Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 648709, 12 pages
http://dx.doi.org/10.1155/2013/648709

Research Article

Approximate Multi-Jensen, Multi-Euler-Lagrange Additive and Quadratic Mappings in n-Banach Spaces

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Received 26 April 2013; Accepted 31 July 2013

Academic Editor: Krzysztof Ciepliński

We prove the generalized Hyers-Ulam stability of multi-Jensen, multi-Euler-Lagrange additive, and quadratic mappings in n-Banach spaces, using the socalled direct method. The corollaries from our main results correct some outcomes from Park (2011).

1. Introduction and Preliminaries

In 2005, Prager and Schwaiger (see [1] and also [2]) introduced the notion of multi-Jensen functions with the connection with generalized polynomials and obtained their general form. In 2008, (see [3]) they also proved the Hyers-Ulam stability of multi-Jensen equation, whereas Ciepliński (see [4, 5]) showed its generalized stability: in the spirit of Bourgin (see [6]) and Găvruţa (see [7]), and in the spirit of Aoki (see [8]) and Rassias (see [9]). Recently, some further results on the stability of multi-Jensen mappings were obtained in [10–14]. We refer the reader to [15–19] for more information on different aspects of stability of functional equations.

In this paper, we deal with the generalized Hyers-Ulam stability of multi-Jensen, multi-Euler-Lagrange additive, and quadratic mappings in n-Banach spaces. The corollaries from our main results correct some outcomes from [20]. The results of Sections 2 and 4 generalize those from [12].

The concept of 2-normed spaces was initially developed by Gähler [21, 22] in the middle of the 1960s, while that of n-normed spaces can be found in [23, 24]. Since then, many others have studied this concept and obtained various results (see [23, 25–27]).

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers and $\mathbb{R}$ represents the set of all real numbers. Moreover, we fix two positive integers k and n.

We recall some basic facts concerning n-normed spaces.

Definition 1. Let $n \in \mathbb{N}$ and let X be a real linear space with $\dim X \geq n$, and let $\|\cdot,\dots,\cdot\| : X^n \to \mathbb{R}$ be a function satisfying the following properties:

(N1) $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent,

(N2) $\|x_1,\dots,x_n\|$ is invariant under permutation,

(N3) $\|\alpha x_1,\dots,x_n\| = |\alpha|\|x_1,\dots,x_n\|$,

(N4) $\|x+y,x_2,\dots,x_n\| \leq \|x,x_2,\dots,x_n\| + \|y,x_2,\dots,x_n\|$

for all $\alpha \in \mathbb{R}$ and $x, y, x_1, x_2, \dots, x_n \in X$. Then the function $\|\cdot,\dots,\cdot\|$ is called an n-norm on X, and the pair $(X, \|\cdot,\dots,\cdot\|)$ is called an n-normed space.

A sequence $\{x_j\}_{j\in\mathbb{N}}$ in an n-normed space X is said to a converge to some $x \in X$ in the n-norm if

$$\lim_{j\to\infty} \left\|x_j - x, y_2, \dots, y_n\right\| = 0, \tag{1}$$

for every $y_2,\dots,y_n \in X$. Every convergent sequence has exactly one limit. If x is the limit of the sequence $\{x_j\}_{j\in\mathbb{N}}$, then we write $\lim_{j\to\infty} x_j = x$. For any convergent sequences $\{x_j\}_{j\in\mathbb{N}}$ and $\{y_j\}_{j\in\mathbb{N}}$ of elements of X, the sequence $\{x_j+y_j\}_{j\in\mathbb{N}}$ is convergent and

$$\lim_{j\to\infty} \left(x_j + y_j\right) = \lim_{j\to\infty} x_j + \lim_{j\to\infty} y_j. \tag{2}$$

If, moreover, $\{\alpha_j\}_{j\in\mathbb{N}}$ is a convergent sequence of real numbers, then the sequence $\{\alpha_j \cdot x_j\}_{j\in\mathbb{N}}$ is also convergent and

$$\lim_{j\to\infty} \left(\alpha_j \cdot x_j\right) = \lim_{j\to\infty} \alpha_j \cdot \lim_{j\to\infty} x_j. \tag{3}$$

A sequence $\{x_j\}_{j\in\mathbb{N}}$ in an n-normed space X is said to be a Cauchy sequence with respect to the n-norm if

$$\lim_{j,l\to\infty}\left\|x_j-x_l,y_2,\dots,y_n\right\|=0, \tag{4}$$

for every $y_2,\dots,y_n\in X$. A linear n-normed space in which every Cauchy sequence is convergent is called an n-Banach space.

Example 2. For $x_1,\dots,x_n\in\mathbb{R}^n$, the Euclidean n-norm $\|x_1,\dots,x_n\|_E$ is defined by

$$\|x_1,\dots,x_n\|_E=\left|\det\left(x_{ij}\right)\right|=\operatorname{abs}\left(\begin{vmatrix}x_{11}&\cdots&x_{1n}\\ \vdots&\ddots&\vdots\\ x_{n1}&\cdots&x_{nn}\end{vmatrix}\right), \tag{5}$$

where $x_i=(x_{i1},\dots,x_{in})\in\mathbb{R}^n$ for each $i=1,\dots,n$.

Example 3. The standard n-norm on X, a real inner product space of dimension $\dim X\ge n$, is as follows:

$$\|x_1,\dots,x_n\|_S=\begin{vmatrix}\langle x_1,x_1\rangle&\cdots&\langle x_1,x_n\rangle\\ \vdots&\ddots&\vdots\\ \langle x_n,x_1\rangle&\cdots&\langle x_n,x_n\rangle\end{vmatrix}^{1/2}, \tag{6}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on X. If $X=\mathbb{R}^n$, then this n-norm is exactly the same as the Euclidean n-norm $\|x_1,\dots,x_n\|_E$ mentioned earlier. For $n=1$, this n-norm is the usual norm $\|x_1\|=\langle x_1,x_1\rangle^{1/2}$.

In what follows, we will also use the following lemma from [19].

Lemma 4. *Let X be an n-normed space. Then,*

(1) *for $x_i\in X$ $(i=1,\dots,n)$ and γ, a real number,*

$$\begin{aligned}&\left\|x_1,\dots,x_i,\dots,x_j,\dots,x_n\right\|\\&\quad=\left\|x_1,\dots,x_i,\dots,x_j+\gamma x_i,\dots,x_n\right\|,\end{aligned} \tag{7}$$

for all $1\le i\ne j\le n$,

(2) $|\|x,y_2,\dots,y_n\|-\|y,y_2,\dots,y_n\||\le\|x-y,y_2,\dots,y_n\|$ *for all $x,y,y_2,\dots,y_n\in X$,*

(3) *if $\|x,y_2,\dots,y_n\|=0$ for all $y_2,\dots,y_n\in X$, then $x=0$,*

(4) *for a convergent sequence $\{x_j\}$ in X,*

$$\lim_{j\to\infty}\left\|x_j,y_2,\dots,y_n\right\|=\left\|\lim_{j\to\infty}x_j,y_2,\dots,y_n\right\|, \tag{8}$$

for all $y_2,\dots,y_n\in X$.

2. Approximate Multi-Jensen Mappings

First, we prove the stability of the system of equations defining multi-Jensen mappings in n-Banach spaces. For a given mapping $f:V^k\to W$, we define the difference operators

$$\begin{aligned}&D_if\left(x_1,\dots,x_{k+1}\right)\\&\quad:=2f\left(x_1,\dots,x_{i-1},\frac{x_i+x_{i+1}}{2},x_{i+2},\dots,x_{k+1}\right)\\&\qquad-f\left(x_1,\dots,x_i,x_{i+2},\dots,x_{k+1}\right)\\&\qquad-f\left(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{k+1}\right),\\&\qquad\left(x_1,\dots,x_{k+1}\right)\in V^{k+1},\quad i\in\{1,\dots,k\}.\end{aligned} \tag{9}$$

Theorem 5. *Let V be a commutative group uniquely divisible by 2, and, W be an n-Banach space. Assume also that for every $i\in\{1,\dots,k\}$, $\varphi_i:V^{k+1}\to[0,\infty)$ is a mapping such that*

$$\begin{aligned}&\sum_{j=0}^{\infty}\frac{1}{3^{j+1}}\left[\varphi_i\left(3^jx_1,x_2,\dots,x_{k+1}\right)\right.\\&\quad+\cdots+\varphi_i\left(x_1,\dots,x_{i-2},3^jx_{i-1},x_i,\dots,x_{k+1}\right)\\&\quad+\varphi_i\left(x_1,\dots,x_{i-1},3^jx_i,3^jx_{i+1},x_{i+2},\dots,x_{k+1}\right)\\&\quad+\varphi_i\left(x_1,\dots,x_{i+1},3^jx_{i+2},x_{i+3},\dots,x_{k+1}\right)\\&\quad\left.+\cdots+\varphi_i\left(x_1,\dots,x_k,3^jx_{k+1}\right)\right]<\infty,\\&\qquad\left(x_1,\dots,x_{k+1}\right)\in V^{k+1}.\end{aligned} \tag{10}$$

