mathbf{x}_{21}^+$ .

*Proof.* It is not difficult to see that the angles of the two-sided cones  $\{\mathbf{x} \mid \mathbf{x}^T \text{diag}(\lambda_1^+, \lambda_1^-)\mathbf{x} \geq 0\}$  and  $\{\mathbf{x} \mid \mathbf{x}^T \text{diag}(\mu_1^+, \mu_1^-)\mathbf{x} \geq 0\}$  are  $2\tan^{-1} \sqrt{-\lambda_1^+/\lambda_1^-}$  and  $2\tan^{-1} \sqrt{-\mu_1^+/\mu_1^-}$ , respectively. These angles are invariant under orthogonal transformation. Moreover, since  $L_1$  and  $L_2$  are symmetric matrices, they possess complete orthonormal set of eigenvectors. Thus,

condition (2) holds

$$\begin{aligned} &\iff \theta > \tan^{-1} \left( \sqrt{-\frac{\lambda_1^+}{\lambda_1^-}} \right) + \tan^{-1} \left( \sqrt{-\frac{\mu_1^+}{\mu_1^-}} \right) \\ &\iff \cos \theta < \cos \left\{ \tan^{-1} \left( \sqrt{-\frac{\lambda_1^+}{\lambda_1^-}} \right) + \tan^{-1} \left( \sqrt{-\frac{\mu_1^+}{\mu_1^-}} \right) \right\} \\ &\iff \cos \theta < \frac{\sqrt{\lambda_1^- \mu_1^-} - \sqrt{\lambda_1^+ \mu_1^+}}{\sqrt{(\lambda_1^+ - \lambda_1^-)(\mu_1^+ - \mu_1^-)}}. \end{aligned} \tag{33}$$

The result then follows.  $\square$

From the previous Theorem 8, it requires not only the relation of eigenvalues having the relation (31) but also the angle of eigenvectors satisfying (32) for two unstable planar systems to match condition (2). However, if both  $L_1$  and  $L_2$  are diagonal matrices or have the same set of eigenvectors with  $\mathbf{x}_{11}^+ = \pm \mathbf{x}_{21}^+$  and  $\mathbf{x}_{21}^+ = \pm \mathbf{x}_{11}^+$ , where  $\mathbf{x}_{11}^-$  and  $\mathbf{x}_{21}^-$  denote unit eigenvectors of  $L_1$  and  $L_2$  associated with stable eigenvalues, then condition (32) is automatically true and condition (30) or (31) is then a necessary and sufficient one for condition (2) as described by the following result.

**Corollary 9.** Suppose that  $L_1$  and  $L_2$  have the same set of eigenvectors with  $\mathbf{x}_{11}^+ = \pm \mathbf{x}_{21}^+$  and  $\mathbf{x}_{21}^+ = \pm \mathbf{x}_{11}^+$ . Then, condition (2) holds if and only if condition (30) or (31) holds.

#### 4.1. Application to Unstabilizable Linear Control Systems.

Consider the two linear control systems

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_i \mathbf{u}_i, \quad i = 1, 2. \tag{34}$$

Here,  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B}_1 \in \mathbb{R}^{n \times m_1}$ , and  $\mathbf{B}_2 \in \mathbb{R}^{n \times m_2}$ , and we assume that neither of the two control systems is stabilizable. Although neither of the two control systems is stabilizable by any linear state feedback, it is still possible to have a set of controllers  $\mathbf{u}_1 = \mathbf{K}_1 \mathbf{x}$  and  $\mathbf{u}_2 = \mathbf{K}_2 \mathbf{x}$  and stabilizing switching laws between the two systems, which makes  $\mathbf{x}(t) \rightarrow 0$  as  $t \rightarrow \infty$ . In this section, we will study the existence condition and implement it through the equivalent relation as described in Section 3.

