Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 658617, 8 pages
http://dx.doi.org/10.1155/2013/658617

Research Article

On the Solvability of Caputo q-Fractional Boundary Value Problem Involving p-Laplacian Operator

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Received 2 April 2013; Accepted 10 June 2013

Academic Editor: Nazim Idrisoglu Mahmudov

We consider the model of a Caputo q-fractional boundary value problem involving p-Laplacian operator. By using the Banach contraction mapping principle, we prove that, under some conditions, the suggested model of the Caputo q-fractional boundary value problem involving p-Laplacian operator has a unique solution for both cases of $0 < p < 1$ and $p > 2$. It is interesting that in both cases solvability conditions obtained here depend on q, p, and the order of the Caputo q-fractional differential equation. Finally, we illustrate our results with some examples.

1. Introduction

In this section we will give some basic definitions and results that will be needed in the sequel. For more details about the theory of q-calculus, fractional calculus, and q-fractional calculus, we refer readers to [1–10].

Let $q \in (0, 1)$ be a fixed real number. Then for any $\alpha \in \mathbb{R}$,

$$[\alpha]_q := \frac{1 - q^\alpha}{1 - q}. \quad (1)$$

The q-binomial function is defined for all $n \in \mathbb{N}$ as

$$(t - s)_q^n = \prod_{k=0}^{n-1} \left(t - q^k s\right),$$
$$(t - s)_q^\beta = t^\beta \prod_{i=0}^{\infty} \left(\frac{1 - (s/t)\, q^i}{1 - (s/t)\, q^{i+\alpha}} \right), \quad (2)$$

where β is not a positive integer. It is easy to see that

$$(at - as)_q^\beta = a^\beta (t - s)_q^\beta. \quad (3)$$

The q-analog of Euler's gamma function is denoted by $\Gamma_q(t)$ and defined as

$$\Gamma_q(t) = \frac{(1 - q)_q^{t-1}}{(1 - q)^{t-1}}, \quad t > 0. \quad (4)$$

The following theorem will be used to compare values of $\Gamma(t)$, the usual gamma function, with values of $\Gamma_q(t)$ for a fixed $q \in (0, 1)$.

Theorem 1 (see [11]). *For $0 < r < q < 1$, one has*

$$\Gamma_r(t) \le \Gamma_q(t) \le \Gamma(t), \quad \textit{for } 0 < t \le 1 \textit{ or } t \ge 2,$$
$$\Gamma(t) \le \Gamma_q(t) \le \Gamma_r(t), \quad \textit{for } 1 \le t \le 2. \quad (5)$$

It is known that for $0 < q < 1$,

$$\mathbb{T}_q = \left\{q^n; n \in \mathbb{Z}\right\} \cup \{0\},$$
$$\mathbb{T}_q^\alpha = \left\{q^{n+\alpha}; n \in \mathbb{Z}\right\} \cup \{0\}, \quad \alpha \in \mathbb{R}^+ \cup \{0\}. \quad (6)$$

The nabla q-derivative of the function $f : \mathbb{T}_q \to \mathbb{R}$ is defined by

$$\nabla_q f(s) = \frac{f(s) - f(qs)}{(1 - q)\, s}, \quad s \in \mathbb{T}_q - \{0\}. \quad (7)$$

The nabla q-integral of f is defined by

$$\int_0^s f(t)\nabla_q t = (1-q)s\sum_{k=0}^{\infty} q^k f(sq^k). \quad (8)$$

Jackson in [12] and Thomae in [13] showed that the q-beta function, which is defined by

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}, \quad (9)$$

has the following q-integral representation:

$$B_q(t,s) = \int_0^1 \tau^{t-1}(1-q\tau)_q^{s-1}\nabla_q\tau, \qquad t,\ s>0. \quad (10)$$

The fundamental theorem of q-calculus states that

$$\nabla_q \int_0^s f(t)\nabla_q t = f(s), \quad (11)$$

and if f is continuous at 0, then

$$\int_0^s \nabla_q f(t)\nabla_q t = f(s) - f(0). \quad (12)$$

Moreover,

$$\nabla_q \int_0^t f(t,s)\nabla_q s = \int_0^t \nabla_q f(t,s)\nabla_q s + f(qt,t), \quad (13)$$

where the derivative is applied with respect to t.

The nabla q-fractional derivative of $(t-s)_q^{\alpha}$ with respect to t and for all $\alpha \in \mathbb{R}$ is given by

$$\nabla_q(t-s)_q^{\alpha} = \frac{1-q^{\alpha}}{1-q}(t-s)_q^{\alpha-1}. \quad (14)$$

Moreover, the q-fractional integral of order $\alpha \neq 0,-1,-2,\ldots$ is defined by

$${}_qI_0^{\alpha} f(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)_q^{\alpha-1} f(s)\nabla_q s. \quad (15)$$

The α-order Caputo q-fractional derivative of a function f is defined by

$${}_qC_0^{\alpha} f(t) = \frac{1}{\Gamma_q(n-\alpha)}\int_0^t (t-qs)^{n-\alpha-1}\nabla_q^n f(s)\nabla_q s, \quad (16)$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the greatest integer less than or equal to α.

