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Research Article

A Coons Patch Spanning a Finite Number of Curves Tested for Variationally Minimizing Its Area

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In surface modeling a surface frequently encountered is a Coons patch that is defined only for a boundary composed of *four* analytical curves. In this paper we extend the range of applicability of a Coons patch by telling how to write it for a boundary composed of an arbitrary number of boundary curves. We partition the curves in a clear and natural way into four groups and then join all the curves in each group into *one* analytic curve by using representations of the unit step function including one that is *fully analytic*. Having a well-parameterized surface, we do some calculations on it that are motivated by differential geometry but give a better optimized and possibly more smooth surface. For this, we use an ansatz consisting of the original surface plus a variational parameter multiplying the numerator part of its mean curvature function and minimize with the respect to it the rms mean curvature and decrease the area of the surface we generate. We do a complete numerical implementation for a boundary composed of five straight lines, that can model a string breaking, and get about 0.82 percent decrease of the area. Given the demonstrated ability of our optimization algorithm to reduce area by as much as 23 percent for a spanning surface not close of being a minimal surface, this much smaller fractional decrease suggests that the Coons patch we have been able to write is already close of being a minimal surface.

1. Introduction

For a closed boundary composed of four curves efficient methods [1–5] of generating a spanning surface are known, and it is also well known how to find different differential geometry-related properties [6] for these surfaces. If needed such a surface can be replaced by other slightly deformed surfaces with the differential geometry properties be closer to some desired values. Specifically we can mention the Coons patch [7, 8] prescription for generating the surface and our variational method [9] (given by (31)) based changes in it that generate a slightly changed surface from it that has lesser rms mean curvature and hence is closer to be a minimal surface. S. A. Coons [7, 8] introduced the Coons patch in 1964. This approach is based on the premise that a patch can be described in terms of four distinct boundary curves. So when the number of boundary curves is four, a Coons patch is clearly a worth analyzing surface spanning it. It is an active area of research and has seen enormous development during recent years that includes the work of Farin and Hansford [1], Hugentobler and Schneider [10], Wang and Tang [4], Szilvási-Nagy and Szabó [2, 11], and Sarkar and Dey [5]. As for the variational methods, generally these try to find the best values of the parameters in a trial function that optimize, subject to some algebraic, integral, or differential constraints, a quantity dependant on the ansatz. Variational methods are one of the active research areas of the optimization theory [12, 13]. For example, work has been done on finding the path of stationary optical length connecting two points, as the Fermat's principle says that the rays of light traverse such a path. In our previous work [9] we tried to find the best values of a parameter in the trial expression for a bounding surface spanning our boundary.

When the number of boundary curves is more than four, methods have been used for N sides surface generation from arbitrary boundary edges. But these are (1) discrete surfaces

[14, 15], (2) the relation of such a surface with Coons patch is at least not clear, (3) such a surface is not studied from the view of differential geometry as we have done [9] for the Coons patch, or (4) it is not told how to replace such a surface with a deformed one closer to being, say, a minimal surface. In the present paper we present a prescription to avoid all these possibly unwanted features; we tell how to generalize both the surface generation equation (11) and the variational improvement equation (31) to closed boundaries composed of an arbitrary finite number of curves. Presently, we are able to fully produce variationally improved surfaces only for a boundary composed of five straight lines, but our algorithms both for the variational improvement equation (31) and, before it, for the surface generation are general. Our previous and present work falls in the category of *Plateau problem* [16, 17] which is finding the surface with minimal area constrained by a given boundary curve, named after the blind Belgian physicist Joseph Antoine Ferdinand Plateau. He showed in 1849 that a minimal surface can be realized in the form of soap films, stretched over various shapes of wire frames. Since then many mathematicians contributed to the theory of minimal surfaces like Schwarz [18] who discovered some triply periodic surfaces, Riemann and Weierstrass [16]. Mathematical solutions for specific boundaries were known for years, but existence of a minimal solution for a given simple closed curve was independently proved in 1931 by Douglas [19] and Radó [20]. They approached to the solution through different methods. Douglas [19] minimized a quantity now called as Douglas Integral for a minimal solution to an arbitrary simple closed curve, while Radó [20] minimized the energy. The work of Radó was built on the previous work of Garnier [21] for minimal solution only for rectifiable simple closed curves. Achievements of Tonelli [22], Courant [23, 24], Morrey [25, 26], McShane [27], Shiffman [28], Morse and Tompkins [29], Osserman [30], Gulliver [31], and Karcher [32] and others contributed with many revolutionary results in the subsequent years.

An arbitrary boundary can be written as a limit of a collection of finite number of arbitrary curves and for a boundary composed of finite number of curves our plan is to first reduce it to only four bounding curves so that we can then apply our Coons-patch-based analysis. For this, techniques are needed to group a finite number of curves into four sets and then within each set combine all the curves as one continuous curve. In this way, one set would become, in the common notation for a Coons patch, $\mathbf{c}_1(u) = \mathbf{x}(u, 0)$ and the other three as $\mathbf{c}_2(u) = \mathbf{x}(u, 1)$, $\mathbf{d}_1(v) = \mathbf{x}(0, v)$, and $\mathbf{d}_2(v) = \mathbf{x}(1, v)$ with $0 \le u, v \le 1$. We introduce our algorithm for grouping in the first paragraph of Section 3. For the combinations, in the remaining portion of Section 3 we present an iterative scheme that uses suitable step function representations to combine an arbitrary number of curves into one continuous curve. In Section 4 we present, in (24) to (26), three analytical representations of the unit step function and give, through Figure 4, geometric description of boundary curves generated by these step function representations. Before these two sections, in Section 2 we recall briefly some basic stuff related to minimization of area and mean curvature and some simple facts about blending-based Coons patch as our initial surface. Having written the expression for the Coons patch spanning an arbitrary discredited boundary, in the Section 5 we describe our variational technique to reduce its rms mean curvature resulting in a decrease of its area as well. So far, our description of algorithms remains general. Then in next Section 6 we apply our step function analytical form, standard Coons patch and our variational techniques to reduce the area of a nonminimal surface spanned by *five* arbitrary straight lines. The resulting percentage decrease in area for few cases is reported in Table 1, and same data points are plotted in Figure 9 along with the curve that gives us percentage decrease in area as numerical function. Section 7 presents results, final remarks and mentions possible future developments.