According to the results of Section 3, there exist  $\mathbf{K}_1$  and  $\mathbf{K}_2$  and a stabilizing switching law between the two control systems given in (34) if there exists  $\beta > 0$  such that

$$(\mathbf{A} + \mathbf{B}_1 \mathbf{K}_1) + \beta (\mathbf{A} + \mathbf{B}_2 \mathbf{K}_2) \text{ is a Hurwitz matrix,} \tag{35}$$

or, equivalently, if there exist  $P > 0$  and  $\beta > 0$  such that

$$\begin{aligned} & \left[ (A + B_1 K_1)^T P + P (A + B_1 K_1) \right] \\ & + \beta \left[ (A + B_2 K_2)^T P + P (A + B_2 K_2) \right] < 0. \end{aligned} \quad (36)$$

If we choose

$$K_1 = -B_1^T P, \quad K_2 = -B_2^T P, \quad (37)$$

then inequality (36) becomes

$$\begin{aligned} & (A^T P + PA - 2PB_1 B_1^T P) \\ & + \beta (A^T P + PA - 2PB_2 B_2^T P) < 0 \end{aligned} \quad (38)$$

or

$$A_\beta^T P + PA_\beta - 2PB_\beta B_\beta^T P < 0, \quad (39)$$

where  $A_\beta := (1 + \beta)A$  and  $B_\beta := (B_1 \dot{\vdots} \sqrt{\beta} B_2)$ . Note the existence of  $P > 0$  and  $\beta > 0$  such that inequality (39) holds if and only if  $(A_\beta, B_\beta)$  is stabilizable. We, thus, have the following result.

**Theorem 10.** Consider the two linear control systems as given by (34). Suppose that neither of the two control systems is stabilizable, but  $(A, B)$  is stabilizable, where  $B = (B_1 \dot{\vdots} B_2)$ . Then, there exist  $K_1 \in \mathbb{R}^{m_1 \times n}$ ,  $K_2 \in \mathbb{R}^{m_2 \times n}$ , and a stabilizing switching law between the two systems  $A + B_1 K_1$  and  $A + B_2 K_2$ . Moreover,  $K_1$  and  $K_2$  can be chosen in the form of (37), and  $P$  is solved by the following Riccati equation:

$$\begin{aligned} & (A^T P + PA - 2PB_1 B_1^T P) \\ & + \beta (A^T P + PA - 2PB_2 B_2^T P) = -H, \end{aligned} \quad (40)$$

where  $H > 0$  and  $\beta > 0$ .

*Proof.* The theorem can be easily verified by noting the following fact that  $(A_\beta, B_\beta)$  is stabilizable if and only if  $(A, B)$  is stabilizable and the discussions before this theorem.  $\square$

After deriving the control laws for the two control systems given in (34), the stabilizing switching laws between the two control systems can then be implemented through an existing algorithm proposed by [1]. To demonstrate the use of the result, we give an example as follows.

#### 4.2. A Numerical Example

*Example 1.* Consider the two linear control systems given by (34) with

$$A = \begin{pmatrix} -32 & -14 & 14 \\ -70 & -27 & 29 \\ -140 & -58 & 60 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2 \\ 5 \\ 10 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}. \quad (41)$$

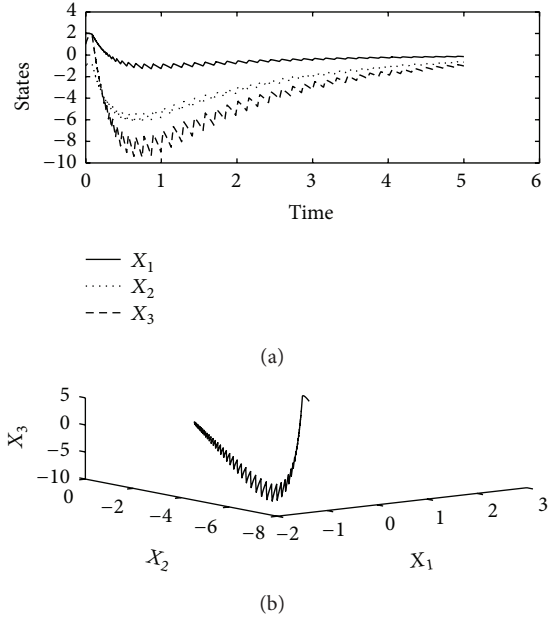


FIGURE 1: (a) Timing response of the three states and (b) state space trajectory.