The following lemma enables us to transfer Caputo q-fractional differential equations into an equivalent q-fractional integral equation.

Lemma 2 (see [3]). *Assume that $\alpha>0$ and f is defined on a suitable domain. Then*

$${}_qI_{0q}^{\alpha}C_0^{\alpha} f(t) = f(t) - \sum_{i=0}^{n-1}\frac{t^k}{\Gamma_q(k+1)}\nabla_q^k f(0) \quad (17)$$

and if $0<\alpha\leq 1$, *then*

$${}_qI_{0q}^{\alpha}C_0^{\alpha} f(t) = f(t) - f(0). \quad (18)$$

On the other hand the operator $\varphi_p(s) = |s|^{p-2}s$, where $p>1$ is called the p-Laplacian operator. It is easy to see that $\varphi_p^{-1} = \varphi_r$, where $(1/p)+(1/r)=1$. The following properties of p-Laplacian operator will be used in the rest of the paper.

(P1) if $1<p<2$, $xy>0$, and $|x|,|y|\geq m>0$, then $|\varphi_p(x)-\varphi_p(y)| \leq (p-1)\,m^{p-2}|x-y|$;

(P2) if $p\geq 2$ and $|x|,|y|\leq M$ then, $|\varphi_p(x)-\varphi_p(y)| \leq (p-1)M^{p-2}|x-y|$.

2. A Model of Caputo q-Fractional Boundary Value Problem Involving p-Laplacian Operator

In this paper, our main aim is to prove the existence and uniqueness of the solution for the following Caputo q-fractional boundary value problem involving the p-Laplacian operator:

$$\begin{gathered}\nabla_q\left(\varphi_p\left({}_qC_0^{\alpha}x(t)\right)\right) = f(t,x(t)),\\ \nabla_q^k x(0) = 0, \quad \text{for } k=2,3,\ldots,n-1,\\ x(0) = a_0 x(1),\\ \nabla_q x(0) = a_1\nabla_q x(1),\end{gathered} \quad (19)$$

where $a_0,\ a_1\neq 1,\ 1<\alpha\in\mathbb{R}$, and $f\in C([0,1]\times\mathbb{R},\mathbb{R})$.

Note that, the boundary value problem given in (19) is antiperiodic for $a_0,\ a_1=-1$.

In the following lemma we obtain a q-integral equation which is equivalent to the Caputo q-fractional boundary value problem given in (19).

Lemma 3. *Assume that $\alpha>1$, a_0, $a_1\neq 1$, and $h\in C([0,1])$. Then*

$$\begin{gathered}\nabla_q\left(\varphi_p\left({}_qC_0^{\alpha}x(t)\right)\right) = h(t),\\ \nabla_q^k x(0) = 0, \quad \textit{for } k=2,3,\ldots,n-1,\\ x(0) = a_0 x(1),\\ \nabla_q x(0) = a_1\nabla_q x(1)\end{gathered} \quad (20)$$

are equivalent to the following q-integral equation:

$$\begin{aligned}x(t) &= b_0^q\int_0^t (t-q\tau)_q^{\alpha-1}\varphi_r\left(\int_0^{\tau} h(s)\nabla_q s\right)\nabla_q\tau\\ &+ b_1^q\int_0^1 (1-q\tau)_q^{\alpha-1}\varphi_r\left(\int_0^{\tau} h(s)\nabla_q s\right)\nabla_q\tau\\ &+ b_2^q(t)\int_0^1 (1-q\tau)_q^{\alpha-2}\varphi_r\left(\int_0^{\tau} h(s)\nabla_q s\right)\nabla_q\tau,\end{aligned} \quad (21)$$

where $b_0^q = 1/(\Gamma_q(\alpha))$, $b_1^q = a_0/(\Gamma_q(\alpha)(1-a_0))$, *and* $b_2^q(t) = (a_1(t+a_0(1-t)))/(\Gamma_q(\alpha-1)(1-a_0)(1-a_1))$.

Proof. Using (20) and the fact that $\varphi_p({}_qC_0^{\alpha}x(0)) = 0$, we have

$$\varphi_p\left({}_qC_0^{\alpha}x(t)\right) = \int_0^t h(s)\nabla_q s, \tag{22}$$

or equivalently,

$${}_qC_0^{\alpha}x(t) = \varphi_r\left(\int_0^t h(s)\nabla_q s\right). \tag{23}$$

Applying q-fractional integral operator ${}_qI_0^{\alpha}$ to both sides and using Lemma 2, we get

$$\begin{aligned} x(t) &- \sum_{k=0}^{n-1}\frac{t^k}{\Gamma_q(k+1)}\nabla_q^k x(0) \\ &= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-q\tau)_q^{\alpha-1}\varphi_r\left(\int_0^{\tau} h(s)\nabla_q s\right)\nabla_q\tau. \end{aligned} \tag{24}$$