If $f:V^k\to W$ is a function satisfying

$$\begin{aligned}&f\left(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_k\right)=0,\\&\left(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k\right)\in V^{k-1},\quad i\in\{1,\dots,k\},\end{aligned} \tag{11}$$

$$\begin{aligned}&\left\|D_if\left(x_1,\dots,x_{k+1}\right),y_2,\dots,y_n\right\|\le\varphi_i\left(x_1,\dots,x_{k+1}\right),\\&\left(x_1,\dots,x_{k+1}\right)\in V^{k+1},\quad i\in\{1,\dots,k\},\quad y_2,\dots,y_n\in W,\end{aligned} \tag{12}$$

then for every $i\in\{1,\dots,k\}$, there exists a multi-Jensen mapping $F_i:V^k\to W$ for which

$$\begin{aligned}&\left\|f\left(x_1,\dots,x_k\right)-F_i\left(x_1,\dots,x_k\right),y_2,\dots,y_n\right\|\\&\le\sum_{j=0}^{\infty}\frac{1}{3^{j+1}}\\&\quad\times\left[\varphi_i\left(x_1,\dots,x_{i-1},3^jx_i,-3^jx_i,x_{i+1},\dots,x_k\right)\right.\\&\qquad\left.+\varphi_i\left(x_1,\dots,x_{i-1},-3^jx_i,3^{j+1}x_i,x_{i+1},\dots,x_k\right)\right],\\&\qquad\left(x_1,\dots,x_k\right)\in V^k,\quad y_2,\dots,y_n\in W.\end{aligned} \tag{13}$$

For every $i \in \{1,\dots,k\}$, the function F_i is given by

$$F_i(x_1,\dots,x_k) := \lim_{j\to\infty} \frac{1}{3^j} f\left(x_1,\dots,x_{i-1},3^j x_i,x_{i+1},\dots,x_k\right), \quad (x_1,\dots,x_k) \in V^k. \tag{14}$$

Proof. Fix $x_1,\dots,x_k \in V$, $y_2,\dots,y_n \in W$ and $i \in \{1,\dots,k\}$. By (12) and (11), we get

$$\begin{aligned} &\|f(x_1,\dots,x_k) + f(x_1,\dots,x_{i-1},-x_i,x_{i+1},\dots,x_k), y_2,\dots,y_n\| \\ &\quad\le \varphi_i(x_1,\dots,x_i,-x_i,x_{i+1},\dots,x_k), \\ &\|2f(x_1,\dots,x_k) - f(x_1,\dots,x_{i-1},-x_i,x_{i+1},\dots,x_k) \\ &\quad - f(x_1,\dots,x_{i-1},3x_i,x_{i+1},\dots,x_k), y_2,\dots,y_n\| \\ &\quad\le \varphi_i(x_1,\dots,x_{i-1},-x_i,3x_i,x_{i+1},\dots,x_k). \end{aligned} \tag{15}$$

Hence,

$$\begin{aligned} &\|3f(x_1,\dots,x_k) - f(x_1,\dots,x_{i-1},3x_i,x_{i+1},\dots,x_k), y_2,\dots,y_n\| \\ &\quad\le \varphi_i(x_1,\dots,x_i,-x_i,x_{i+1},\dots,x_k) \\ &\qquad + \varphi_i(x_1,\dots,x_{i-1},-x_i,3x_i,x_{i+1},\dots,x_k), \end{aligned} \tag{16}$$

and consequently for any nonnegative integers l and m such that $l < m$, we obtain

$$\begin{aligned} &\Big\|\frac{1}{3^l} f(x_1,\dots,x_{i-1},3^l x_i,x_{i+1},\dots,x_k) \\ &\quad - \frac{1}{3^m} f(x_1,\dots,x_{i-1},3^m x_i,x_{i+1},\dots,x_k), y_2,\dots,y_n\Big\| \\ &\le \sum_{j=l}^{m-1} \frac{1}{3^{j+1}} \left[\varphi_i(x_1,\dots,x_{i-1},3^j x_i,-3^j x_i,x_{i+1},\dots,x_k) \right. \\ &\qquad \left. + \varphi_i(x_1,\dots,x_{i-1},-3^j x_i,3^{j+1} x_i,x_{i+1},\dots,x_k)\right]. \end{aligned} \tag{17}$$

Therefore, from (10), it follows that $\{(1/3^j) f(x_1,\dots,x_{i-1}, 3^j x_i,x_{i+1},\dots,x_k)\}_{j\in\mathbb{N}}$ is a Cauchy sequence. Since W is an n-Banach space, this sequence is convergent and we define $F_i : V^k \to W$ by (14). Putting $l = 0$, letting $m \to \infty$ in (17), and using Lemma 4 and (10), we see that (13) holds.

Finally, fix $x_i' \in V$, $j \in \mathbb{N}$, and note that according to (12), we have

$$\begin{aligned} &\Big\|\frac{1}{3^j} D_i f\left(x_1,\dots,x_{i-1},3^j x_i,3^j x_i',x_{i+1},\dots,x_k\right), y_2,\dots,y_n\Big\| \\ &\quad\le \frac{1}{3^j}\varphi_i\left(x_1,\dots,x_{i-1},3^j x_i,3^j x_i',x_{i+1},\dots,x_k\right). \end{aligned} \tag{18}$$

Next, fix $s \in \{1,\dots,k\} \setminus \{i\}$, $x_s' \in V$, and assume that $s < i$ (the same arguments apply to the case where $s > i$). From (12), it follows that

$$\begin{aligned} &\Big\|\frac{1}{3^j} D_s f\left(x_1,\dots,x_s,x_s',x_{s+1},\dots,x_{i-1}, 3^j x_i,x_{i+1},\dots,x_k\right), y_2,\dots,y_n\Big\| \\ &\quad\le \frac{1}{3^j}\varphi_s\left(x_1,\dots,x_s,x_s',x_{s+1},\dots, x_{i-1},3^j x_i,x_{i+1},\dots,x_k\right). \end{aligned} \tag{19}$$

Letting $j \to \infty$ in the above two inequalities and using (10) and Lemma 4, we see that the mapping F_i is multi-Jensen. □

Theorem 6. *Let V be a real linear space and, W be an n-Banach space. Assume also that for every $i \in \{1,\dots,k\}$, $\varphi_i : V^{k+1} \to [0,\infty)$ is a mapping such that*

$$\begin{aligned} &\sum_{j=0}^{\infty} 3^j \left[\varphi_i\left(\frac{x_1}{3^j},x_2,\dots,x_{k+1}\right) \right. \\ &\quad + \cdots + \varphi_i\left(x_1,\dots,x_{i-2},\frac{x_{i-1}}{3^j},x_i,\dots,x_{k+1}\right) \\ &\quad + \varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{3^j},\frac{x_{i+1}}{3^j},x_{i+2},\dots,x_{k+1}\right) \\ &\quad + \varphi_i\left(x_1,\dots,x_{i+1},\frac{x_{i+2}}{3^j},x_{i+3},\dots,x_{k+1}\right) \\ &\quad \left. + \cdots + \varphi_i\left(x_1,\dots,x_k,\frac{x_{k+1}}{3^j}\right)\right] < \infty, \\ &\qquad (x_1,\dots,x_{k+1}) \in V^{k+1}. \end{aligned} \tag{20}$$

If $f : V^k \to W$ is a function satisfying conditions (11) *and* (12), *then for every $i \in \{1,\dots,k\}$ there exists a multi-Jensen mapping $F_i : V^k \to W$ for which*

$$\begin{aligned} &\|f(x_1,\dots,x_k) - F_i(x_1,\dots,x_k), y_2,\dots,y_n\| \\ &\le \sum_{j=0}^{\infty} 3^j \left[\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{3^{j+1}},-\frac{x_i}{3^{j+1}},x_{i+1},\dots,x_k\right) \right. \\ &\qquad \left. + \varphi_i\left(x_1,\dots,x_{i-1},-\frac{x_i}{3^{j+1}},\frac{x_i}{3^j},x_{i+1},\dots,x_k\right)\right], \\ &\qquad (x_1,\dots,x_k) \in V^k,\ y_2,\dots,y_n \in W. \end{aligned} \tag{21}$$

