Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 672069, 8 pages
http://dx.doi.org/10.1155/2013/672069

Research Article

One-Local Retract and Common Fixed Point in Modular Metric Spaces

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Received 8 July 2013; Accepted 19 August 2013

Academic Editor: Mohamed A. Khamsi

The notion of a modular metric on an arbitrary set and the corresponding modular spaces, generalizing classical modulars over linear spaces like Orlicz spaces, were recently introduced. In this paper we introduced and study the concept of one-local retract in modular metric space. In particular, we investigate the existence of common fixed points of modular nonexpansive mappings defined on nonempty ω-closed ω-bounded subset of modular metric space.

1. Introduction

The purpose of this paper is to give an outline of a common fixed-point theory for nonexpansive mappings (i.e., mappings with the modular Lipschitz constant 1) on some subsets of modular metric spaces which are natural generalization of classical modulars over linear spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii, and many other spaces. Modular metric spaces were introduced in [1, 2]. The main idea behind this new concept is the physical interpretation of the modular. Informally speaking, whereas a metric on a set represents nonnegative finite distances between any two points of the set, a modular on a set attributes a nonnegative (possibly, infinite valued) "field of (generalized) velocities" to each "time" $\lambda > 0$ (the absolute value of) an average velocity $\omega_\lambda(x, y)$ is associated in such a way that in order to cover the "distance" between points $x, y \in X$ it takes time λ to move from x to y with velocity $\omega_\lambda(x, y)$. But the way we approached the concept of modular metric spaces is different. Indeed we look at these spaces as the nonlinear version of the classical modular spaces introduced by Nakano [3] on vector spaces and Musielak-Orlicz spaces introduced by Musielak [4] and Orlicz [5].

In recent years, there was an uptake interest in the study of electrorheological fluids, sometimes referred to as "smart fluids" (for instance, lithium polymethacrylate). For these fluids, modeling with sufficient accuracy using classical Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, where p is a fixed constant is not adequate, but rather the exponent p should be able to vary [6, 7]. One of the most interesting problems in this setting is the famous Dirichlet energy problem [8, 9]. The classical technique used so far in studying this problem is to convert the energy function, naturally defined by a modular, to a convoluted and complicated problem which involves a norm (the Luxemburg norm). The modular metric approach is more natural and has not been used extensively.

In many cases, particularly in applications to integral operators, approximation, and fixed point results, modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts. In recent years, there was a strong interest to study the fixed point property in modular function spaces after the first paper [10] was published in 1990. More recently, the authors presented a fixed point result for pointwise nonexpansive and asymptotic pointwise nonexpansive acting in modular functions spaces [11]. The theory of nonexpansive mappings defined on convex subsets of Banach spaces has been well developed since the 1960s (see, e.g., Belluce and Kirk [12], Browder [13], Bruck [14], and Lim [15]), and generalized to other metric spaces (see e.g., [16–18]), and modular function spaces (see e.g., [10]). The corresponding fixed-point results were then extended to larger classes of mappings like pointwise contractions, asymptotic pointwise contractions [18–22], and asymptotic pointwise nonexpansive mappings

[11]. In [23], Penot presented an abstract version of Kirk's fixed point theorem [24] for nonexpansive mappings. Many results of fixed point in metric spaces were developed after Penot's formulation. Using Penot's work, the author in [25] proved some results in metric spaces with uniform normal structure similar to the ones known in Banach spaces. In [26], Khamsi introduced the concept of one-local retract in metric spaces and proved that any commutative family of nonexpansive mappings defined on a metric space with a compact and normal convexity structure has a common fixed point. Recently in [27], the authors introduced the concept of one-local retract in modular function spaces and proved the existence of common fixed points for commutative mappings.

In this paper, we study the concept of one-local retract in more general setting in modular metric space; therefore, we prove the existence of common fixed points for a family of modular nonexpansive mappings defined on nonempty ω-closed ω-bounded subsets in modular metric space.

For more on metric fixed point theory, the reader may consult the book [28] and for modular function spaces the book [29].

2. Basic Definitions and Properties

Let X be a nonempty set. Throughout this paper for a function $\omega : (0,\infty) \times X \times X \to [0,\infty]$, we will write

$$\omega_\lambda (x, y) = \omega (\lambda, x, y), \tag{1}$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1 (see [1, 2]). A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be modular metric on X if it satisfies the following axioms:

(i) $x = y$ if and only if $\omega_\lambda(x, y) = 0$, for all $\lambda > 0$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$, for all $\lambda > 0$, and $x, y \in X$;

(iii) $\omega_{\lambda+\mu}(x, y) \le \omega_\lambda(x, z) + \omega_\mu(z, y)$, for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If, instead of (i), we have only the condition (i′)

$$\omega_\lambda (x, x) = 0, \quad \forall\lambda > 0,\ x \in X, \tag{2}$$

then ω is said to be a pseudomodular (metric) on X. A modular metric ω on X is said to be regular if the following weaker version of (i) is satisfied:

$$x = y \quad \text{iff } \omega_\lambda (x, y) = 0, \quad \text{for some } \lambda > 0. \tag{3}$$