Using $\nabla_q^k x(0) = 0$, for $k = 2, 3, \ldots, [\alpha]-1$ in (24), we obtain

$$\begin{aligned} x(t) &= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-q\tau)_q^{\alpha-1}\varphi_r\left(\int_0^{\tau} h(s)\nabla_q s\right)\nabla_q\tau \\ &\quad + x(0) + t\nabla_q x(0). \end{aligned} \tag{25}$$

According to (13) and (14), we have

$$\begin{aligned} \nabla_q x(t) &= \frac{1}{\Gamma_q(\alpha-1)}\int_0^t (t-q\tau)_q^{\alpha-2}\varphi_r\left(\int_0^{\tau} h(s)\nabla_q s\right)\nabla_q\tau \\ &\quad + \nabla_q x(0). \end{aligned} \tag{26}$$

Taking $t = 1$ in both sides of (25) and (26), we get

$$\begin{aligned} x(1) &= \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-q\tau)_q^{\alpha-1}\varphi_r\left(\int_0^{\tau} h(s)\nabla_q s\right)\nabla_q\tau \\ &\quad + x(0) + \nabla_q x(0), \\ \nabla_q x(1) &= \frac{1}{\Gamma_q(\alpha-1)}\int_0^1 (1-q\tau)_q^{\alpha-2}\varphi_r\left(\int_0^{\tau} h(s)\nabla_q s\right)\nabla_q\tau \\ &\quad + \nabla_q x(0). \end{aligned} \tag{27}$$

Solving equations obtained by the given boundary value conditions $x(0) = a_0x(1)$ and $\nabla_q x(0) = a_1\nabla_q x(1)$, it follows that

$$\begin{aligned} \nabla_q x(0) &= \frac{a_1}{\Gamma_q(\alpha-1)(1-a_1)} \\ &\quad\times\int_0^1 (1-q\tau)_q^{\alpha-2}\varphi_r\left(\int_0^{\tau} h(s)\nabla_q s\right)\nabla_q\tau, \\ x(0) &= \frac{a_0}{\Gamma_q(\alpha)(1-a_0)}\int_0^1 (1-q\tau)_q^{\alpha-1}\varphi_r\left(\int_0^{\tau} h(s)\nabla_q s\right)\nabla_q\tau \\ &\quad + \frac{a_0}{\Gamma_q(\alpha-1)(1-a_0)(1-a_1)} \\ &\quad\times\int_0^1 (1-q\tau)_q^{\alpha-2}\varphi_r\left(\int_0^{\tau} h(s)\nabla_q s\right)\nabla_q\tau. \end{aligned} \tag{28}$$

Substituting (28) into (25) gives (21) which completes the proof. □

3. Solvability of the Caputo q-Fractional Boundary Value Problem

This section is devoted to the solvability of the Caputo q-fractional boundary value problem given in (19). In the first part we shall prove the existence and uniqueness of the solution, and then we shall illustrate our main results with some examples.

Recall that $C[0,1]$ is a Banach space with the norm $\|x\| = \max_{t\in[0,1]}|x(t)|$. Now consider $T_i : C[0,1] \to C[0,1]$, $i = 0, 1$ with

$$\begin{aligned} T_0x(t) &:= \varphi_r\left(\int_0^t f(s,x(s))\nabla_q s\right), \\ T_1x(t) &= b_0^q\int_0^t (t-q\tau)_q^{\alpha-1}x(\tau)\nabla_q\tau \\ &\quad + b_1^q\int_0^1 (1-q\tau)_q^{\alpha-1}x(\tau)\nabla_q\tau \\ &\quad + b_2^q(t)\int_0^1 (1-q\tau)_q^{\alpha-2}x(\tau)\nabla_q\tau. \end{aligned} \tag{29}$$

Then $T = T_1 \circ T_0$ is a continuous and compact operator.

Theorem 4. *Suppose that $1 < r < 2$, a_0, $a_1 \neq 1$, $q \in (0,1)$ is fixed, and the following conditions hold: $\exists\lambda > 0$, $0 < \delta < 2/(2-r)$ and d with*

$$\begin{aligned} 0 &< d \\ &< \lambda^{2-r}\frac{\Gamma_q(\delta(r-2)+2+\alpha)}{(r-1)\Gamma_q(\delta(r-2)+2)} \\ &\quad\times\left[\frac{|1-a_0||1-a_1|}{(|1-a_0|+|a_0|)(|1-a_1|+|a_1|[\delta(r-2)+\alpha+1]_q)}\right] \end{aligned} \tag{30}$$

such that

$$[\delta]_q\lambda t^{\delta-1} \leq f(t,x), \quad \text{for any } (t,x) \in (0,1]\times\mathbb{R}, \tag{31}$$

$$|f(t,x) - f(t,y)| \leq d|x-y|, \quad \text{for } t \in [0,1],\ x, y \in \mathbb{R}. \tag{32}$$