By direct checking, neither  $(A, B_1)$  nor  $(A, B_2)$  is stabilizable, but  $(A, B)$  is stabilizable, where  $B = (B_1 \dot{\vdots} B_2)$ . The solution of the Riccati equation given by (40) with  $\beta = 2$  and  $H$  being the identity matrix is found to be

$$P = \begin{pmatrix} 36.4 & 18.1 & -18.2 \\ 18.1 & 21.1 & -15.1 \\ -18.2 & -15.1 & 12.1 \end{pmatrix}. \quad (42)$$

Choose  $K_1$  and  $K_2$  in the form of (37); then we have two linear systems with system matrices  $A_i = A + B_i K_i$ ,  $i = 1, 2$ , where

$$\begin{aligned} A_1 &= \begin{pmatrix} 5.7 & 4.1 & -5.3 \\ 24.2 & 18.3 & -19.3 \\ 48.5 & 32.6 & -36.7 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} -32.0 & -14.0 & 14.0 \\ -69.9 & -33.0 & 31.9 \\ -139.9 & -64.0 & 62.9 \end{pmatrix}. \end{aligned} \quad (43)$$

Clearly, neither of the two systems is stable, while, by Theorem 10, they will satisfy condition (3). A stabilizing switching law can then be constructed through the algorithm proposed by [1]. Figure 1 demonstrates the timing response of the three states and phase trajectory with initial  $(2, -1)^T$ . The three states are observed to converge to zero, which agrees with the theoretical results.

## 5. Switching among $N$ Linear Systems

In this section, we extend the derivation stated above to the equivalence of two sufficient conditions for the existence of

quadratically stabilizing switching laws among  $N$  linear systems. Consider the  $N$   $n$ -dimensional unstable linear systems

$$\dot{\mathbf{x}} = A_i \mathbf{x}, \quad i = 1, 2, \dots, N, \quad (44)$$

where  $\mathbf{x} \in \mathbb{R}^n$  and  $A_i \in \mathbb{R}^{n \times n}$ , and we assume that all  $A_i, i = 1, \dots, N$ , are unstable matrices; that is, none of the  $N$  matrices is a Hurwitz matrix. It is known that a sufficient condition for the existence of quadratically stabilizing switching laws among the  $N$  unstable systems can be stated as in condition (45) as follows:

$$\begin{aligned} &\exists P > 0 \\ &\text{such that } \bigcup_{i=1}^N \{ \mathbf{x} \mid \mathbf{x}^T (A_i^T P + P A_i) \mathbf{x} < 0 \} = \mathbb{R}^n \setminus \{0\}. \end{aligned} \quad (45)$$

However, condition (45) does not provide a means to obtain such a matrix  $P$ . On the other hand, Decarlo et al. [11] verified that the matrix pencil condition as given by (4) is also a sufficient condition for the existence of stabilizing switching laws. They also proposed an algorithm to determine the existence of  $\alpha_i$  and to explicitly construct a class of quadratically stabilizing switching laws if condition (4) holds. However, it is not difficult to prove that condition (4) is equivalent to condition (45).

**5.1. Implementation for Switching among  $N$  Systems.** To achieve the second objective as described in Section 2, an LMI condition is proposed in this section to implement the existence of  $\alpha_i$  given in (4). To this end, if we can find  $\alpha_i > 0$  and  $\sum_{i=1}^N \alpha_i = 1$  such that  $A_{eq} := \sum_{i=1}^N \alpha_i A_i^s$  is a Hurwitz matrix, where  $A_i^s := (A_i + A_i^T)/2$ , then it can be inferred by Lemma 2 below that the quadratically stabilizing switching for systems (44) is feasible.