Finally, ω is said to be convex if, for $\lambda, \mu > 0$ and $x, y, z \in X$, it satisfies the inequality

$$\omega_{\lambda+\mu} (x, y) \le \frac{\lambda}{\lambda + \mu}\omega_\lambda (x, z) + \frac{\mu}{\lambda + \mu}\omega_\mu (z, y). \tag{4}$$

Note that, for a metric pseudomodular ω on a set X, and any $x, y \in X$, the function $\lambda \to \omega_\lambda(x, y)$ is nonincreasing on $(0,\infty)$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda (x, y) \le \omega_{\lambda-\mu} (x, x) + \omega_\mu (x, y) = \omega_\mu (x, y). \tag{5}$$

Definition 2 (see [1, 2]). Let ω be a pseudomodular on X. Fix $x_0 \in X$. The two sets:

$$X_\omega = X_\omega (x_0) = \{x \in X : \omega_\lambda (x, x_0) \longrightarrow 0 \text{ as } \lambda \longrightarrow \infty\},$$
$$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \exists\lambda = \lambda (x) > 0 \text{ such that } \omega_\lambda (x, x_0) < \infty\} \tag{6}$$

are said to be modular spaces (around x_0).

It is clear that $X_\omega \subset X^*_\omega$ but this inclusion may be proper in general. It follows from [1, 2] that if ω is a modular on X, then the modular space X_ω can be equipped with a (nontrivial) metric, generated by ω and given by

$$d_\omega (x, y) = \inf \{\lambda > 0 : \omega_\lambda (x, y) \le \lambda\}, \tag{7}$$

for any $x, y \in X_\omega$. If ω is a convex modular on X, according to [1, 2] the two modular spaces coincide, that is $X^*_\omega = X_\omega$, and this common set can be endowed with the metric d^*_ω given by

$$d^*_\omega (x, y) = \inf \{\lambda > 0 : \omega_\lambda (x, y) \le 1\}, \tag{8}$$

for any $x, y \in X_\omega$. These distances will be called Luxemburg distances (see example below for the justification).

Definition 3. Let X_ω be a modular metric space.

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ if and only if $\omega_1(x_n, x) \to 0$, as $n \to \infty$. x will be called the ω-limit of (x_n).

(2) The sequence $(x_n)_{n\in N}$ in X_ω is said to be ω-Cauchy if $\omega_1(x_m, x_n) \to 0$, as $m, n \to \infty$.

(3) A subset C of X_ω is said to be ω-closed if the ω-limit of a ω-convergent sequence of C always belongs to C.

(4) A subset C of X_ω is said to be ω-complete if any ω-Cauchy sequence in C is a ω-convergent sequence and its ω-limit is in C.

(5) Let $x \in X_\omega$ and $C \subset X_\omega$. The ω-distance between x and C is defined as

$$d_\omega (x, C) = \inf \{\omega_1 (x, y) ; y \in C\}. \tag{9}$$

(6) A subset C of X_ω is said to be ω-bounded if we have

$$\delta_\omega (C) = \sup \{\omega_1 (x, y) ; x, y \in C\} < \infty. \tag{10}$$

In general if $\lim_{n\to\infty}\omega_\lambda(x_n, x) = 0$, for some $\lambda > 0$, then we may not have $\lim_{n\to\infty}\omega_\lambda(x_n, x) = 0$, for all $\lambda > 0$. Therefore, as it is done in modular function spaces, we will say that ω satisfies Δ_2 condition if this is the case; that is $\lim_{n\to\infty}\omega_\lambda(x_n, x) = 0$, for some $\lambda > 0$ implies $\lim_{n\to\infty}\omega_\lambda(x_n, x) = 0$, for all $\lambda > 0$. In [1, 2], one will find a discussion about the connection between ω-convergence and metric convergence with respect to the Luxemburg distances. In particular, we have

$$\lim_{n\to\infty} d_\omega (x_n, x) = 0 \quad \text{iff } \lim_{n\to\infty} \omega_\lambda (x_n, x) = 0, \quad \forall\lambda > 0, \tag{11}$$

for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$. And in particular we have that ω-convergence and d_ω-convergence are equivalent if and only if the modular ω satisfies the Δ_2-condition. Moreover, if the modular ω is convex, then we know that d_ω^* and d_ω are equivalent which implies that

$$\lim_{n\to\infty} d_\omega^*(x_n, x) = 0 \quad \text{iff} \lim_{n\to\infty} \omega_\lambda(x_n, x) = 0, \quad \forall\lambda > 0, \tag{12}$$

for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$ [1, 2]. Another question that arises in this setting is the uniqueness of the ω-limit. Assume ω is regular, and let $\{x_n\} \in X_\omega$ be a sequence such that $\{x_n\}$ ω-converges to $a \in X_\omega$ and $b \in X_\omega$. Then we have

$$\omega_2(a, b) \le \omega_1(a, x_n) + \omega_1(x_n, b), \tag{13}$$

for any $n \ge 1$. Our assumptions will imply $\omega_2(a, b) = 0$. Since ω is regular, we get $a = b$; that is, the ω-limit of a sequence is unique.