Then the boundary value problem (19) *has a unique solution.*

Proof. Inequality given in (31) implies that

$$\lambda t^{\delta} \leq \int_0^t f(s,x)\nabla_q s, \quad \text{for any } (t,x)\in[0,1]\times\mathbb{R}. \tag{33}$$

On the other hand using (P1) and (32), we have

$$\begin{aligned}
&|T_0x(t)-T_0y(t)| \\
&= \left|\varphi_r\left(\int_0^t f(s,x(s))\nabla_q s\right)-\varphi_r\left(\int_0^t f(s,y(s))\nabla_q s\right)\right| \\
&\leq (r-1)\left(\lambda t^{\delta}\right)^{r-2}\left|\int_0^t f(s,x(s))\nabla_q s-\int_0^t f(s,y(s))\nabla_q s\right| \\
&\leq (r-1)\left(\lambda t^{\delta}\right)^{r-2}\int_0^t \left|f(s,x(s))-f(s,y(s))\right|\nabla_q s \\
&\leq d(r-1)\left(\lambda t^{\delta}\right)^{r-2}\int_0^t |x(s)-y(s)|\nabla_q s \\
&\leq d(r-1)\lambda^{r-2}t^{\delta(r-2)+1}\|x-y\|.
\end{aligned} \tag{34}$$

Similarly,

$$\begin{aligned}
&|Tx(t)-Ty(t)| \\
&= |T_1(T_0x(t))-T_1(T_0y(t))| \\
&= \left|b_0^q\int_0^t (t-q\tau)_q^{\alpha-1}\left((T_0x)(\tau)-(T_0y)(\tau)\right)\nabla_q\tau\right. \\
&\quad + b_1^q\int_0^1 (1-q\tau)_q^{\alpha-1}\left((T_0x)(\tau)-(T_0y)(\tau)\right)\nabla_q \\
&\quad \left.+b_2^q(t)\int_0^1 (1-q\tau)_q^{\alpha-2}\left((T_0x)(\tau)-(T_0y)(\tau)\right)\nabla_q\tau\right|.
\end{aligned} \tag{35}$$

Finally using (34) in (35), we get

$$\begin{aligned}
&|Tx(t)-Ty(t)| \\
&\quad\leq d(r-1)\lambda^{r-2}\|x-y\| \\
&\quad\quad\times\left[b_0^q\int_0^t (t-q\tau)_q^{\alpha-1}\tau^{\delta(r-2)+1}\nabla_q\tau\right. \\
&\quad\quad\quad + |b_1^q|\int_0^1 (1-q\tau)_q^{\alpha-1}\tau^{\delta(r-2)+1}\nabla_q\tau \\
&\quad\quad\quad \left.+|b_2^q(t)|\int_0^1 (1-q\tau)_q^{\alpha-2}\tau^{\delta(r-2)+1}\nabla_q\tau\right].
\end{aligned} \tag{36}$$

Since

$$\begin{aligned}
&\int_0^t (t-q\tau)_q^{\alpha-1}\tau^{\delta(r-2)+1}\nabla_q\tau \\
&\quad = \int_0^1 (1-q\tau)_q^{\alpha-1}t^{\delta(r-2)+\alpha+1}\tau^{\delta(r-2)+1}\nabla_q\tau,
\end{aligned} \tag{37}$$

we have

$$\begin{aligned}
&|Tx(t)-Ty(t)| \\
&\quad\leq d(r-1)\lambda^{r-2}\|x-y\| \\
&\quad\quad\times\left[b_0^q\int_0^1 (1-q\tau)_q^{\alpha-1}t^{\delta(r-2)+\alpha+1}\tau^{\delta(r-2)+1}\nabla_q\tau\right. \\
&\quad\quad\quad + |b_1^q|\int_0^1 (1-q\tau)_q^{\alpha-1}\tau^{\delta(r-2)+1}\nabla_q\tau \\
&\quad\quad\quad \left.+|b_2^q(t)|\int_0^1 (1-q\tau)_q^{\alpha-2}\tau^{\delta(r-2)+1}\nabla_q\tau\right] \\
&\quad = d(r-1)\lambda^{r-2}\|x-y\| \\
&\quad\quad\times\left[b_0^q t^{\delta(r-2)+\alpha+1}B_q(\delta(r-2)+2,\alpha)\right. \\
&\quad\quad\quad + |b_1^q|B_q(\delta(r-2)+2,\alpha) \\
&\quad\quad\quad \left.+|b_2^q(t)|B_q(\delta(r-2)+2,\alpha-1)\right] \\
&\quad = d(r-1)\lambda^{r-2}\|x-y\| \\
&\quad\quad\times\left[b_0^q t^{\delta(r-2)+\alpha+1}B_q(\delta(r-2)+2,\alpha)\right. \\
&\quad\quad\quad + |b_1^q|B_q(\delta(r-2)+2,\alpha)+|b_2^q(t)| \\
&\quad\quad\quad \left.\times\frac{[\delta(r-2)+\alpha+1]_q}{[\alpha-1]_q}B_q(\delta(r-2)+2,\alpha)\right] \\
&\quad\leq d(r-1)\lambda^{r-2}\|x-y\|B_q(\delta(r-2)+2,\alpha) \\
&\quad\quad\times\left[b_0^q t^{\delta(r-2)+\alpha+1}+|b_1^q|+|b_2^q(t)|\frac{[\delta(r-2)+\alpha+1]_q}{[\alpha-1]_q}\right].
\end{aligned} \tag{38}$$