For every $i \in \{1,\dots,k\}$, the function F_i is given by

$$F_i(x_1,\dots,x_k) := \lim_{j\to\infty} 3^j f\left(x_1,\dots,x_{i-1},\frac{x_i}{3^j},x_{i+1},\dots,x_k\right), \quad (x_1,\dots,x_k) \in V^k. \tag{22}$$

Proof. Fix $x_1, \ldots, x_k \in V$, $y_2, \ldots, y_n \in W$, $j \in \mathbb{N} \cup \{0\}$ and $i \in \{1, \ldots, k\}$. By (12) and (11), we get

$$\begin{aligned}
&\Big\| 3^{j+1} f\left(x_1, \ldots, x_{i-1}, \frac{x_i}{3^{j+1}}, x_{i+1}, \ldots, x_k\right) \\
&\quad -3^j f\left(x_1, \ldots, x_{i-1}, \frac{x_i}{3^j}, x_{i+1}, \ldots, x_k\right), y_2, \ldots, y_n \Big\| \\
&\leq 3^j \left[\varphi_i\left(x_1, \ldots, x_{i-1}, \frac{x_i}{3^{j+1}}, -\frac{x_i}{3^{j+1}}, x_{i+1}, \ldots, x_k\right)\right. \\
&\quad \left. +\varphi_i\left(x_1, \ldots, x_{i-1}, -\frac{x_i}{3^{j+1}}, \frac{x_i}{3^j}, x_{i+1}, \ldots, x_k\right)\right],
\end{aligned} \tag{23}$$

and consequently for any non-negative integers l and m such that $l < m$, we obtain

$$\begin{aligned}
&\Big\| 3^l f\left(x_1, \ldots, x_{i-1}, \frac{x_i}{3^l}, x_{i+1}, \ldots, x_k\right) \\
&\quad -3^m f\left(x_1, \ldots, x_{i-1}, \frac{x_i}{3^m}, x_{i+1}, \ldots, x_k\right), y_2, \ldots, y_n \Big\| \\
&\leq \sum_{j=l}^{m-1} 3^j \left[\varphi_i\left(x_1, \ldots, x_{i-1}, \frac{x_i}{3^{j+1}}, -\frac{x_i}{3^{j+1}}, x_{i+1}, \ldots, x_k\right)\right. \\
&\quad \left. +\varphi_i\left(x_1, \ldots, x_{i-1}, -\frac{x_i}{3^{j+1}}, \frac{x_i}{3^j}, x_{i+1}, \ldots, x_k\right)\right].
\end{aligned} \tag{24}$$

Therefore, from (20), it follows that $\{3^j f(x_1, \ldots, x_{i-1}, x_i/3^j, x_{i+1}, \ldots, x_k)\}_{j \in \mathbb{N}}$ is a Cauchy sequence. Since W is an n-Banach space, this sequence is convergent and we define $F_i : V^k \rightarrow W$ by (22). Putting $l = 0$, letting $m \rightarrow \infty$ in (24), and using Lemma 4 and (20), we see that (21) holds.

Finally, fix $x_i' \in V$, and note that according to (12), we have

$$\begin{aligned}
&\Big\| 3^j D_i f\left(x_1, \ldots, x_{i-1}, \frac{x_i}{3^j}, \frac{x_i'}{3^j}, x_{i+1}, \ldots, x_k\right), y_2, \ldots, y_n \Big\| \\
&\leq 3^j \varphi_i\left(x_1, \ldots, x_{i-1}, \frac{x_i}{3^j}, \frac{x_i'}{3^j}, x_{i+1}, \ldots, x_k\right).
\end{aligned} \tag{25}$$

Next, fix $s \in \{1, \ldots, k\} \setminus \{i\}$, $x_s' \in V$, and assume that $s < i$ (the same arguments apply to the case where $s > i$). From (12), it follows that

$$\begin{aligned}
&\Big\| 3^j D_s f\left(x_1, \ldots, x_s, x_s', x_{s+1}, \ldots, x_{i-1}, \frac{x_i}{3^j}, x_{i+1}, \ldots, x_k\right), \\
&\quad y_2, \ldots, y_n \Big\| \\
&\leq 3^j \varphi_s\left(x_1, \ldots, x_s, x_s', x_{s+1}, \ldots, x_{i-1}, \frac{x_i}{3^j}, x_{i+1}, \ldots, x_k\right).
\end{aligned} \tag{26}$$

Letting $j \rightarrow \infty$ in the previous two inequalities and using (20) and Lemma 4, we see that the mapping F_i is multi-Jensen. □

As applications of Theorems 5 and 6 we get the following corollaries.

Corollary 7. *Let V be a real normed linear space and, W be an n-Banach space. Assume also that $\theta \in [0, \infty)$ and $r \in (0, \infty)$ are such that $r \neq 1$. If $f : V^k \rightarrow W$ is a function satisfying* (11) *and*

$$\begin{aligned}
&\| D_i f(x_1, \ldots, x_{k+1}), y_2, \ldots, y_n \| \\
&\leq \theta \left[\|x_1\|^r \cdots \|x_{i-1}\|^r \left(\|x_i\|^r + \|x_{i+1}\|^r\right) \right. \\
&\quad \left. \times \|x_{i+2}\|^r \cdots \|x_{k+1}\|^r \right], \\
&(x_1, \ldots, x_{k+1}) \in V^{k+1}, \quad i \in \{1, \ldots, k\}, \\
&y_2, \ldots, y_n \in W,
\end{aligned} \tag{27}$$

then for every $i \in \{1, \ldots, k\}$ there exists a multi-Jensen mapping $F_i : V^k \rightarrow W$ for which

$$\begin{aligned}
&\| f(x_1, \ldots, x_k) - F_i(x_1, \ldots, x_k), y_2, \ldots, y_n \| \\
&\leq \frac{\theta \|x_1\|^r \cdots \|x_k\|^r (3 + 3^r)}{|3 - 3^r|},
\end{aligned} \tag{28}$$

for all $x_1, \ldots, x_k \in V$, $y_2, \ldots, y_n \in W$.

Corollary 8. *Let V be a real normed linear space and let W be an n-Banach space. Assume also that $\theta \in [0, \infty)$ and $r, p, q \in (0, \infty)$ are such that $r, p + q \in (0, 1)$ or $r, p + q \in (1, \infty)$. If $f : V^k \rightarrow W$ is a function satisfying* (11) *and*

$$\begin{aligned}
&\| D_i f(x_1, \ldots, x_{k+1}), y_2, \ldots, y_n \| \\
&\leq \theta \|x_1\|^r \cdots \|x_{i-1}\|^r \left(\|x_i\|^p \|x_{i+1}\|^q\right) \\
&\quad \times \|x_{i+2}\|^r \cdots \|x_{k+1}\|^r, \\
&(x_1, \ldots, x_{k+1}) \in V^{k+1}, \quad i \in \{1, \ldots, k\}, \\
&y_2, \ldots, y_n \in W,
\end{aligned} \tag{29}$$

then for every $i \in \{1, \ldots, k\}$, there exists a multi-Jensen mapping $F_i : V^k \rightarrow W$ for which

$$\begin{aligned}
&\| f(x_1, \ldots, x_k) - F_i(x_1, \ldots, x_k), y_2, \ldots, y_n \| \\
&\leq \frac{\theta \|x_1\|^r \cdots \|x_{i-1}\|^r \|x_i\|^{p+q} \|x_{i+1}\|^r \cdots \|x_k\|^r (1 + 3^q)}{|3 - 3^{p+q}|},
\end{aligned} \tag{30}$$

for all $x_1, \ldots, x_k \in V$, $y_2, \ldots, y_n \in W$.

From Corollary 8, we obtain the following corollary which corrects Theorems 3.1 and 3.2 from [20].

Corollary 9. *Let V be a real normed linear space and W be an n-Banach space. Assume also that $\theta \in [0, \infty)$ and $p, q \in (0, \infty)$ are such that $p + q \neq 1$. If $f : V \rightarrow W$ is a function satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\Big\| 2f\left(\frac{x_1 + x_2}{2}\right) - f(x_1) - f(x_2), y \Big\| \\
&\leq \theta \|x_1\|^p \|x_2\|^q, \quad x_1, x_2 \in V, \; y \in W,
\end{aligned} \tag{31}$$

then there exists a Jensen mapping $F : V \to W$ *for which*

$$\|f(x) - F(x), y\| \leq \frac{\theta\|x\|^{p+q}(1+3^q)}{|3-3^{p+q}|}, \quad x \in V,\ y \in W. \tag{32}$$

3. Approximate Multi-Euler-Lagrange Additive Mappings

In this section, we prove the stability of the system of equations defining multi-Euler-Lagrange additive mappings.