Let (X, ω) be a modular metric space. Throughout the rest of this work, we will assume that ω satisfies the Fatou property; that is, if $\{x_n\}$ ω-converges to x and $\{y_n\}$ ω-converges to y, then we must have

$$\omega_1(x, y) \le \liminf_{n\to\infty} \omega_1(x_n, y_n). \tag{14}$$

For any $x \in X_\omega$ and $r \ge 0$, we define the modular ball

$$B_\omega(x, r) = \{y \in X_\omega; \omega_1(x, y) \le r\}. \tag{15}$$

Note that if ω satisfies the Fatou property, then modular balls (ω-balls) are ω-closed. An admissible subset of X_ω is defined as an intersection of modular balls. We say A is an admissible subset of C if

$$A = \bigcap_{i\in I} B_\omega(b_i, r_i) \cap C, \tag{16}$$

where $b_i \in C$, $r_i \ge 0$, and I is an arbitrary index set. Denote by $\mathscr{A}_\omega(X_\omega)$ the family of admissible subsets of X_ω. Note that $\mathscr{A}_\omega(X_\omega)$ is stable by intersection. At this point we will need to define the concept of Chebyshev center and radius in modular metric spaces. Let $A \subset X$ be a nonempty ω-bounded subset. For any $x \in A$, define

$$r_x(A) = \sup\{\omega_1(x, y);\ y \in A\}. \tag{17}$$

The Chebyshev radius of A is defined by

$$R_\omega(A) = \inf\{r_x(A);\ x \in A\}. \tag{18}$$

Obviously we have $R_\omega(A) \le r_x(A) \le \delta_\omega(A)$, for any $x \in A$. The Chebyshev center of A is defined as

$$\mathscr{C}_\omega(A) = \{x \in A;\ r_x(A) = R_\omega(A)\}. \tag{19}$$

Definition 4. Let (X, ω) be a modular metric space. Let C be a nonempty subset of X_ω.

(i) We will say that $\mathscr{A}_\omega(C)$ is compact if any family $(A_\alpha)_{\alpha\in\Gamma}$ of elements of $\mathscr{A}_\omega(C)$ has a nonempty intersection provided $\cap_{\alpha\in F} A_\alpha \ne \emptyset$, for any finite subset $F \subset \Gamma$.

(ii) We will say that $\mathscr{A}_\omega(C)$ is normal if for any $A \in \mathscr{A}_\omega(C)$, not reduced to one point, ω-bounded, we have $R_\omega(A) < \delta_\omega(A)$.

Remark 5. Note that if $\mathscr{A}_\omega(X_\omega)$ is compact, then X_ω is ω-complete.

Definition 6. Let (X, ω) be a modular metric space. Let C be a nonempty subset of X_ω. A mapping $T : C \to C$ is said to be ω-nonexpansive if

$$\omega_1(T(x), T(y)) \le \omega_1(x, y) \quad \text{for any } x, y \in C. \tag{20}$$

For such mapping we will denote by $\mathrm{Fix}(T)$ the set of its fixed points; that is, $\mathrm{Fix}(T) = \{x \in C; T(x) = x\}$.

In [1, 2] the author defined Lipschitzian mappings in modular metric spaces and proved some fixed point theorems. Our definition is more general. Indeed, in the case of modular function spaces, it is proved in [10] that

$$\omega_\lambda(T(x), T(y)) \le \omega_\lambda(x, y), \quad \text{for any } \lambda > 0 \tag{21}$$

if and only if $d_\omega(T(x), T(y)) \le d_\omega(x, y)$, for any $x, y \in C$. Next we give an example, which first appeared in [10], of a mapping which is ω-nonexpansive in our sense but fails to be nonexpansive with respect to d_ω.

Example 7. Let $X = (0, \infty)$. Define the Musielak-Orlicz function modular on the space of all Lebesgue measurable functions by

$$\rho(f) = \frac{1}{e^2}\int_0^\infty |f(x)|^{x+1} dm(x). \tag{22}$$

Let B be the set of all measurable functions $f : (0, \infty) \to \mathbb{R}$ such that $0 \le f(x) \le 1/2$. Consider the map

$$T(f)(x) = \begin{cases} f(x-1), & \text{for } x \ge 1 \\ 0, & \text{for } x \in [0, 1]. \end{cases} \tag{23}$$

Clearly, $T(B) \subset B$. In [10], it was proved that, for every $\lambda \le 1$ and for all $f, g \in B$, we have

$$\rho(\lambda(T(f) - T(g))) \le \lambda\rho(\lambda(f - g)). \tag{24}$$

This inequality clearly implies that T is ω-nonexpansive. On the other hand, if we take $f = 1_{[0,1]}$, then

$$\|T(f)\|_\rho > e \ge \|f\|_\rho, \tag{25}$$

which clearly implies that T is not d_ω-nonexpansive.