**Lemma 2** (see [47]). *For any  $A_i \in \mathbb{R}^{n \times n}, i = 1, 2, \dots, N$ , there exists  $A_i = (A_i + A_i^H)/2 + (A_i - A_i^H)/2$ , where  $(A_i + A_i^H)/2 \equiv A_i^s$  means the symmetric part of  $A_i$ . By the Rayleigh quotient of  $A_i^s$ , one can derive that*

$$\text{Re} [\lambda (A_i)] \subseteq [\lambda_{\min} (A_i^s), \lambda_{\max} (A_i^s)]. \quad (46)$$

*Proof.* It is trivial; thus, it is omitted here.  $\square$

According to (46), condition (4) can be turned into the following LMIs:

$$A_{eq}^s = \alpha_1 A_1^s + \alpha_2 A_2^s + \dots + \alpha_N A_N^s < 0. \quad (47)$$

From Lemma 2 and letting  $\beta_k := \alpha_{k+1}/\alpha_1, k = 1, 2, \dots, N - 1$ , if we can find positive constants  $\beta_1, \beta_2, \dots, \beta_{N-1}$  such that

$$A_{eq}^s = A_1^s + \beta_1 A_2^s + \dots + \beta_{N-1} A_N^s < 0, \quad (48)$$

then condition (4) automatically holds. Although condition (48) is less general than condition (4), it can be solved by LMI toolbox in MATLAB to carry out the implementation.

**5.2. Switching among  $N$  Unstabilizable Control Systems.** Consider the  $N$  linear control systems

$$\dot{\mathbf{x}} = A \mathbf{x} + B_i \mathbf{u}_i, \quad i = 1, 2, \dots, N, \quad (49)$$

where  $A \in \mathbb{R}^{n \times n}$  and  $B_i \in \mathbb{R}^{n \times m_i}$ , and we assume that each of  $(A, B_i)$  is unstabilizable. Although neither of the  $(A, B_i)$  is stabilizable by any linear state feedback, it is still possible to have a set of controllers  $\mathbf{u}_i(\mathbf{x}) = K_i \mathbf{x}$  and quadratically stabilizing switching laws among the  $N$  systems, which makes  $\mathbf{x}(t) \rightarrow \mathbf{0}$  as  $t \rightarrow \infty$ . In this section, we will study the existence condition and implement it through the equivalent relation of condition (4) and condition (2).

According to the abovementioned equivalence result, there exist  $K_1, \dots, K_N$  and stabilizing switching laws among the  $N$  control systems given in (49) if there exist positive constants  $\beta_1, \dots, \beta_{N-1}$  such that

$$(A + B_1 K_1) + \sum_{k=1}^{N-1} \beta_k (A + B_k K_k) \quad (50)$$

is a Hurwitz matrix or, equivalently, if there exist  $P > 0$  and positive constants  $\beta_1, \dots, \beta_{N-1}$  such that

$$\begin{aligned} &[(A + B_1 K_1)^T P + P (A + B_1 K_1)] \\ &+ \sum_{k=1}^{N-1} \beta_k [(A + B_k K_k)^T P + P (A + B_k K_k)] < 0. \end{aligned} \quad (51)$$

If we choose

$$K_i = -B_i^T P, \quad (52)$$

then inequality (51) becomes

$$\begin{aligned} &(A^T P + P A - 2 P B_1 B_1^T P) \\ &+ \sum_{k=1}^{N-1} \beta_k (A^T P + P A - 2 P B_k B_k^T P) < 0, \end{aligned} \quad (53)$$

or

$$A_\beta^T P + P A_\beta - 2 P B_\beta B_\beta^T P < 0, \quad (54)$$

where

$$A_\beta := \left( 1 + \sum_{k=1}^{N-1} \beta_k \right) A, \quad (55)$$

$$B_\beta = \left( B_1 \cdot \sqrt{\beta_1} F_2 \cdot \dots \cdot \sqrt{\beta_{N-1}} B_N \right). \quad (56)$$

Note the existence of  $P > 0$  and positive constants  $\beta_1, \dots, \beta_{N-1}$  such that inequality (56) holds if and only if  $(A_\beta, B_\beta)$  is stabilizable. We, thus, have the following result.