Throughout this section, let V be a real linear space and let W be an n-Banach space, and $a, b \in \mathbb{R} \setminus \{0\}$ are fixed with $\lambda := a + b \neq 0, \pm 1$.

A mapping $f : V^k \to W$ is called a multiEuler-Lagrange additive mapping as follows if it satisfies the Euler-Lagrange additive equations in each of their k arguments as follows:

$$\begin{aligned} &f\left(x_1, \ldots, x_{i-1}, ax_i + bx_i', x_{i+1}, \ldots, x_k\right) \\ &\quad + f\left(x_1, \ldots, x_{i-1}, bx_i + ax_i', x_{i+1}, \ldots, x_k\right) \\ &= (a+b)\left[f\left(x_1, \ldots, x_k\right) \right. \\ &\qquad \left. + f\left(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_k\right)\right], \end{aligned} \tag{33}$$

for all $i \in \{1, \ldots, k\}$ and all $x_1, \ldots, x_{i-1}, x_i, x_i', x_{i+1}, \ldots, x_k \in V$. If $a = b = 1$, then the multi-Euler-Lagrange additive mapping is multiadditive (see [28]). For a given mapping $f : V^k \to W$, we define the difference operators

$$\begin{aligned} &\widetilde{D}_i f\left(x_1, \ldots, x_{k+1}\right) \\ &\quad := f\left(x_1, \ldots, x_{i-1}, ax_i + bx_{i+1}, x_{i+2}, \ldots, x_{k+1}\right) \\ &\qquad + f\left(x_1, \ldots, x_{i-1}, bx_i + ax_{i+1}, x_{i+2}, \ldots, x_{k+1}\right) \\ &\qquad - (a+b)\left[f\left(x_1, \ldots, x_i, x_{i+2}, \ldots, x_{k+1}\right) \right. \\ &\qquad\qquad \left. + f\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1}\right)\right], \\ &\qquad\qquad \left(x_1, \ldots, x_{k+1}\right) \in V^{k+1}, \quad i \in \{1, \ldots, k\}. \end{aligned} \tag{34}$$

Theorem 10. *Assume that for every* $i \in \{1, \ldots, k\}$, $\varphi_i : V^{k+1} \to [0, \infty)$ *is a mapping such that*

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{1}{|\lambda|^j}\left[\varphi_i\left(\lambda^j x_1, \ldots, x_{k+1}\right) \right. \\ &\quad + \cdots + \varphi_i\left(x_1, \ldots, x_{i-2}, \lambda^j x_{i-1}, x_i, \ldots, x_{k+1}\right) \\ &\quad + \varphi_i\left(x_1, \ldots, x_{i-1}, \lambda^j x_i, \lambda^j x_{i+1}, x_{i+2}, \ldots, x_{k+1}\right) \\ &\quad + \varphi_i\left(x_1, \ldots, x_{i+1}, \lambda^j x_{i+2}, x_{i+3}, \ldots, x_{k+1}\right) \\ &\quad \left. + \cdots + \varphi_i\left(x_1, \ldots, x_k, \lambda^j x_{k+1}\right)\right] < \infty, \\ &\qquad \left(x_1, \ldots, x_{k+1}\right) \in V^{k+1}. \end{aligned} \tag{35}$$

If $f : V^k \to W$ *is a function satisfying*

$$\begin{aligned} &\left\|\widetilde{D}_i f\left(x_1, \ldots, x_{k+1}\right), y_2, \ldots, y_n\right\| \leq \varphi_i\left(x_1, \ldots, x_{k+1}\right), \\ &\left(x_1, \ldots, x_{k+1}\right) \in V^{k+1}, \quad i \in \{1, \ldots, k\}, \quad y_2, \ldots, y_n \in W, \end{aligned} \tag{36}$$

then for every $i \in \{1, \ldots, k\}$, *there exists a unique multi-Euler-Lagrange additive mapping* $A_i : V^k \to W$ *for which*

$$\begin{aligned} &\left\|f\left(x_1, \ldots, x_k\right) - A_i\left(x_1, \ldots, x_k\right), y_2, \ldots, y_n\right\| \\ &\quad \leq \frac{1}{2}\sum_{j=0}^{\infty} \frac{1}{|\lambda|^{j+1}} \\ &\qquad \times \varphi_i\left(x_1, \ldots, x_{i-1}, \lambda^j x_i, \lambda^j x_i, x_{i+1}, \ldots, x_k\right), \\ &\qquad \left(x_1, \ldots, x_k\right) \in V^k, \quad y_2, \ldots, y_n \in W. \end{aligned} \tag{37}$$

For every $i \in \{1, \ldots, k\}$, *the function* A_i *is given by*

$$\begin{aligned} &A_i\left(x_1, \ldots, x_k\right) \\ &\quad := \lim_{j \to \infty} \frac{1}{\lambda^j} f\left(x_1, \ldots, x_{i-1}, \lambda^j x_i, x_{i+1}, \ldots, x_k\right), \\ &\qquad \left(x_1, \ldots, x_k\right) \in V^k. \end{aligned} \tag{38}$$

Proof. Fix $x_1, \ldots, x_k \in V$, $y_2, \ldots, y_n \in W$, $j \in \mathbb{N} \cup \{0\}$ and $i \in \{1, \ldots, k\}$. By (36), we get

$$\begin{aligned} &\left\|f\left(x_1, \ldots, x_k\right) - \frac{1}{\lambda} f\left(x_1, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_k\right), \right. \\ &\quad \left. y_2, \ldots, y_n\right\| \\ &\quad \leq \frac{1}{|2\lambda|}\varphi_i\left(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_k\right), \end{aligned} \tag{39}$$

whence

$$\begin{aligned} &\left\|\frac{1}{\lambda^j} f\left(x_1, \ldots, x_{i-1}, \lambda^j x_i, x_{i+1}, \ldots, x_k\right) \right. \\ &\quad \left. - \frac{1}{\lambda^{j+1}} f\left(x_1, \ldots, x_{i-1}, \lambda^{j+1} x_i, x_{i+1}, \ldots, x_k\right), y_2, \ldots, y_n\right\| \\ &\quad \leq \frac{1}{2|\lambda|^{j+1}}\varphi_i\left(x_1, \ldots, x_{i-1}, \lambda^j x_i, \lambda^j x_i, x_{i+1}, \ldots, x_k\right). \end{aligned} \tag{40}$$

For any nonnegative integers l and m with $l < m$, using (40) we get

$$\begin{aligned} &\left\|\frac{1}{\lambda^l} f\left(x_1, \ldots, x_{i-1}, \lambda^l x_i, x_{i+1}, \ldots, x_k\right) \right. \\ &\quad \left. - \frac{1}{\lambda^m} f\left(x_1, \ldots, x_{i-1}, \lambda^m x_i, x_{i+1}, \ldots, x_k\right), y_2, \ldots, y_n\right\| \\ &\quad \leq \sum_{j=l}^{m-1} \frac{1}{2|\lambda|^{j+1}} \\ &\qquad \times \varphi_i\left(x_1, \ldots, x_{i-1}, \lambda^j x_i, \lambda^j x_i, x_{i+1}, \ldots, x_k\right), \end{aligned} \tag{41}$$

which tends to zero as l tends to infinity. Therefore, from (35) it follows that $\{(1/\lambda^j) f(x_1, \ldots, x_{i-1}, \lambda^j x_i, x_{i+1}, \ldots, x_k)\}_{j\in\mathbb{N}}$ is a Cauchy sequence in n-Banach space W and it thus converges. Hence, we can define $A_i : V^k \to W$ by

$$A_i(x_1, \ldots, x_k) := \lim_{j\to\infty} \frac{1}{\lambda^j} f(x_1, \ldots, x_{i-1}, \lambda^j x_i, x_{i+1}, \ldots, x_k). \tag{42}$$

Putting $l = 0$, letting $m \to \infty$ in (41), and using (35), we see that (37) holds.

Now, fix also $x_i' \in V$, and from (36), we have

$$\left\| \frac{1}{\lambda^j} \widetilde{D}_i f(x_1, \ldots, x_{i-1}, \lambda^j x_i, \lambda^j x_i', x_{i+1}, \ldots, x_k), y_2, \ldots, y_n \right\| \le \frac{1}{|\lambda|^j} \varphi_i(x_1, \ldots, x_{i-1}, \lambda^j x_i, \lambda^j x_i', x_{i+1}, \ldots, x_k). \tag{43}$$

Next, fix $s \in \{1, \ldots, k\} \setminus \{i\}$, $x_s' \in V$, and assume that $s < i$ (the same arguments apply to the case where $s > i$). From (36) it follows that

$$\left\| \frac{1}{\lambda^j} \widetilde{D}_s f(x_1, \ldots, x_s, x_s', x_{s+1}, \ldots, x_{i-1}, \lambda^j x_i, x_{i+1}, \ldots, x_k), y_2, \ldots, y_n \right\| \le \frac{1}{|\lambda|^j} \varphi_s(x_1, \ldots, x_s, x_s', x_{s+1}, \ldots, x_{i-1}, \lambda^j x_i, x_{i+1}, \ldots, x_k). \tag{44}$$

Letting $j \to \infty$ in the above two inequalities and using (35) and Lemma 4, we see that the mapping A_i is multi-Euler-Lagrange additive.