Next we present the analog of Kirk's fixed point theorem [24] in modular metric spaces.

Theorem 8 (see [30]). *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that the family $\mathscr{A}_\omega(C)$ is normal and compact. Let $T : C \to C$ be ω-nonexpansive. Then T has a fixed point.*

3. One-Local Retract Subsets in Modular Metric Spaces

Let C be a nonempty subset of X_ω. A nonempty subset D of C is said to be a one-local retract of C if, for every family $\{B_i; i \in I\}$ of ω-balls centered in D such that $C \cap (\cap_{i\in I} B_i) \neq \emptyset$, it is the case that $D \cap (\cap_{i\in I} B_i) \neq \emptyset$. It is immediate that each ω-nonexpansive retract of X_ω is a one-local retract (but not conversely). Recall that $D \subset C$ is a ω-nonexpansive retract of C if there exists a ω-nonexpansive map $R : C \to D$ such that $R(x) = x$, for every $x \in D$.

The result in [26] may be stated in modular metric spaces as follows.

Theorem 9. *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that $\mathscr{A}_\omega(C)$ is normal and compact. Then, for any ω-nonexpansive mapping $T : C \to C$, the fixed point set $\operatorname{Fix}(T)$ is nonempty one-local retract of C.*

Proof. Theorem 8 shows that $\operatorname{Fix}(T)$ is nonempty. Let us complete the proof by showing that it is a one-local retract of C. Let $\{B_\omega(x_i, r_i)\}_{i\in I}$ be any family of ω-closed balls such that $x_i \in \operatorname{Fix}(T)$, for any $i \in I$, and

$$C_0 = C \cap \left(\bigcap_{i\in I} B_\omega(x_i, r_i) \right) \neq \emptyset. \tag{26}$$

Let us prove that $\operatorname{Fix}(T) \cap (\cap_{i\in I} B_\omega(x_i, r_i)) \neq \emptyset$. Since $\{x_i\}_{i\in I} \subset \operatorname{Fix}(T)$ and T is ω-nonexpansive, then $T(C_0) \subset C_0$. Clearly, $C_0 \in \mathscr{A}_\omega(C)$ and is nonempty. Then we have $\mathscr{A}_\omega(C_0) \subset \mathscr{A}_\omega(C)$. Therefore, $\mathscr{A}_\omega(C_0)$ is compact and normal. Theorem 8 will imply that T has a fixed point in C_0 which will imply

$$\operatorname{Fix}(T) \cap \left(\bigcap_{i\in I} B_\omega(x_i, r_i) \right) \neq \emptyset. \tag{27}$$

□

Now, we discuss some properties of one-local retract subsets.

Theorem 10. *Let (X, ω) be a modular metric space. Let C be a nonempty ω-closed ω-bounded subset of X_ω. Let D be a nonempty subset of C. The following are equivalent.*

(i) *D is a one-local retract of C.*

(ii) *D is a ω-nonexpansive retract of $D \cup \{x\} \to D$, for every $x \in C$.*

Proof. Let us prove (i) ⇒ (ii). Let $x \in C$. We may assume that x does not belong to D. In order to construct a ω-nonexpansive retract $R : D \cup \{x\} \to D$, we only need to find $R(x) \in D$ such that

$$\omega_1(R(x), y) \leq \omega_1(x, y), \quad \text{for every } y \in D. \tag{28}$$

Since $x \in \cap_{y\in D} B_\omega(y, \omega_1(x, y))$ and $x \in C$, then

$$C \cap \left(\bigcap_{y\in D} B_\omega(y, \omega_1(x, y)) \right) \neq \emptyset. \tag{29}$$

Since D is one-local retract of C, we get

$$D_0 = D \cap \left(\bigcap_{y\in D} B_\omega(y, \omega_1(x, y)) \right) \neq \emptyset. \tag{30}$$

Any point in D_0 will work as $R(x)$. □

Next, we prove that (ii) ⇒ (i). In order to prove that D is a one-local retract of C, let $\{B_\omega(x_i, r_i)\}_{i\in I}$ be any family of ω-closed balls such that $x_i \in D$, for any $i \in I$, and

$$C_0 = C \cap \left(\bigcap_{i\in I} B_\omega(x_i, r_i) \right) \neq \emptyset. \tag{31}$$

Let us prove that $D \cap (\cap_{i\in I} B_\omega(x_i, r_i)) \neq \emptyset$. Let $x \in C_0$. If $x \in D$, we have nothing to prove. Assume otherwise that x does not belong to D. Property (ii) implies the existence of a ω-nonexpansive retract $R : D \cup \{x\} \to C$. It is easy to check that $R(x) \in D \cap (\cap_{i\in I} B_\omega(x_i, r_i)) = \emptyset$, which completes the proof of our theorem.