**Theorem 3.** *Consider a class of  $N$  linear control systems as given by (49). Suppose that none of the  $N$  control systems is stabilizable, but  $(A, B_i)$  is stabilizable, where  $B = (B_1 \cdot B_2 \cdot \dots \cdot B_N)$ .*

Then, there exist  $K_i \in \mathbb{R}^{m_i \times n}$  and a quadratically stabilizing switching law among systems  $A + B_k K_k$ ,  $k = 1, 2, \dots, N$ . Moreover,  $K_k$  can be chosen in the form of (52), and  $P$  is solved by the following Riccati equation:

$$A_\beta^T P + PA_\beta - 2PB_\beta B_\beta^T P = -H, \quad (57)$$

where  $H > 0$  and  $\beta > 0$ .

*Proof.* To prove the theorem, we just need to prove that  $(A_\alpha, B_\alpha)$  is stabilizable if and only if  $(A, B)$  is stabilizable, where  $A_\alpha = (\sum_{i=1}^N \alpha_i)A = \alpha A$ ,  $B_\alpha = (\sqrt{\alpha_1}B_1 \dots \sqrt{\alpha_N}B_N)$ , and  $B = (B_1 \dots B_N)$ . If  $(A, B)$  is stabilizable, then  $\bar{K} = (\bar{K}_1 \dots \bar{K}_N)^T$  must exist such that  $A + B\bar{K}$  is Hurwitz and  $A + B\bar{K} = A + \sum_{i=1}^N B_i \bar{K}_i$ . We select  $K = ((\alpha/\sqrt{\alpha_1})\bar{K}_1 \dots (\alpha/\sqrt{\alpha_N})\bar{K}_N)^T$ , and then we can get

$$\begin{aligned} A_\alpha + B_\alpha K &= A_\alpha + \sum_{i=1}^N \sqrt{\alpha_i} B_i \frac{\alpha}{\sqrt{\alpha_i}} \bar{K}_i \\ &= \alpha A + \sum_{i=1}^N \alpha B_i \bar{K}_i = \alpha \left( A + \sum_{i=1}^N B_i \bar{K}_i \right). \end{aligned} \quad (58)$$

Since  $A + \sum_{i=1}^N B_i \bar{K}_i$  is a Hurwitz matrix and  $\alpha > 0$ , therefore,  $(A_\alpha, B_\alpha)$  is stabilizable, and then we can find  $P$  by (57).  $\square$

After the determination of  $K_i$ , the stabilizing switching law can be implemented by the results of Section 5.1.

### 5.3. Numerical Examples

*Example 1.* Consider the three unstable matrices in the form of (44) with

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & -10 & -10 \\ 2 & 0 & 6 \\ -9 & -4 & 0 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} -10 & 0 & -4 \\ -1 & -10 & -9 \\ -1 & -10 & 5 \end{pmatrix}, \\ A_3 &= \begin{pmatrix} 7 & -8 & -2 \\ 1 & -1 & 4 \\ 7 & 3 & -9 \end{pmatrix}. \end{aligned} \quad (59)$$

Indeed, we have that  $\lambda(A_1) = \{10.3039, -5.1520 \pm 5.7990i\}$ ,  $\lambda(A_2) = \{-9.5994, -15.0750, 9.6744\}$ , and  $\lambda(A_3) = \{-1.7049, 3.8525 \pm 4.0791i\}$ . By using the results of Section 3, we can easily find that  $\beta_1 = 4.2349$  and  $\beta_2 = 4.1726$  via solving (48). It follows from Lemma 1 that  $A_{eq} := A_1 + \beta_1 A_2 + \beta_2 A_3$  is a Hurwitz matrix and

$$A_{eq} = \begin{pmatrix} -13.1408 & -43.3808 & -35.2848 \\ 1.9377 & -46.5215 & -15.4237 \\ 15.9733 & -33.8312 & -16.3788 \end{pmatrix}. \quad (60)$$