Now, let us finally assume that $A_i' : V^k \to W$ is another multi-Euler-Lagrange additive mapping satisfying (37). Then we have

$$\begin{aligned} &\left\| A_i(x_1, \ldots, x_k) - A_i'(x_1, \ldots, x_k), y_2, \ldots, y_n \right\| \\ &= \lim_{m\to\infty} \frac{1}{|\lambda|^m} \times \left\| A_i(x_1, \ldots, x_{i-1}, \lambda^m x_i, x_{i+1}, \ldots, x_k) - A_i'(x_1, \ldots, x_{i-1}, \lambda^m x_i, x_{i+1}, \ldots, x_k), y_2, \ldots, y_n \right\| \\ &\le \lim_{m\to\infty} \frac{1}{|\lambda|^m} \times \left[\left\| A_i(x_1, \ldots, x_{i-1}, \lambda^m x_i, x_{i+1}, \ldots, x_k) - f(x_1, \ldots, x_{i-1}, \lambda^m x_i, x_{i+1}, \ldots, x_k), y_2, \ldots, y_n \right\| \right. \\ &\qquad \left. + \left\| f(x_1, \ldots, x_{i-1}, \lambda^m x_i, x_{i+1}, \ldots, x_k) - A_i(x_1, \ldots, x_{i-1}, \lambda^m x_i, x_{i+1}, \ldots, x_k), y_2, \ldots, y_n \right\| \right] \\ &\le \lim_{m\to\infty} \frac{1}{|\lambda|} \times \sum_{j=0}^{\infty} \frac{1}{|\lambda|^{m+j}} \times \varphi_i(x_1, \ldots, x_{i-1}, \lambda^{m+j} x_i, \lambda^{m+j} x_i, x_{i+1}, \ldots, x_k) \\ &= 0, \end{aligned} \tag{45}$$

and therefore $A_i = A_i'$. □

Theorem 11. *Assume that for every $i \in \{1, \ldots, k\}$, $\varphi_i : V^{k+1} \to [0, \infty)$ is a mapping such that*

$$\begin{aligned} \sum_{j=0}^{\infty} |\lambda|^j \Big[&\varphi_i\left(\frac{x_1}{\lambda^j}, x_2, \ldots, x_{k+1}\right) + \cdots + \varphi_i\left(x_1, \ldots, x_{i-2}, \frac{x_{i-1}}{\lambda^j}, x_i, \ldots, x_{k+1}\right) \\ &+ \varphi_i\left(x_1, \ldots, x_{i-1}, \frac{x_i}{\lambda^j}, \frac{x_{i+1}}{\lambda^j}, x_{i+2}, \ldots, x_{k+1}\right) + \varphi_i\left(x_1, \ldots, x_{i+1}, \frac{x_{i+2}}{\lambda^j}, x_{i+3}, \ldots, x_{k+1}\right) \\ &+ \cdots + \varphi_i\left(x_1, \ldots, x_k, \frac{x_{k+1}}{\lambda^j}\right) \Big] < \infty, \quad (x_1, \ldots, x_{k+1}) \in V^{k+1}. \end{aligned} \tag{46}$$

If $f : V^k \to W$ is a function satisfying (36)*, then for every $i \in \{1, \ldots, k\}$, there exists a unique multi-Euler-Lagrange additive mapping $A_i : V^k \to W$ for which*

$$\begin{aligned} &\left\| f(x_1, \ldots, x_k) - A_i(x_1, \ldots, x_k), y_2, \ldots, y_n \right\| \\ &\le \frac{1}{2} \sum_{j=1}^{\infty} |\lambda|^{j-1} \times \varphi_i\left(x_1, \ldots, x_{i-1}, \frac{x_i}{\lambda^j}, \frac{x_i}{\lambda^j}, x_{i+1}, \ldots, x_k\right), \\ &(x_1, \ldots, x_k) \in V^k, \quad y_2, \ldots, y_n \in W. \end{aligned} \tag{47}$$

For every $i \in \{1, \ldots, k\}$, the function A_i is given by

$$A_i(x_1, \ldots, x_k) := \lim_{j\to\infty} \lambda^j f\left(x_1, \ldots, x_{i-1}, \frac{x_i}{\lambda^j}, x_{i+1}, \ldots, x_k\right), \quad (x_1, \ldots, x_k) \in V^k. \tag{48}$$

Proof. Fix $x_1, \ldots, x_k \in V$, $y_2, \ldots, y_n \in W$, $j \in \mathbb{N} \cup \{0\}$ and $i \in \{1, \ldots, k\}$. By (36) we get

$$\left\| f(x_1, \ldots, x_k) - \lambda f\left(x_1, \ldots, x_{i-1}, \frac{x_i}{\lambda}, x_{i+1}, \ldots, x_k\right), y_2, \ldots, y_n \right\| \leq \frac{1}{2}\varphi_i\left(x_1, \ldots, x_{i-1}, \frac{x_i}{\lambda}, \frac{x_i}{\lambda}, x_{i+1}, \ldots, x_k\right). \tag{49}$$

For any non-negative integers l and m with $0 \leq l < m$, using (49), we get

$$\begin{aligned} &\left\| \lambda^l f\left(x_1, \ldots, x_{i-1}, \frac{x_i}{\lambda^l}, x_{i+1}, \ldots, x_k\right) - \lambda^m f\left(x_1, \ldots, x_{i-1}, \frac{x_i}{\lambda^m}, x_{i+1}, \ldots, x_k\right), y_2, \ldots, y_n \right\| \\ &\quad \leq \sum_{j=l}^{m-1} \frac{|\lambda|^j}{2} \times \varphi_i\left(x_1, \ldots, x_{i-1}, \frac{x_i}{\lambda^{j+1}}, \frac{x_i}{\lambda^{j+1}}, x_{i+1}, \ldots, x_k\right), \end{aligned} \tag{50}$$

which tends to zero as l tends to infinity. Therefore from (46), it follows that $\{\lambda^j f(x_1, \ldots, x_{i-1}, x_i/\lambda^j, x_{i+1}, \ldots, x_k)\}_{j\in\mathbb{N}}$ is a Cauchy sequence in n-Banach space W and it thus converges. Hence, we can define $A_i : V^k \to W$ by

$$A_i(x_1, \ldots, x_k) := \lim_{j\to\infty} \lambda^j f\left(x_1, \ldots, x_{i-1}, \frac{x_i}{\lambda^j}, x_{i+1}, \ldots, x_k\right). \tag{51}$$

Putting $l = 0$, letting $m \to \infty$ in (50), and using (46), we see that (47) holds. The further part of the proof is similar to the proof of Theorem 10. □

As applications of Theorems 10 and 11, we get the following corollaries.

Corollary 12. *Let V be a real normed linear space and, W be an n-Banach space. Assume also that $\theta \in [0, \infty)$ and $r \in (0, \infty)$ are such that $r \neq 1$. If $f : V^k \to W$ is a function satisfying*

$$\begin{aligned} &\left\| \widetilde{D}_i f(x_1, \ldots, x_{k+1}), y_2, \ldots, y_n \right\| \\ &\quad \leq \theta\left[\|x_1\|^r \cdots \|x_{i-1}\|^r \left(\|x_i\|^r + \|x_{i+1}\|^r\right) \times \|x_{i+2}\|^r \cdots \|x_{k+1}\|^r\right], \\ &(x_1, \ldots, x_{k+1}) \in V^{k+1}, \quad i \in \{1, \ldots, k\}, \quad y_2, \ldots, y_n \in W, \end{aligned} \tag{52}$$

then for every $i \in \{1, \ldots, k\}$ there exists a unique multi-Euler-Lagrange additive mapping $A_i : V^k \to W$ for which

$$\left\| f(x_1, \ldots, x_k) - A_i(x_1, \ldots, x_k), y_2, \ldots, y_n \right\| \leq \frac{\theta\|x_1\|^r \cdots \|x_k\|^r}{\left||\lambda| - |\lambda|^r\right|}, \tag{53}$$

for all $x_1, \ldots, x_k \in V$, $y_2, \ldots, y_n \in W$.