For the rest of this work, we will need the following technical result.

Lemma 11. *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Let D be a nonempty one-local retract of C. Set $\operatorname{co}_C(D) = C \cap (\cap\{A; A \in \mathscr{A}_\omega(C)$ and $D \subset A\})$. Then*

(i) *$r_x(D) = r_x(\operatorname{co}_C(D))$, for any $x \in C$;*

(ii) *$R_\omega(\operatorname{co}_C(D)) = R_\omega(D)$;*

(iii) *$\delta_\omega(\operatorname{co}_C(D)) = \delta_\omega(D)$.*

Proof. Let us first prove (i). Fix $x \in C$. Since $D \subset \operatorname{co}_C(D)$, we get $r_x(D) \leq r_x(\operatorname{co}_C(D))$. On the other hand we have $D \subset B_\omega(x, r_x(D)) \in \mathscr{A}_\omega(C)$. The definition of $\operatorname{co}_C(D)$ implies $\operatorname{co}_C(D) \subset B_\omega(x, r_x(D))$. Hence $r_x(\operatorname{co}_C(D)) \leq r_x(D)$, which implies

$$r_x(\operatorname{co}_C(D)) = r_x(D). \tag{32}$$

Next, we prove (ii). Let $x \in D$. We have $x \in \operatorname{co}_C(D)$. Using (i), we get

$$r_x(\operatorname{co}_C(D)) = r_x(D) \geq R_\omega(\operatorname{co}_C(D)). \tag{33}$$

Hence, $R_\omega(D) \geq R_\omega(\operatorname{co}_C(D))$. Next, let $x \in \operatorname{co}_C(D)$. We have $D \subset \operatorname{co}_C(D) \subset B_\omega(x, r_x(\operatorname{co}_C(D)))$. Hence, $x \in \cap_{y\in D} B_\omega(y, r_x(\operatorname{co}_C(D)))$. Hence

$$C \cap \left(\bigcap_{y\in D} B_\omega(y, r_x(\operatorname{co}_C(D))) \right) = \emptyset. \tag{34}$$

Since D is a one-local retract of C, we get

$$D_0 = D \cap \left(\bigcap_{y\in D} B_\omega(y, r_x(\operatorname{co}_C(D))) \right) = \emptyset. \tag{35}$$

Let $y \in D_0$. Then it is easy to see that $r_y(D) \leq r_x(\mathrm{co}_C(D))$. Hence $R_\omega(D) \leq r_x(\mathrm{co}_C(D))$. Since x was arbitrary taken in $\mathrm{co}_C(D)$, we get

$$R_\omega(D) \leq R_\omega(\mathrm{co}_C(D)), \tag{36}$$

which implies

$$R_\omega(D) = R_\omega(\mathrm{co}_C(D)). \tag{37}$$

Finally, let us prove (iii). Since $D \subset \mathrm{co}_C(D)$, we get

$$\delta_\omega(D) \leq \delta_\omega(\mathrm{co}_C(D)). \tag{38}$$

Now, for any $x \in D$, we have

$$D \subset B_\omega(x, \delta_\omega(D)). \tag{39}$$

Hence

$$\mathrm{co}_C(D) \subset B_\omega(x, \delta_\omega(D)). \tag{40}$$

This implies

$$x \in \bigcap_{y \in \mathrm{co}_C(D)} B_\omega(y, \delta_\omega(D)). \tag{41}$$

Since x was taken arbitrary in D, we get

$$D \subset \bigcap_{y \in \mathrm{co}_C(D)} B_\omega(y, \delta_\omega(D)). \tag{42}$$

The definition of $\mathrm{co}_C(D)$ implies

$$\mathrm{co}_C(D) \in \bigcap_{y \in \mathrm{co}_C(D)} B_\omega(y, \delta_\omega(D)). \tag{43}$$

So for any $x, y \in \mathrm{co}_C(D)$, we have

$$\omega_1(x, y) \leq \delta_\omega(D). \tag{44}$$

Hence

$$\delta_\omega(\mathrm{co}_C(D)) \leq \delta_\omega(D), \tag{45}$$

which implies

$$\delta_\omega(\mathrm{co}_C(D)) = \delta_\omega(D). \tag{46}$$

□

As an application of this lemma we have the following result.