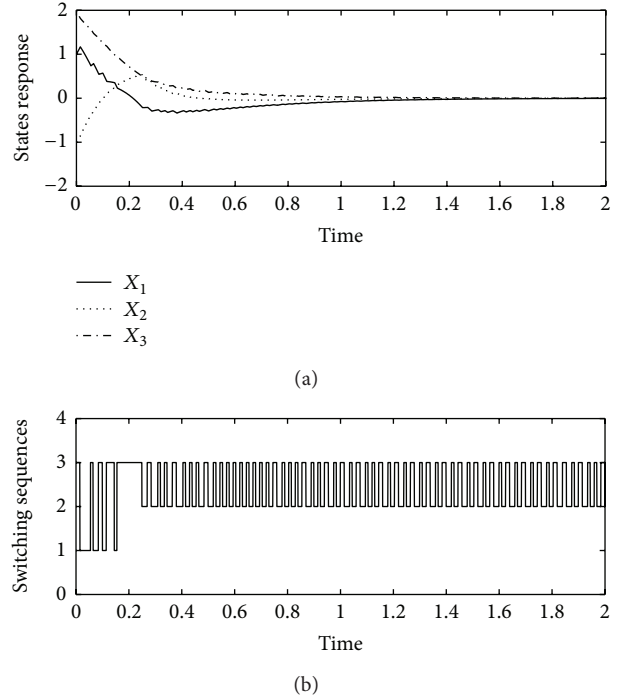


FIGURE 2: (a) Timing response of the three states and (b) switching sequences.

By solving the Lyapunov equation  $A_{eq}^T P + PA_{eq} + I = 0$ , we have that

$$P = \begin{pmatrix} 0.0605 & 0.0073 & -0.0173 \\ 0.0073 & 0.0194 & -0.0253 \\ -0.0173 & -0.0253 & 0.0660 \end{pmatrix}. \quad (61)$$

The quadratically stabilizing switching law is then easily derived by the algorithm presented by [1]. Figure 2(a) describes the three states response with initial  $(2, -1, 1)^T$ . The three states are observed to converge to zero. Figure 2(b) shows the switching sequences. Figure 3 describes the phase plane trajectory of the switched systems. The trajectory is observed to converge to the origin. The simulation results agree with the theoretical results.

*Example 2.* Consider the three linear control systems in the form of (49) with

$$\begin{aligned} A &= \begin{pmatrix} 33 & -32 & -32 \\ 8 & -7 & -8 \\ 16 & -16 & -15 \end{pmatrix}, & B_1 &= \begin{pmatrix} -5 \\ 4 \\ -6 \end{pmatrix}, \\ B_2 &= \begin{pmatrix} 11 \\ 5 \\ 1 \end{pmatrix}, & B_3 &= \begin{pmatrix} -21 \\ -1 \\ -16 \end{pmatrix}. \end{aligned} \quad (62)$$

By direct checking, neither of  $(A, B_1)$ ,  $(A, B_2)$ , and  $(A, B_3)$  is stabilizable, but  $(A, B)$  is stabilizable, where  $B = (B_1 \dots B_2 \dots B_3)$ .



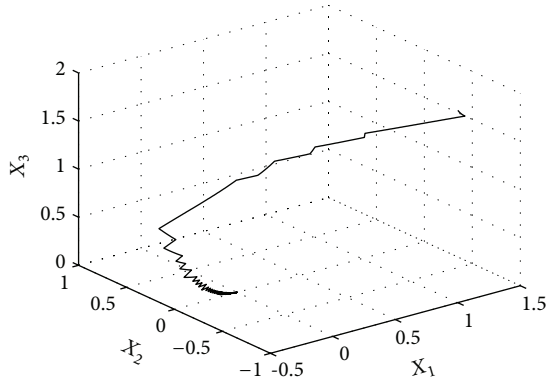


FIGURE 3: Phase plane trajectory of the three systems.

The solution of the Riccati equation given by (57) with  $\beta = 2$  and  $H$  being the identity matrix is found to be

$$P = \begin{pmatrix} 125.814 & -153.305 & -178.213 \\ -153.305 & 307.177 & 224.444 \\ -178.213 & 224.444 & 294.794 \end{pmatrix}. \quad (63)$$