Corollary 13. *Let V be a real normed linear space and let W be an n-Banach space. Assume also that $\theta \in [0, \infty)$ and $r, p, q \in (0, \infty)$ are such that $r, p + q \in (0, 1)$ or $r, p + q \in (1, \infty)$. If $f : V^k \to W$ is a function satisfying*

$$\begin{aligned} &\left\| \widetilde{D}_i f(x_1, \ldots, x_{k+1}), y_2, \ldots, y_n \right\| \\ &\quad \leq \theta\|x_1\|^r \cdots \|x_{i-1}\|^r \left(\|x_i\|^p \|x_{i+1}\|^q\right) \times \|x_{i+2}\|^r \cdots \|x_{k+1}\|^r, \\ &(x_1, \ldots, x_{k+1}) \in V^{k+1}, \quad i \in \{1, \ldots, k\}, \quad y_2, \ldots, y_n \in W, \end{aligned} \tag{54}$$

then for every $i \in \{1, \ldots, k\}$, there exists a unique multi-Euler-Lagrange additive mapping $A_i : V^k \to W$ for which

$$\left\| f(x_1, \ldots, x_k) - A_i(x_1, \ldots, x_k), y_2, \ldots, y_n \right\| \leq \frac{\theta\|x_1\|^r \cdots \|x_{i-1}\|^r \|x_i\|^{p+q} \|x_{i+1}\|^r \cdots \|x_k\|^r}{2\left||\lambda| - |\lambda|^{p+q}\right|}, \tag{55}$$

for all $x_1, \ldots, x_k \in V$, $y_2, \ldots, y_n \in W$.

From Corollary 13 we obtain the following corollary which corrects Theorems 2.1 and 2.2 from [20].

Corollary 14. *Let V be a real normed linear space and W be an 2-Banach space. Assume also that $\theta \in [0, \infty)$ and $p, q \in (0, \infty)$ are such that $p + q \neq 1$. If $f : V \to W$ is a function satisfying*

$$\left\| f(x_1 + x_2) - f(x_1) - f(x_2), y \right\| \leq \theta\|x_1\|^p \|x_2\|^q, \quad x_1, x_2 \in V, \quad y \in W, \tag{56}$$

then there exists a unique additive mapping $A : V \to W$ for which

$$\left\| f(x) - A(x), y \right\| \leq \frac{\theta\|x\|^{p+q}}{\left|2 - 2^{p+q}\right|}, \quad x \in V, \ y \in W. \tag{57}$$

4. Approximate Multi-Euler-Lagrange Quadratic Mappings

In this section, we prove the stability of the system of equations defining multi-Euler-Lagrange quadratic mappings.

Throughout this section, let V be a real linear space and let W be an n-Banach space, and $a, b \in \mathbb{R} \setminus \{0\}$ are fixed with $\lambda := a^2 + b^2 \neq 1$.

Rassias [29] introduced the notion of a generalized Euler-Lagrange-type quadratic mapping, and investigated its generalized stability.

A mapping $f : V^k \to W$ is called a multi-Euler-Lagrange quadratic mapping, if it satisfies the Euler-Lagrange quadratic equations in each of their k arguments:

$$\begin{aligned} f&\left(x_1,\dots,x_{i-1},ax_i+bx_i',x_{i+1},\dots,x_k\right) \\ &+ f\left(x_1,\dots,x_{i-1},bx_i-ax_i',x_{i+1},\dots,x_k\right) \\ &= \left(a^2+b^2\right) \\ &\quad\times\left[f\left(x_1,\dots,x_k\right)+f\left(x_1,\dots,x_{i-1},x_i',x_{i+1},\dots,x_k\right)\right], \end{aligned} \tag{58}$$

for all $i \in \{1,\dots,k\}$ and all $x_1,\dots,x_{i-1},x_i,x_i',x_{i+1},\dots,x_k \in V$.

If $a = b = 1$, then the multi-Euler-Lagrange quadratic mapping is multiquadratic (see [30]). Letting $x_i = x_i' = 0$ in (58), we get $f(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_k) = 0$. Putting $x_i' = 0$ in (58), we have

$$\begin{aligned} f&\left(x_1,\dots,x_{i-1},ax_i,x_{i+1},\dots,x_k\right) \\ &+ f\left(x_1,\dots,x_{i-1},bx_i,x_{i+1},\dots,x_k\right) \\ &= \lambda f\left(x_1,\dots,x_k\right). \end{aligned} \tag{59}$$

Replacing x_i by ax_i and x_i' by bx_i in (58), respectively, we obtain

$$\begin{aligned} f&\left(x_1,\dots,x_{i-1},\lambda x_i,x_{i+1},\dots,x_k\right) \\ &= \lambda\left[f\left(x_1,\dots,x_{i-1},ax_i,\mathrm{x}_{i+1},\dots,x_k\right)\right. \\ &\quad \left.+f\left(x_1,\dots,x_{i-1},bx_i,x_{i+1},\dots,x_k\right)\right]. \end{aligned} \tag{60}$$

From (59) and (60), one gets

$$f\left(x_1,\dots,x_{i-1},\lambda x_i,x_{i+1},\dots,x_k\right) = \lambda^2 f\left(x_1,\dots,x_k\right), \tag{61}$$

for all $i \in \{1,\dots,k\}$ and all $x_1,\dots,x_k \in V$.

For a given mapping $f : V^k \to W$, we define the difference operators

$$\begin{aligned} &\widehat{D}_i f\left(x_1,\dots,x_{k+1}\right) \\ &:= f\left(x_1,\dots,x_{i-1},ax_i+bx_{i+1},x_{i+2},\dots,x_{k+1}\right) \\ &\quad + f\left(x_1,\dots,x_{i-1},bx_i-ax_{i+1},x_{i+2},\dots,x_{k+1}\right)-\left(a^2+b^2\right) \\ &\quad \times\left[f\left(x_1,\dots,x_i,x_{i+2},\dots,x_{k+1}\right)\right. \\ &\qquad \left.+f\left(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k\right)\right], \\ &\left(x_1,\dots,x_{k+1}\right) \in V^{k+1}, \quad i \in \{1,\dots,k\}. \end{aligned} \tag{62}$$

Theorem 15. *Assume that for every $i \in \{1,\dots,k\}$, $\varphi_i : V^{k+1} \to [0,\infty)$ is a mapping such that*

$$\begin{aligned} &\sum_{j=0}^{\infty}\frac{1}{\lambda^{2j}} \\ &\quad\times\left[\varphi_i\left(\lambda^j x_1,x_2,\dots,x_{k+1}\right)\right. \\ &\qquad +\cdots+\varphi_i\left(x_1,\dots,x_{i-2},\lambda^j x_{i-1},x_i,\dots,x_{k+1}\right) \\ &\qquad +\varphi_i\left(x_1,\dots,x_{i-1},\lambda^j x_i,\lambda^j x_{i+1},x_{i+2},\dots,x_{k+1}\right) \\ &\qquad +\varphi_i\left(x_1,\dots,x_{i+1},\lambda^j x_{i+2},x_{i+3},\dots,x_{k+1}\right) \\ &\qquad \left.+\cdots+\varphi_i\left(x_1,\dots,x_k,\lambda^j x_{k+1}\right)\right] < \infty, \\ &\qquad\qquad \left(x_1,\dots,x_{k+1}\right) \in V^{k+1}. \end{aligned} \tag{63}$$

If $f : V^k \to W$ is a function satisfying condition (11) *and*

$$\begin{aligned} &\left\|\widehat{D}_i f\left(x_1,\dots,x_{k+1}\right),y_2,\dots,y_n\right\| \\ &\quad\le \varphi_i\left(x_1,\dots,x_{k+1}\right), \\ &\qquad \left(x_1,\dots,x_{k+1}\right) \in V^{k+1}, \quad i \in \{1,\dots,k\}, \\ &\qquad\qquad y_2,\dots,y_n \in W, \end{aligned} \tag{64}$$

then for every $i \in \{1,\dots,k\}$, there exists a unique multi-Euler-Lagrange quadratic mapping $Q_i : V^k \to W$ for which

$$\begin{aligned} &\left\|f\left(x_1,\dots,x_k\right)-Q_i\left(x_1,\dots,x_k\right),y_2,\dots,y_n\right\| \\ &\quad\le\sum_{j=0}^{\infty}\left[\frac{1}{\lambda^{2j+1}}\varphi_i\left(x_1,\dots,x_{i-1},\lambda^j x_i,0,x_{i+1},\dots,x_k\right)\right. \\ &\qquad +\frac{1}{\lambda^{2j+2}}\varphi_i\left(x_1,\dots,x_{i-1},a\lambda^j x_i,\right. \\ &\qquad\qquad \left.\left. b\lambda^j x_i,x_{i+1},\dots,x_k\right)\right], \\ &\qquad \left(x_1,\dots,x_k\right) \in V^k, \qquad y_2,\dots,y_n \in W. \end{aligned} \tag{65}$$