Theorem 12. *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that $\mathscr{A}_\omega(C)$ is normal and compact. If D is a nonempty one-local retract of C, then $\mathscr{A}_\omega(D)$ is compact and normal.*

Proof. Using the definition of one-local retract, it is easy to see that $\mathscr{A}_\omega(D)$ is compact. Let us show that $\mathscr{A}_\omega(D)$ is normal. Let $A_0 \in \mathscr{A}_\omega(D)$ be nonempty and reduced to one point. Set

$$\mathrm{co}_C(A_0) = C \cap (\cap \{A; A \in \mathscr{A}_\omega(C) \text{ and } A_0 \subset A\}). \tag{47}$$

Then from Lemma 11, we get

$$\begin{aligned} R_\omega(\mathrm{co}_C(A_0)) &= R_\omega(A_0), \\ \delta_\omega(\mathrm{co}_C(A_0)) &= \delta_\omega(A_0). \end{aligned} \tag{48}$$

Since $\mathrm{co}_C(A_0) \in \mathscr{A}_\omega(C)$, then we must have

$$R_\omega(\mathrm{co}_C(A_0)) < \delta_\omega(\mathrm{co}_C(A_0)), \tag{49}$$

because $\mathscr{A}_\omega(C)$ is normal. Therefore, we have

$$R_\omega(A_0) < \delta_\omega(A_0), \tag{50}$$

which completes the proof of our claim. □

The following result has found many application in metric spaces. Most of the ideas in its proof go back to Baillon's work [31].

Theorem 13. *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that $\mathscr{A}_\omega(C)$ is normal and compact. Let $(C_\beta)_{\beta \in \Gamma}$ be a decreasing family of one-local retracts of C, where $(\Gamma, \prec)$ is totally ordered. Then $\cap_{\beta \in \Gamma} C_\beta$ is not empty and is one-local retract of C.*

Proof. Consider the family

$$\mathscr{F} = \left\{ \prod_{\beta \in \Gamma} A_\beta : A_\beta \in \mathscr{A}_\omega(C_\beta), (A_\beta) \text{ is decreasing} \right\}. \tag{51}$$

$\mathscr{F}$ is not empty since $\prod_{\beta\in\Gamma} C_\beta \in \mathscr{F}$. $\mathscr{F}$ will be ordered by inclusion; that is, $\prod_{\beta\in\Gamma} A_\beta \subset \prod_{\beta\in\Gamma} B_\beta$ if and only if $A_\beta \subset B_\beta$ for any $\beta \in \Gamma$. From Theorem 12, we know that $\mathscr{A}_\omega(C_\beta)$ is compact, for every $\beta \in \Gamma$. Therefore, $\mathscr{F}$ satisfies the hypothesis of Zorn's Lemma. Hence for every $D \in \mathscr{F}$, there exists a minimal element $A \in \mathscr{F}$ such that $A \subset D$. We claim that if $A = \prod_{\beta\in\Gamma} A_\beta$ is minimal, then there exists $\beta_0 \in \Gamma$ such that $\delta_\omega(A_\beta) = 0$, for every $\beta > \beta_0$. Assume not, that is, $\delta_\omega(A_\beta) > 0$, for every $\beta \in \Gamma$. Fix $\beta \in \Gamma$. For every $K \subset C$, set

$$\mathrm{co}_\beta(K) = \bigcap_{x \in C_\beta} B_\omega(x, r_x(K)). \tag{52}$$

Consider, $A' = \prod_{\alpha\in\Gamma} A'_\alpha$ where

$$\begin{aligned} A'_\alpha &= \mathrm{co}_\beta(A_\beta) \cap A_\alpha \quad \text{if } \alpha \leq \beta, \\ A'_\alpha &= A_\alpha \quad \text{if } \alpha \geq \beta. \end{aligned} \tag{53}$$

The family $(A'_{\alpha\geq\beta})$ is decreasing since $A \in \mathscr{F}$. Let $\alpha \leq \gamma \leq \beta$. Then $A'_\gamma \subset A'_\alpha$, since $A_\gamma \subset A_\alpha$ and $A_\beta = \mathrm{co}_\beta(A_\beta) \cap A_\beta$. Hence the family (A'_α) is decreasing. On the other hand if $\alpha \prec \beta$, then $\mathrm{co}_\beta(A_\beta) \cap A_\alpha \in \mathscr{A}_\omega(C_\alpha)$ since $C_\beta \subset C_\alpha$. Hence $A'_\alpha \in \mathscr{A}_\omega(C_\alpha)$. Therefore, we have $A' \in \mathscr{F}$. Since A is minimal, then $A = A'$. Hence

$$A_\alpha = \mathrm{co}_\beta(A_\beta) \cap A_\alpha, \quad \text{for every } \alpha < \beta. \tag{54}$$

Let $x \in C_\beta$ and $\alpha < \beta$. Since $A_\beta \subset A_\alpha$, then

$$r_x\left(A_\beta\right) \le r_x\left(A_\alpha\right). \tag{55}$$

Because $\mathrm{co}_\beta(A_\beta) = \cap_{y\in C_\beta} B_\omega(y, r_y(A_\beta))$, then we have

$$\mathrm{co}_\beta\left(A_\beta\right) \subset B_\omega\left(y, r_y\left(A_\beta\right)\right), \tag{56}$$

which implies

$$r_y\left(A_\beta\right) \le r_y\left(A_\alpha\right). \tag{57}$$

Since $A_\alpha \subset \mathrm{co}_\beta(A_\beta)$, then

$$r_y\left(A_\beta\right) \le r_y\left(A_\alpha\right) \le r_y\left(\mathrm{co}_\beta\left(A_\beta\right)\right) \le r_y\left(A_\beta\right). \tag{58}$$