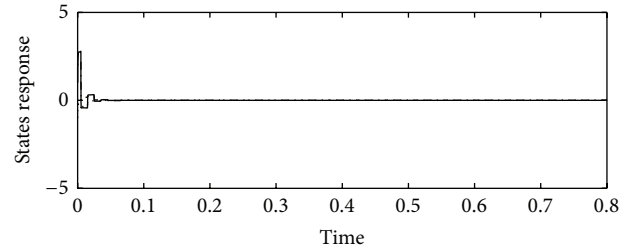
Choose  $K_1, K_2,$  and  $K_3$  in the form of (52); then we have three linear systems with system matrices  $A_i = A + B_i K_i, i = 1, 2,$  and  $3,$  where

$$\begin{aligned} E_1 &= 10^3 * \begin{pmatrix} -0.8320 & 3.2108 & 0.0684 \\ 0.7000 & -2.6013 & -0.0883 \\ -1.0220 & 3.8754 & 0.1055 \end{pmatrix}, \\ E_2 &= 10^3 * \begin{pmatrix} -4.7984 & -0.8457 & 5.9446 \\ -2.1881 & -0.3769 & 2.7087 \\ -0.4232 & -0.0900 & 0.5283 \end{pmatrix}, \\ E_3 &= 10^4 * \begin{pmatrix} -0.3483 & -0.1674 & -1.2177 \\ -0.0133 & -0.0073 & -0.0494 \\ -0.0494 & -0.1067 & -0.7788 \end{pmatrix}. \end{aligned} \quad (64)$$

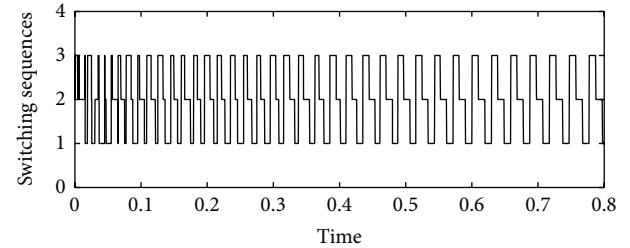
Of course, all of the three matrices  $A_1, A_2,$  and  $A_3$  are not stable, while, by Theorem 3, they satisfy condition (4). A quadratically stabilizing switching law can then be constructed through the algorithm proposed by [1]. Figure 4(a) demonstrates the three states response which are observed to converge to zero. Figure 4(b) shows the switching sequences. Figure 5 describes the phase plane trajectory with initial  $(2, -1, 1)^T$ . The phase plane trajectory is observed to converge to the origin. The simulation results agree with the theoretical results.

### 6. Conclusions

In this study, we propose a way to show the equivalence of two sufficient conditions for the existence of stabilizing switching laws between two unstable linear systems. One of the conditions is powerful for theoretical derivation, while the other is easily implemented through an existing algorithm. By the use of the equivalent relation, a simple existence condition of controllers and stabilizing switching laws between two



(a)



(b)

FIGURE 4: (a) Timing response of three states and (b) switching sequences.

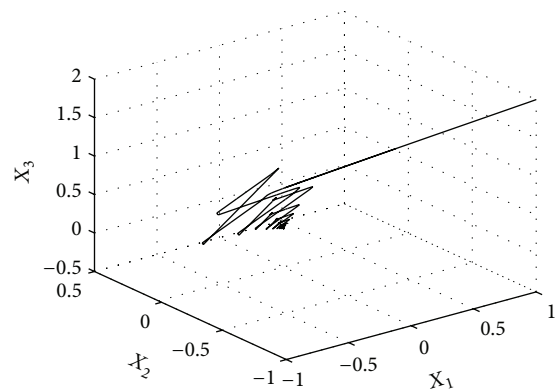


FIGURE 5: Phase plane trajectory of the three control systems.

unstabilizable linear control systems is then proposed and found to be easily implemented. An illustrative example has demonstrated the use and benefit of the equivalent relation. In addition, the existence conditions are also specialized to planar systems to facilitate the understanding of their geometrical interpretations. It requires not only the relation between the eigenvalues of the associated matrices but also the angles between the unstable eigenvectors to match the existence condition. Moreover, we have discussed a feasible way to construct the quadratically stabilizing switching laws among  $N$  unstable linear systems and extended the result to organize the controllers and the quadratically stabilizing switching laws of a class of  $N$  unstabilizable linear control systems. The illustrative examples and simulation results

demonstrate the feasibility and effectiveness of the proposed method for switching control issues.

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