For every $i \in \{1,\dots,k\}$, the function Q_i is given by

$$\begin{aligned} &Q_i\left(x_1,\dots,x_k\right) \\ &\quad := \lim_{j\to\infty}\frac{1}{\lambda^{2j}} \\ &\qquad\quad \times f\left(x_1,\dots,x_{i-1},\lambda^j x_i,x_{i+1},\dots,x_k\right), \\ &\qquad\qquad \left(x_1,\dots,x_k\right) \in V^k. \end{aligned} \tag{66}$$

Proof. Fix $x_1,\dots,x_k \in V$, $y_2,\dots,y_n \in W$, $j \in \mathbb{N}\cup\{0\}$ and $i \in \{1,\dots,k\}$. By (64), we get

$$\begin{aligned}
&\| f(x_1,\dots,x_{i-1},ax_i,x_{i+1},\dots,x_k) \\
&\quad + f(x_1,\dots,x_{i-1},bx_i,x_{i+1},\dots,x_k) \\
&\quad -\lambda f(x_1,\dots,x_k), y_2,\dots,y_n\| \\
&\qquad \le \varphi_i(x_1,\dots,x_i,0,x_{i+1},\dots,x_k), \\
&\| f(x_1,\dots,x_{i-1},\lambda x_i,x_{i+1},\dots,x_k) \\
&\quad - \lambda f(x_1,\dots,x_{i-1},ax_i,x_{i+1},\dots,x_k) \\
&\quad - \lambda f(x_1,\dots,x_{i-1},bx_i,x_{i+1},\dots,x_k), \\
&\quad y_2,\dots,y_n\| \\
&\qquad \le \varphi_i(x_1,\dots,x_{i-1},ax_i,bx_i,x_{i+1},\dots,x_k).
\end{aligned} \tag{67}$$

From (67), we obtain

$$\begin{aligned}
&\Big\| \frac{1}{\lambda^2} f(x_1,\dots,x_{i-1},\lambda x_i,x_{i+1},\dots,x_k) \\
&\quad - f(x_1,\dots,x_k), y_2,\dots,y_n \Big\| \\
&\qquad \le \frac{1}{\lambda}\varphi_i(x_1,\dots,x_i,0,x_{i+1},\dots,x_k) + \frac{1}{\lambda^2} \\
&\qquad\quad \times \varphi_i(x_1,\dots,x_{i-1},ax_i,bx_i,x_{i+1},\dots,x_k),
\end{aligned} \tag{68}$$

and consequently for any non-negative integers l and m such that $l < m$, we get

$$\begin{aligned}
&\Big\| \frac{1}{\lambda^{2l}} f\left(x_1,\dots,x_{i-1},\lambda^l x_i,x_{i+1},\dots,x_k\right) - \frac{1}{\lambda^{2m}} \\
&\quad \times f(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k), y_2,\dots,y_n \Big\| \\
&\le \sum_{j=l}^{m-1} \Big\| \frac{1}{\lambda^{2(j+1)}} f\left(x_1,\dots,x_{i-1},\lambda^{j+1} x_i,x_{i+1},\dots,x_k\right) \\
&\qquad - \frac{1}{\lambda^{2j}} f\left(x_1,\dots,x_{i-1},\lambda^{j} x_i,x_{i+1},\dots,x_k\right), \\
&\qquad y_2,\dots,y_n \Big\| \\
&\le \sum_{j=l}^{m-1} \Big[\frac{1}{\lambda^{2j+1}} \varphi_i\left(x_1,\dots,x_{i-1},\lambda^j x_i,0,x_{i+1},\dots,x_k\right) \\
&\qquad + \frac{1}{\lambda^{2j+2}} \\
&\qquad \times \varphi_i\left(x_1,\dots,x_{i-1},a\lambda^j x_i,b\lambda^j x_i,x_{i+1},\dots,x_k\right) \Big].
\end{aligned} \tag{69}$$

Therefore from (63), it follows that $\{(1/\lambda^{2j}) f(x_1,\dots,x_{i-1},\lambda^j x_i,x_{i+1},\dots,x_k)\}_{j\in\mathbb{N}}$ is a Cauchy sequence. Since W is an n-Banach space, this sequence is convergent and we define $Q_i : V^k \to W$ by (66). Putting $l = 0$, letting $m \to \infty$ in (69) and using Lemma 4 and (63), we see that (65) holds.

Now, fix also $x_i' \in V$ and note that according to (64), we have

$$\begin{aligned}
&\Big\| \frac{1}{\lambda^{2j}} \widehat{D}_i f\left(x_1,\dots,x_{i-1},\lambda^j x_i,\lambda^j x_i', x_{i+1},\dots,x_k\right), \\
&\quad y_2,\dots,y_n \Big\| \\
&\qquad \le \frac{1}{\lambda^{2j}} \varphi_i\left(x_1,\dots,x_{i-1},\lambda^j x_i,\lambda^j x_i', x_{i+1},\dots,x_k\right).
\end{aligned} \tag{70}$$

Next, fix $s \in \{1,\dots,k\}\setminus\{i\}$, $x_s' \in V$, and assume that $s < i$ (the same arguments apply to the case where $s > i$). From (64), it follows that

$$\begin{aligned}
&\Big\| \frac{1}{\lambda^{2j}} \widehat{D}_s f\left(x_1,\dots,x_s,x_s',x_{s+1},\dots,x_{i-1},\right. \\
&\qquad \left.\lambda^j x_i,x_{i+1},\dots,x_k\right), y_2,\dots,y_n \Big\| \\
&\quad \le \frac{1}{\lambda^{2j}} \varphi_s\left(x_1,\dots,x_s,x_s',x_{s+1},\dots,x_{i-1},\right. \\
&\qquad \left.\lambda^j x_i,x_{i+1},\dots,x_k\right).
\end{aligned} \tag{71}$$

Letting $j \to \infty$ in the above two inequalities and using (63), and Lemma 4 we see that the mapping Q_i is multi-Euler-Lagrange quadratic.

Now, let us finally assume that $Q_i' : V^k \to W$ is another multi-Euler-Lagrange quadratic mapping satisfying (65) and note that according to (61) and using Lemma 4, and (63) we have

$$\begin{aligned}
&\| Q_i(x_1,\dots,x_k) - Q_i'(x_1,\dots,x_k), y_2,\dots,y_n \| \\
&= \lim_{m\to\infty} \frac{1}{\lambda^{2m}} \| Q_i(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k) \\
&\qquad - Q_i'(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k), \\
&\qquad y_2,\dots,y_n \| \\
&\le \lim_{m\to\infty} \frac{1}{\lambda^{2m}} [\| Q_i(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k) \\
&\qquad - f(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k), \\
&\qquad y_2,\dots,y_n \| \\
&\qquad + \| f(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k) \\
&\qquad\quad - Q_i(x_1,\dots,x_{i-1},\lambda^m x_i,x_{i+1},\dots,x_k), \\
&\qquad\quad y_2,\dots,y_n \|]
\end{aligned}$$

$$\leq \lim_{m\to\infty}\sum_{j=0}^{\infty}\Big[\frac{1}{\lambda^{2(m+j)+1}}\times\varphi_i\left(x_1,\dots,x_{i-1},\lambda^{m+j}x_i,0,x_{i+1},\dots,x_k\right)+\frac{1}{\lambda^{2(m+j)+2}}\times\varphi_i\left(x_1,\dots,x_{i-1},a\lambda^{m+j}x_i,b\lambda^{m+j}x_i,x_{i+1},\dots,x_k\right)\Big]=0. \quad (72)$$

Therefore, by Lemma 4, we can conclude that $Q_i = Q_i'$. □

Similar to Theorem 15, one can get the following.