Therefore, we have

$$r_y\left(A_\alpha\right) \le r_y\left(A_\beta\right), \quad \text{for every } \; y \in C_\beta. \tag{59}$$

Using the definition of Chebyshev radius R_ω, we get

$$R_\omega\left(A_\alpha\right) \le R_\omega\left(A_\beta\right). \tag{60}$$

Let $x \in A_\alpha$ and set $s = r_x(A_\alpha)$. Then $x \in \mathrm{co}_\beta(A_\beta)$ since $A_\alpha \subset \mathrm{co}_\beta(A_\beta)$. Hence,

$$x \in \left(\bigcap_{y\in A_\beta} B_\omega\left(y,s\right)\right) \cap \mathrm{co}_\beta\left(A_\beta\right). \tag{61}$$

Since C_β is one-local retract of C, then

$$S_\beta = C_\beta \cap \left(\bigcap_{y\in A_\beta} B_\omega\left(y,s\right)\right) \cap \mathrm{co}_\beta\left(A_\beta\right) \ne \emptyset. \tag{62}$$

Since $A_\beta = C_\beta \cap \mathrm{co}_\beta(A_\beta)$, then we have

$$S_\beta = A_\beta \cap \left(\bigcap_{y\in A_\beta} B_\omega\left(y,s\right)\right). \tag{63}$$

Let $h \in S_\beta$, then $h \in \cap_{y\in A_\beta} B_\omega(y,s)$. Hence, $r_h(A_\beta) \le s$, which implies

$$R_\omega\left(A_\beta\right) \le s = r_x\left(A_\alpha\right), \quad \text{for every } x \in A_\alpha. \tag{64}$$

Hence, $R_\omega(A_\beta) \le R_\omega(A_\alpha)$. Therefore, we have

$$R_\omega\left(A_\beta\right) = R_\omega\left(A_\alpha\right), \quad \text{for every } \alpha, \beta \in \Gamma. \tag{65}$$

Since $\delta_\omega(A_\beta) > 0$, for every $\beta \in \Gamma$, set A''_β to the Chebyshev center of A_β, that is, $A''_\beta = C_\omega(A_\beta)$, for every $\beta \in \Gamma$. Since $R_\omega(A_\beta) = R_\omega(A_\alpha)$, for every $\alpha, \beta \in \Gamma$, then the family (A''_β) is decreasing. Indeed, let $\alpha \prec \beta$ and $x \in A''_\beta$. Then we have $r_x(A_\beta) = R_\omega(A_\beta)$. Since we proved that

$$r_y\left(A_\beta\right) = r_y\left(A_\alpha\right), \quad \text{for every } a \in C_\beta, \tag{66}$$

then

$$r_x\left(A_\alpha\right) = r_x\left(A_\beta\right) = R_\omega\left(A_\beta\right) = R_\omega\left(A_\alpha\right), \tag{67}$$

which implies that $x \in A''_\alpha$. Therefore, we have $A'' = \prod_{\beta\in\Gamma} A''_\beta \in \mathcal{F}$. Since $A'' \subset A$ and A is minimal, we get $A = A''$. Therefore, we have $C_\omega(A_\beta) = A_\beta$ for every $\beta \in \Gamma$. This contradicts the fact that $\mathscr{A}_\omega(C_\beta)$ is normal for every $\beta \in \Gamma$. Hence there exists $\beta_0 \in \Gamma$ such that

$$\delta_\omega\left(A_\beta\right) = 0, \quad \text{for every } \beta \succ \beta_0. \tag{68}$$

The proof of our claim is therefore complete. Then we have $A_\beta = \{x\}$, for every $\beta \succ \beta_0$. This clearly implies that $x \in \cap_{\beta\in\Gamma} C_\beta \ne \emptyset$. In order to complete the proof, we need to show that $S = \cap_{\beta\in\Gamma} C_\beta$ is one-local retract of C. Let $(B_i)_{i\in I}$ be a family of ω-balls centered in S such that $\cap_{i\in I}(B_i) \ne \emptyset$. Set

$$D_\beta = \left(\bigcap_{i\in I} B_i\right) \cap C_\beta, \quad \text{for any } \beta \in \Gamma. \tag{69}$$

Since C_β is a one-local retract of C and the family (B_i) is centered in C_β, then D_β is not empty and $D_\beta \in \mathscr{A}_\omega(C_\beta)$. Therefore,

$$D = \prod_{\beta\in\Gamma} D_\beta \in \mathcal{F}. \tag{70}$$

Let $A = \prod_{\beta\in\Gamma} A_\beta \subset D$ be a minimal element of $\mathcal{F}$. The above proof shows that

$$\bigcap_{\beta\in\Gamma} A_\beta \subset \bigcap_{\beta\in\Gamma} D_\beta \ne \emptyset. \tag{71}$$

The proof of our theorem is complete. □

The next theorem will be useful to prove the main result of the next section.