In other words,

$$\begin{aligned}
&|Tx(t)-Ty(t)| \\
&\quad\leq d(r-1)\lambda^{r-2}\|x-y\|\frac{\Gamma_q(\delta(r-2)+2)\Gamma_q(\alpha)}{\Gamma_q(\delta(r-2)+2+\alpha)} \\
&\quad\quad\times\left[\frac{1}{\Gamma_q(\alpha)}t^{\delta(r-2)+\alpha+1}+\left|\frac{a_0}{\Gamma_q(\alpha)(1-a_0)}\right|\right. \\
&\quad\quad\quad \left.+\left|\frac{a_1(t+a_0(1-t))}{\Gamma_q(\alpha-1)(1-a_0)(1-a_1)}\right|\frac{[\delta(r-2)+\alpha+1]_q}{[\alpha-1]_q}\right] \\
&\quad\leq d(r-1)\lambda^{r-2}\|x-y\|\frac{\Gamma_q(\delta(r-2)+2)\Gamma_q(\alpha)}{\Gamma_q(\delta(r-2)+2+\alpha)} \\
&\quad\quad\times\left[\frac{1}{\Gamma_q(\alpha)}+\frac{|a_0|}{\Gamma_q(\alpha)|1-a_0|}\right. \\
&\quad\quad\quad \left.+\frac{|a_1|(|a_0|+|1-a_0|)[\delta(r-2)+\alpha+1]_q}{\Gamma_q(\alpha)|1-a_0||1-a_1|}\right]
\end{aligned}$$

$$\leq d(r-1)\lambda^{r-2}\|x-y\|\frac{\Gamma_q(\delta(r-2)+2)\Gamma_q(\alpha)}{\Gamma_q(\delta(r-2)+2+\alpha)}$$
$$\times\Big[\big(|1-a_0||1-a_1|+|a_0||1-a_1|$$
$$+|a_1|(|a_0|+|1-a_0|)[\delta(r-2)+\alpha+1]_q\big)$$
$$\times\big(\Gamma_q(\alpha)|1-a_0||1-a_1|\big)^{-1}\Big]$$
$$=d(r-1)\lambda^{r-2}\frac{\Gamma_q(\delta(r-2)+2)}{\Gamma_q(\delta(r-2)+2+\alpha)}$$
$$\times\left[\frac{(|1-a_0|+|a_0|)(|1-a_1|+|a_1|[\delta(r-2)+\alpha+1]_q)}{|1-a_0||1-a_1|}\right]$$
$$\times\|x-y\|=K\|x-y\|, \tag{39}$$

where $K=d(r-1)\lambda^{r-2}(\Gamma_q(\delta(r-2)+2)/\Gamma_q(\delta(r-2)+2+\alpha))[((|1-a_0|+|a_0|)(|1-a_1|+|a_1|[\delta(r-2)+\alpha+1]_q))/(|1-a_0||1-a_1|)]$.

By condition (30), we get $0<K<1$, which implies that T is a contraction. As a consequence of the Banach contraction mapping theorem and Lemma 3, the boundary value problem given in (19) has a unique solution. □

Theorem 5. *Suppose that $1<r<2$, $a_0, a_1\neq 1$, and the following conditions hold for a fixed $q\in(0,1)$, $\exists\lambda>0$, $0<\delta<2/(2-r)$, and d with*

$$0<d$$
$$<\lambda^{2-r}\frac{\Gamma_q(\delta(r-2)+2+\alpha)}{(r-1)\Gamma_q(\delta(r-2)+2)}$$
$$\times\left[\frac{|1-a_0||1-a_1|}{(|1-a_0|+|a_0|)(|1-a_1|+|a_1|[\delta(r-2)+\alpha+1]_q)}\right] \tag{40}$$

such that

$$f(t,x)\leq-[\delta]_q\lambda t^{\delta-1},\quad \textit{for any } (t,x)\in(0,1]\times\mathbb{R},$$
$$|f(t,x)-f(t,y)|\leq d|x-y|,\quad \textit{for } t\in[0,1],\ x,y\in\mathbb{R}. \tag{41}$$

Then the boundary value problem (19) *has a unique solution.*

Remark 6. When $q\to 1$, Theorems 4 and 5 reduce to Theorems 3.1 and 3.2 of [14].