Theorem 16. *Assume that for every $i \in \{1,\dots,k\}$, $\varphi_i : V^{k+1} \to [0,\infty)$ is a mapping such that*

$$\sum_{j=0}^{\infty}\lambda^{2j}\Big[\varphi_i\left(\frac{x_1}{\lambda^j},x_2,\dots,x_{k+1}\right)+\cdots+\varphi_i\left(x_1,\dots,x_{i-2},\frac{x_{i-1}}{\lambda^j},x_i,\dots,x_{k+1}\right)+\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^j},\frac{x_{i+1}}{\lambda^j},x_{i+2},\dots,x_{k+1}\right)+\varphi_i\left(x_1,\dots,x_{i+1},\frac{x_{i+2}}{\lambda^j},x_{i+3},\dots,x_{k+1}\right)+\cdots+\varphi_i\left(x_1,\dots,x_k,\frac{x_{k+1}}{\lambda^j}\right)\Big]<\infty,\quad (x_1,\dots,x_{k+1})\in V^{k+1}. \quad (73)$$

If $f : V^k \to W$ is a function satisfying condition (11) *and*

$$\left\|\widehat{D}_i f\left(x_1,\dots,x_{k+1}\right),y_2,\dots,y_n\right\|\leq\varphi_i\left(x_1,\dots,x_{k+1}\right),\quad (x_1,\dots,x_{k+1})\in V^{k+1},\quad i\in\{1,\dots,k\},\quad y_2,\dots,y_n\in W, \quad (74)$$

then for every $i \in \{1,\dots,k\}$, there exists a unique multi-Euler-Lagrange quadratic mapping $Q_i : V^k \to W$ for which

$$\left\|f\left(x_1,\dots,x_k\right)-Q_i\left(x_1,\dots,x_k\right),y_2,\dots,y_n\right\|\leq\sum_{j=0}^{\infty}\Big[\lambda^{2j+1}\times\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^{j+1}},0,x_{i+1},\dots,x_k\right)+\lambda^{2j}\varphi_i\left(x_1,\dots,x_{i-1},\frac{ax_i}{\lambda^{j+1}},\frac{bx_i}{\lambda^{j+1}},x_{i+1},\dots,x_k\right)\Big],\quad (x_1,\dots,x_k)\in V^k,\quad y_2,\dots,y_n\in W. \quad (75)$$

For every $i \in \{1,\dots,k\}$ the function Q_i is given by

$$Q_i\left(x_1,\dots,x_k\right):=\lim_{j\to\infty}\lambda^{2j}f\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^j},x_{i+1},\dots,x_k\right),\quad (x_1,\dots,x_k)\in V^k. \quad (76)$$

Proof. Fix $x_1,\dots,x_k \in V$, $y_2,\dots,y_n \in W$, $j \in \mathbb{N}\cup\{0\}$ and $i \in \{1,\dots,k\}$. By (74), we obtain

$$\left\|\lambda^2 f\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda},x_{i+1},\dots,x_k\right)-f\left(x_1,\dots,x_k\right),y_2,\dots,y_n\right\|\leq\lambda\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda},0,x_{i+1},\dots,x_k\right)+\varphi_i\left(x_1,\dots,x_{i-1},\frac{ax_i}{\lambda},\frac{bx_i}{\lambda},x_{i+1},\dots,x_k\right), \quad (77)$$

and consequently for any non-negative integers l and m such that $l < m$, we get

$$\left\|\lambda^{2l}f\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^l},x_{i+1},\dots,x_k\right)-\lambda^{2m}f\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^m},x_{i+1},\dots,x_k\right),y_2,\dots,y_n\right\|$$
$$\leq\sum_{j=l}^{m-1}\left\|\lambda^{2(j+1)}f\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^{j+1}},x_{i+1},\dots,x_k\right)-\lambda^{2j}f\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^j},x_{i+1},\dots,x_k\right),y_2,\dots,y_n\right\|$$
$$\leq\sum_{j=l}^{m-1}\Big[\lambda^{2j+1}\varphi_i\left(x_1,\dots,x_{i-1},\frac{x_i}{\lambda^{j+1}},0,x_{i+1},\dots,x_k\right)+\lambda^{2j}\varphi_i\left(x_1,\dots,x_{i-1},\frac{ax_i}{\lambda^{j+1}},\frac{bx_i}{\lambda^{j+1}},x_{i+1},\dots,x_k\right)\Big]. \quad (78)$$

Therefore, from (73), it follows that $\{\lambda^{2j}f(x_1,\dots,x_{i-1},x_i/\lambda^j,x_{i+1},\dots,x_k)\}_{j\in\mathbb{N}}$ is a Cauchy sequence. Since W is an n-Banach space, this sequence is convergent and we define $Q_i : V^k \to W$ by (76). Putting $l = 0$, letting $m \to \infty$ in (78), and using Lemma 4 and (73) we see that (75) holds. The further part of the proof is similar to the proof of Theorem 15. □

As applications of Theorems 15 and 16, we get the following corollaries.

Corollary 17. *Let V be a real normed linear space and, W be an n-Banach space. Assume also that $\theta \in [0,\infty)$ and $r \in (0,\infty)$ are such that $r \neq 1$. If $f : V^k \to W$ is a function satisfying*

$$\left\|\widehat{D}_i f\left(x_1,\dots,x_{k+1}\right),y_2,\dots,y_n\right\|\leq\theta\left[\left\|x_1\right\|^r\cdots\left\|x_{i-1}\right\|^r\left(\left\|x_i\right\|^r+\left\|x_{i+1}\right\|^r\right)\times\left\|x_{i+2}\right\|^r\cdots\left\|x_{k+1}\right\|^r\right],$$

$$(x_1, \ldots, x_{k+1}) \in V^{k+1}, \quad i \in \{1, \ldots, k\},$$
$$y_2, \ldots, y_n \in W, \tag{79}$$

then for every $i \in \{1, \ldots, k\}$, there exists a unique multi-Euler-Lagrange quadratic mapping $Q_i : V^k \to W$ for which

$$\|f(x_1, \ldots, x_k) - Q_i(x_1, \ldots, x_k), y_2, \ldots, y_n\| \leq \frac{\theta \|x_1\|^r \cdots \|x_k\|^r (\lambda + |a|^r + |b|^r)}{|\lambda^2 - \lambda^r|}, \tag{80}$$

for all $x_1, \ldots, x_k \in V$, $y_2, \ldots, y_n \in W$.

Corollary 18. *Let V be a real normed linear space and let W be an n-Banach space. Assume also that $\theta \in [0, \infty)$ and $r, p, q \in (0, \infty)$ are such that $r, p + q \in (0, 2)$ or $r, p + q \in (2, \infty)$. If $f : V^k \to W$ is a function satisfying*

$$\begin{aligned}
&\left\|\widehat{D}_i f(x_1, \ldots, x_{k+1}), y_2, \ldots, y_n\right\| \\
&\quad \leq \theta \|x_1\|^r \cdots \|x_{i-1}\|^r \left(\|x_i\|^p \|x_{i+1}\|^q\right) \\
&\quad\quad \times \|x_{i+2}\|^r \cdots \|x_{k+1}\|^r, \\
&(x_1, \ldots, x_{k+1}) \in V^{k+1}, \quad i \in \{1, \ldots, k\}, \\
&\quad y_2, \ldots, y_n \in W,
\end{aligned} \tag{81}$$

then for every $i \in \{1, \ldots, k\}$, there exists a unique multi-Euler-Lagrange quadratic mapping $Q_i : V^k \to W$ for which

$$\begin{aligned}
&\|f(x_1, \ldots, x_k) - Q_i(x_1, \ldots, x_k), y_2, \ldots, y_n\| \\
&\quad \leq \frac{|a|^p |b|^q \theta \|x_1\|^r \cdots \|x_{i-1}\|^r \|x_i\|^{p+q} \|x_{i+1}\|^r \cdots \|x_k\|^r}{|\lambda^2 - \lambda^{p+q}|},
\end{aligned} \tag{82}$$

for all $x_1, \ldots, x_k \in V$, $y_2, \ldots, y_n \in W$.

For $a = b = 1$, Corollary 18 yields the following corollary which corrects Theorems 4.1 and 4.2 from [20].

Corollary 19. *Let V be a real normed linear space and let W be an 2-Banach space. Assume also that $\theta \in [0, \infty)$ and $p, q \in (0, \infty)$ are such that $p + q \neq 2$. If $f : V \to W$ is a function satisfying*

$$\begin{aligned}
&\|f(x_1 + x_2) - f(x_1) - f(x_2), y\| \\
&\quad \leq \theta \|x_1\|^p \|x_2\|^q, \quad x_1, x_2 \in V, \quad y \in W,
\end{aligned} \tag{83}$$

then there exists a unique quadratic mapping $Q : V \to W$ for which

$$\|f(x) - Q(x), y\| \leq \frac{\theta \|x\|^{p+q}}{|4 - 2^{p+q}|}, \quad x \in V, \ y \in W. \tag{84}$$

Acknowledgments

This project was supported by the National Natural Science Foundation of China (NNSFC) (Grant no. 11171022). The author would like to thank Professor Krzysztof Ciepliński and anonymous referees for their valuable comments and suggestions.

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