Theorem 14. *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that $\mathscr{A}_\omega(C)$ is normal and compact. Let $(C_\beta)_{\beta\in\Gamma}$ be a family of one-local retracts of C such that for any finite subset I of Γ. Then $\cap_{\beta\in\Gamma} C_\beta$ is not empty and is one-local retract of C.*

Proof. Consider the family $\mathcal{F}$ of subsets $I \subset \Gamma$ such that, for any finite subset $J \subset \Gamma$ (empty or not), we have $\cap_{\alpha\in I\cup J} C_\alpha$ that is nonempty one-local retract of C. Note that $\mathcal{F}$ is not empty since any finite subset of Γ is in $\mathcal{F}$. Using Theorem 13, we can show that $\mathcal{F}$ satisfies the hypothesis of Zorn's lemma. Hence $\mathcal{F}$ has a maximal element $I \subset \Gamma$. Assume $I \ne \Gamma$. Let $\alpha \in \Gamma \setminus I$. Obviously we have $I \cup \{\alpha\} \in \mathcal{F}$. This is a clear contradiction with the maximality of I. Therefore we have $I = \Gamma \in \mathcal{F}$; that is, $\cap_{\beta\in\Gamma} C_\beta$ is not empty and is a one-local retract of C. □

4. Common Fixed Point Result

In this section we discuss the existence of a common fixed point of a family of commutative ω-nonexpansive mappings in modular metric space which either generalize or improve the corresponding recent common fixed point results of [26, 27].

First, we will need to discuss the case of finite families.

Theorem 15. *Let (X, ω) be a modular metric space and C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that $\mathscr{A}_\omega(C)$ is normal and compact. Let $\mathscr{F} = \{T_1, T_2, \ldots, T_n\}$ be a family of commutative ω-nonexpansive mappings defined on C. Then the family $\mathscr{F}$ has a common fixed point. Moreover, the common fixed point set $\mathrm{Fix}\,(\mathscr{F})$ is a one-local retract of C.*

Proof. First, let us prove that $\mathrm{Fix}\,(\mathscr{F})$ is not empty. Using Theorem 9, $\mathrm{Fix}\,(T_1)$ is nonempty one-local retract of C, and then Theorem 12 implies that $\mathscr{A}_\omega(\mathrm{Fix}\,(T_1))$ is compact and normal. On the other hand since T_1 and T_2 are commutative, we have

$$T_2\left(\mathrm{Fix}\,(T_1)\right) \subset \mathrm{Fix}\,(T_1). \tag{72}$$

Hence T_2 has a fixed point in $\mathrm{Fix}\,(T_1)$. If we restrict ourselves to $\mathrm{Fix}\,(T_1, T_2)$, the common fixed point set of T_1 and T_2, then one can prove in an identical argument that T_3 has a fixed point in $\mathrm{Fix}\,(T_1, T_2)$. Step by step, we can prove that the common fixed point set $\mathrm{Fix}\,(\mathscr{F})$ of $T_1, T_2, \ldots, T_n$ is not empty. The same argument used to prove that the fixed point set of ω-nonexpansive map is a one-local retract can be reduced here to prove that $\mathrm{Fix}\,(\mathscr{F})$ is one-local retract. □

The following result extends [26, Theorem 8] to the setting of modular metric space.

Theorem 16. *Let (X, ω) be a modular metric space and let C be a nonempty ω-closed ω-bounded subset of X_ω. Assume that $\mathscr{A}_\omega(C)$ is normal and compact. Let $\mathscr{F} = (T_i)_{i\in I}$ be a family of commutative ω-nonexpansive mappings defined on C. Then the family $\mathscr{F}$ has a common fixed point. Moreover, the common fixed point set $\mathrm{Fix}\,(\mathscr{F})$ is a one-local retract of C.*

Proof. Let $\Gamma = \{\beta : \beta$ be a nonempty finite subset of $I\}$. Theorem 15 implies that, for every $\beta \in \Gamma$, the set $F_\beta = \bigcap_{i\in\beta} \mathrm{Fix}\,(T_i)$ of common fixed point set of the mappings T_i, $i \in \beta$, is nonempty one-local retract of C. Clearly the family $(F_\beta)_{\beta\in\Gamma}$ is decreasing and satisfies the assumptions of Theorem 14. Therefore, we deduced that $\bigcap_{\beta\in\Gamma} F_\beta$ is nonempty and is a one-local retract of C. □

Acknowledgment

The author would like to thank Professor Mohamed A. Khamsi with whom the author had many fruitful discussions regarding this work.

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