Theorem 7. *Suppose that $r>2$, $a_0, a_1\neq 1$, and the following conditions hold for a fixed $q\in(0,1)$. There exists a nonnegative function $g(x)\in L[0,1]$ with $M:=\int_0^1 g(\tau)\nabla_q\tau\geq 0$ such that*

$$|f(t,x)|\leq g(t),\quad \textit{for any } (t,x)\in[0,1]\times\mathbb{R}, \tag{42}$$

and there exists a constant d with

$$0<d$$
$$<\frac{\Gamma_q(\alpha+2)}{(r-1)M^{r-2}}$$
$$\times\left[\frac{|1-a_0||1-a_1|}{(|1-a_0|+|a_0|)(|1-a_1|+|a_1|[\alpha+1]_q)}\right], \tag{43}$$

$$|f(t,x)-f(t,y)|\leq d|x-y|,\quad \textit{for } t\in[0,1],\ x,y\in\mathbb{R}. \tag{44}$$

Then the boundary value problem (19) *has a unique solution.*

Proof. By (42), we can get that

$$\int_0^t|f(\tau,x(\tau))|\nabla_q\tau\leq\int_0^1 g(\tau)\nabla_q\tau=M \tag{45}$$

for all $t\in[0,1]$. By the definition of T_0, we have

$$|T_0x(t)-T_0y(t)|$$
$$=\left|\varphi_r\left(\int_0^t f(s,x(s))\nabla_q s\right)-\varphi_r\left(\int_0^t f(s,y(s))\nabla_q s\right)\right|. \tag{46}$$

Using (P2) and (45) gives

$$|T_0x(t)-T_0y(t)|$$
$$\leq(r-1)M^{r-2}$$
$$\times\left|\int_0^t f(s,x(s))\nabla_q s-\int_0^t f(s,y(s))\nabla_q s\right|$$
$$\leq(r-1)M^{r-2}$$
$$\times\int_0^t|f(s,x(s))-f(s,y(s))|\nabla_q s \tag{47}$$
$$\leq d(r-1)M^{r-2}\int_0^t|x(s)-y(s)|\nabla_q s$$
$$\leq d(r-1)M^{r-2}t\|x-y\|.$$

Therefore,

$$|Tx(t)-Ty(t)|$$
$$=|T_1(T_0x(t))-T_1(T_0y(t))|$$
$$=\Big|b_0^q\int_0^t(t-q\tau)_q^{\alpha-1}((T_0x)(\tau)-(T_0y)(\tau))\nabla_q\tau$$
$$+b_1^q\int_0^1(1-q\tau)_q^{\alpha-1}((T_0x)(\tau)-(T_0y)(\tau))\nabla_q$$
$$+b_2^q(t)\int_0^1(1-q\tau)_q^{\alpha-2}((T_0x)(\tau)-(T_0y)(\tau))\nabla_q\tau\Big|$$

$$\leq d(r-1)M^{r-2}\|x-y\| \times\left[\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-q\tau)_q^{\alpha-1}\tau\nabla_q\tau + \frac{|a_0|}{\Gamma_q(\alpha)|1-a_0|}\int_0^1(1-q\tau)_q^{\alpha-1}\tau\nabla_q\tau + \frac{|a_1(t+a_0(1-t))|}{\Gamma_q(\alpha-1)|1-a_0||1-a_1|}\int_0^1(1-q\tau)_q^{\alpha-2}\tau\nabla_q\tau\right]. \tag{48}$$

Since

$$\int_0^t(t-q\tau)_q^{\alpha-1}\tau\nabla_q\tau = \int_0^1 t^{\alpha+1}(1-q\tau)_q^{\alpha-1}\tau\nabla_q\tau \tag{49}$$

we have

$$|Tx(t)-Ty(t)| \leq d(r-1)M^{r-2}\|x-y\| \times\left[\frac{1}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha+1}(1-q\tau)_q^{\alpha-1}\tau\nabla_q\tau + \frac{|a_0|}{\Gamma_q(\alpha)|1-a_0|}\int_0^1(1-q\tau)_q^{\alpha-1}\tau\nabla_q\tau + \frac{|a_1|(|a_0|+t|1-a_0|)}{\Gamma_q(\alpha-1)|1-a_0||1-a_1|}\int_0^1(1-q\tau)_q^{\alpha-2}\tau\nabla_q\tau\right]. \tag{50}$$

Using q-Beta function and the fact that $t\in[0,1]$, we get

$$\begin{aligned}|Tx(t)-Ty(t)| &\leq d(r-1)M^{r-2}\|x-y\| \times\left[\frac{1}{\Gamma_q(\alpha)}B_q(2,\alpha)+\frac{|a_0|}{\Gamma_q(\alpha)|1-a_0|}B_q(2,\alpha) + \frac{|a_1|(|a_0|+|1-a_0|)}{\Gamma_q(\alpha-1)|1-a_0||1-a_1|}\frac{[\alpha+1]_q}{[\alpha-1]_q}B_q(2,\alpha)\right]\\ &\leq d(r-1)M^{r-2}B_q(2,\alpha)\|x-y\| \times\left[\frac{1}{\Gamma_q(\alpha)}+\frac{|a_0|}{\Gamma_q(\alpha)|1-a_0|} + \frac{|a_1|(|a_0|+|1-a_0|)[\alpha+1]_q}{\Gamma_q(\alpha)|1-a_0||1-a_1|}\right]\\ &\leq d(r-1)M^{r-2}B_q(2,\alpha)\|x-y\| \times\left[\left(|1-a_0||1-a_1|+|a_0||1-a_1| + |a_1|(|a_0|+|1-a_0|)[\alpha+1]_q\right) \times\left(\Gamma_q(\alpha)|1-a_0||1-a_1|\right)^{-1}\right]\\ &\leq \frac{d(r-1)M^{r-2}}{\Gamma_q(\alpha+2)} \times\left[\frac{(|1-a_0|+|a_0|)(|1-a_1|+|a_1|[\alpha+1]_q)}{|1-a_0||1-a_1|}\right] \times\|x-y\| \leq K\|x-y\|,\end{aligned} \tag{51}$$

where

$$K=\frac{d(r-1)M^{r-2}}{\Gamma_q(\alpha+2)} \times\left[\frac{(|1-a_0|+|a_0|)(|1-a_1|+|a_1|[\alpha+1]_q)}{|1-a_0||1-a_1|}\right]. \tag{52}$$

By condition (43), we get $K<1$ which implies that T is a contraction; therefore boundary value problem given in (19) has a unique solution. □

Next, we give some examples to illustrate our results.

Example 8. Consider the following Boundary value problem

$$\begin{gathered}\nabla_q\left(\varphi_{7/3}\left({}_qC_0^{3/2}x(t)\right)\right) = 4t^2\left(2+\cos\left(\frac{\sqrt{\pi}x}{24}+\omega\right)\right), \quad t\in(0,1),\\ \nabla_q^k x(0)=0, \quad \text{for } k=2,3,\ldots,n-1,\\ x(0)=\frac{1}{2}x(1),\\ \nabla_q x(0)=\frac{1}{2}\nabla_q x(1),\end{gathered} \tag{53}$$

where

$$p=\frac{7}{3},\qquad \alpha=\frac{3}{2},\qquad \delta=4,\qquad a_0=\frac{1}{2},\qquad a_1=\frac{1}{2}. \tag{54}$$

Then $r=7/4$, and take $\delta=4$, $\lambda=1$, and $d=\sqrt{\pi}/6$. Using Theorem 1 and the fact that $[3/2]_q>1$ for any fixed $q\in(0,1)$, we have

$$\begin{aligned}
&\lambda^{2-r}\frac{\Gamma_q(\delta(r-2)+2+\alpha)}{(r-1)\Gamma_q(\delta(r-2)+2)} \\
&\quad\times\Big[(|1-a_0||1-a_1|) \\
&\quad\quad\times\big((|1-a_0|+|a_0|) \\
&\quad\quad\quad\times(|1-a_1|+|a_1|[\delta(r-2)+\alpha+1]_q)\big)^{-1}\Big] \\
&\quad=\frac{2}{3}\left[\frac{[3/2]_q\Gamma_q(3/2)}{(1+[3/2]_q)}\right]>\frac{2}{3}\left[\frac{[3/2]_q\Gamma_q(3/2)}{([3/2]_q+[3/2]_q)}\right] \\
&\quad>\frac{1}{3}\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{6}=d>0, \\
&K=d(r-1)\lambda^{r-2}\frac{\Gamma_q(\delta(r-2)+2)}{\Gamma_q(\delta(r-2)+2+\alpha)} \\
&\quad\times\left[\frac{(|1-a_0|+|a_0|)(|1-a_1|+|a_1|[\delta(r-2)+\alpha+1]_q)}{|1-a_0||1-a_1|}\right] \\
&\quad=\frac{\sqrt{\pi}}{4\Gamma_q(5/2)}\left[\left(1+\left[\frac{3}{2}\right]_q\right)\right] \\
&\quad<\frac{\sqrt{\pi}}{4[3/2]_q\Gamma_q(3/2)}\left[\left(\left[\frac{3}{2}\right]_q+\left[\frac{3}{2}\right]_q\right)\right]\le\frac{\sqrt{\pi}}{2\Gamma(3/2)}=1.
\end{aligned}\tag{55}$$

Moreover, it can be easily seen that

$$\begin{aligned}
[\delta]_q\lambda t^{\delta-1}&=[4]_q t^3\le 4t^2\left(2+\cos\left(\frac{\sqrt{\pi}x}{24}+\omega\right)\right) \\
&=f(t,x).
\end{aligned}\tag{56}$$

Finally,

$$\begin{aligned}
&|f(t,x)-f(t,y)| \\
&\quad=\left|4t^2\left(2+\cos\left(\frac{\sqrt{\pi}x}{24}+\omega\right)\right)\right. \\
&\quad\quad\left.-4t^2\left(2+\cos\left(\frac{\sqrt{\pi}y}{24}+\omega\right)\right)\right| \\
&\quad=4t^2\left|\cos\left(\frac{\sqrt{\pi}x}{24}+\omega\right)-\cos\left(\frac{\sqrt{\pi}y}{24}+\omega\right)\right| \\
&\quad\le 4\left|\left(\frac{\sqrt{\pi}x}{24}+\omega\right)-\left(\frac{\sqrt{\pi}y}{24}+\omega\right)\right| \\
&\quad=\frac{\sqrt{\pi}}{6}|x-y|.
\end{aligned}\tag{57}$$

Therefore as a consequence of Theorem 4, boundary value problem given in (53) has a unique solution.

Example 9. Consider the following boundary value problem:

$$\begin{aligned}
&\nabla_q\left(\varphi_{9/4}\left(\varphi_{31/15}\left({}_qC_0^{3/2}x(t)\right)\right)\right) \\
&\quad=4t^2\left(2+\cos\left(\frac{\sqrt{\pi}x}{24}+\omega\right)\right),\quad t\in(0,1), \\
&\nabla_q^k x(0)=0,\quad \text{for } k=2,3,\ldots,n-1, \\
&x(0)=\frac{1}{2}x(1), \\
&\nabla_q x(0)=\frac{1}{2}\nabla_q x(1).
\end{aligned}\tag{58}$$

Then

$$\begin{aligned}
\varphi_{9/4}(\varphi_{31/15}(s))&=\varphi_{9/4}\left(|s|^{1/15}s\right)=\left||s|^{1/15}s\right|^{(9/4)-2}|s|^{1/15}s \\
&=|s|^{(1/15)(1/4)}|s|^{1/4}|s|^{1/15}s \\
&=|s|^{(1/60)+(1/4)+(1/15)}s=|s|^{1/3}s \\
&=|s|^{(7/3)-2}s=\varphi_{7/3}(s).
\end{aligned}\tag{59}$$

Therefore boundary value problem given in (58) reduces to the boundary value problem given in (53), and it has a unique solution.

Example 10. Now consider the following antiperiodic boundary value problem:

$$\begin{aligned}
&\nabla_q\left(\varphi_{7/4}\left({}_qC_0^{3/2}x(t)\right)\right) \\
&\quad=\left(\sin^2\left(\frac{\sqrt{\pi}x}{40}+\omega\right)\right),\quad t\in(0,1), \\
&\nabla_q^k x(0)=0,\quad \text{for } k=2,3,\ldots,n-1, \\
&x(0)=-x(1), \\
&\nabla_q x(0)=-\nabla_q x(1),
\end{aligned}\tag{60}$$

where

$$p=\frac{7}{4},\qquad \alpha=\frac{3}{2},\qquad a_0=-1,\qquad a_1=-1.\tag{61}$$

Then $r = 7/3$, and take $d = \sqrt{\pi}/20$. Using Theorem 1 and taking $g(t) = 1$, we get that

$$\begin{aligned} M &= 1, \\ \frac{\Gamma_q(\alpha+2)}{(r-1)M^{r-2}} & \\ &\times \left[\frac{|1-a_0|\,|1-a_1|}{\left(|1-a_0|+|a_0|\right)\left(|1-a_1|+|a_1|\,[\alpha+1]_q\right)} \right] \\ &= \left[\frac{\Gamma_q(7/2)}{\left(2+[5/2]_q\right)} \right] > \left[\frac{[5/2]_q[3/2]_q\Gamma_q(3/2)}{\left(2[5/2]_q+[5/2]_q\right)} \right] \\ &= \frac{[3/2]_q\Gamma_q(3/2)}{3} > \frac{\Gamma(3/2)}{3} = \frac{\sqrt{\pi}}{6} > \frac{\sqrt{\pi}}{20} = d. \end{aligned} \quad (62)$$

On the other hand,

$$\begin{aligned} |f(t,x)-f(t,y)| &\le \left| \sin^2\left(\frac{\sqrt{\pi}x}{40}+\omega\right) - \sin^2\left(\frac{\sqrt{\pi}y}{40}+\omega\right) \right| \\ &\le \frac{\sqrt{\pi}}{20}|x-y|, \quad \text{for } t\in[0,1],\ x,y\in\mathbb{R}, \\ K &= \frac{d(r-1)M^{r-2}}{\Gamma_q(\alpha+2)} \\ &\quad\times \left[\frac{\left(|1-a_0|+|a_0|\right)\left(|1-a_1|+|a_1|[\alpha+1]_q\right)}{|1-a_0||1-a_1|} \right] \\ &= \frac{\sqrt{\pi}}{20\Gamma_q(7/2)}\left[2+\left[\frac{5}{2}\right]_q\right] \\ &< \frac{\sqrt{\pi}}{20[5/2]_q[3/2]_q\Gamma_q(3/2)}\left[3\left[\frac{5}{2}\right]_q\right] \\ &= \frac{3\sqrt{\pi}}{20[3/2]_q\Gamma_q(3/2)} < \frac{3\sqrt{\pi}}{20\Gamma(3/2)} = \frac{3}{10} < 1. \end{aligned} \quad (63)$$

Therefore by Theorem 7, the antiperiodic boundary value problem given in (60) has a unique solution.

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