Table 1: Selected out put values for x_i.

x_i	Percentage decrease in area A
2	0.08214
4	0.05076
5	0.04301
6	0.03781
8	0.03208
10	0.04983
12	0.03208
14	0.03781
15	0.04301
16	0.05077
18	0.08215

2. Coons Patch as Initial Surface Composed of Four Continuous Curves and Related Quantities

Considering a surface $\mathbf{x}(u, v)$ for $0 \le u \le 1$ and $0 \le v \le 1$ defined as a map over a domain $D \subset R^2$ spanning a fixed boundary $\mathbf{x}(\partial D) = \Gamma$, we can evaluate some quantities for it that are of interest of differential geometry. We choose the area and rms mean curvature and in the later part of the paper report efforts to decrease both. Area is evaluated by the area functional:

$$A(\mathbf{x}) = \int\int_D \left|\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)\right| du\, dv, \quad (1)$$

with $\mathbf{x}_u(u, v)$ and $\mathbf{x}_v(u, v)$ being partial derivatives of $\mathbf{x}(u, v)$ with respect to u and v. It is known [6] that the first variation of $A(\mathbf{x})$ vanishes everywhere if and only if the mean curvature H of $\mathbf{x}(u, v)$ is zero everywhere in it. Thus a surface of the least area is also a surface of the least (zero) rms mean curvature spanning the given boundary. Amongst the two, the mean square mean curvature (termed μ^2 later) has not a square root in its integrand, and its minimization is easier than that of the area directly. For a locally parameterized surface $\mathbf{x} = \mathbf{x}(u, v)$, the mean curvature H may be given by

$$H = \frac{Ge - 2Ff + Eg}{EG - F^2}, \quad (2)$$

where

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \qquad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \qquad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle \quad (3)$$

are the first fundamental coefficients, and

$$e = \langle \mathbf{N}, \mathbf{x}_{uu} \rangle, \qquad f = \langle \mathbf{N}, \mathbf{x}_{uv} \rangle, \qquad g = \langle \mathbf{N}, \mathbf{x}_v \rangle \quad (4)$$

are the second fundamental coefficients with

$$\mathbf{N}(u, v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}, \quad (5)$$

being the unit normal to the surface $\mathbf{x}(u, v)$. For a minimal surface [6, 33] the mean curvature (2) is identically zero. For minimization we use only the numerator part of mean curvature H given by (2), as done in [34] following [16] which explicitly mentions that "for a locally parameterized surface, the mean curvature vanishes when the numerator part of the mean curvature is equal to zero." We call the numerator part of (2) as H_0 corresponding the *initial* surface $\mathbf{x}_0(u, v)$ that is used in the ansatz equation (31) to get first-order variationally improved surface $\mathbf{x}_1(u, v)$ of lesser area; we chose our initial surface to be a Coons patch described later. This process could be continued as an iterative process until a minimal surface is achieved. But due to complexity of the calculations required for obtaining the second-order improvement $\mathbf{x}_2(u, v)$, we have been able to calculate the first-order surface $\mathbf{x}_1(u, v)$ only. The numerator part H_0 is denoted by

$$H_0 = e_0 G_0 - 2F_0 f_0 + g_0 E_0, \quad (6)$$

where E_0, F_0, G_0, e_0, f_0 and g_0 denote the fundamental magnitudes, given by (3) and (4), for the initial surface $\mathbf{x}_0(u, v)$, and $N_0(u, v)$ is the unit normal, given by (5), to this initial surface. We call the root mean square (rms) of this H_0, for $0 \le u \le 1$ and $0 \le v \le 1$, as μ_0. That is,

$$\mu_0 = \left(\int_0^1 \int_0^1 H_0^2 \, du \, dv \right)^{1/2}. \quad (7)$$

In the notation of (2) to (4), equation (1) becomes, for $\mathbf{x}_0(u, v)$,

$$A_0 = \int_0^1 \int_0^1 \sqrt{E_0 G_0 - F_0^2} \, du \, dv. \quad (8)$$

If the boundary Γ is composed of four continuous curves $\mathbf{c}_1(u) = \mathbf{x}(u, 0)$, $\mathbf{c}_2(u) = \mathbf{x}(u, 1)$, $\mathbf{d}_1(v) = \mathbf{x}(0, v)$ and $\mathbf{d}_2(v) = \mathbf{x}(1, v)$, a surface $\mathbf{x}(u, v)$ spanning it can be the Coons patch [7]. (As mentioned previously we choose this to be our initial surface for the variational process we report.) Using blending functions $f_1(u), f_2(u), g_1(v)$ and $g_2(v)$ satisfying the conditions that

$$\sum_{i=1}^{2} f_i(u) = \sum_{i=1}^{2} g_i(v) = 1, \quad (9)$$

that is, $f_1(u) + f_2(u) = 1$, $g_1(v) + g_2(v) = 1$ for nonbarycentric combination of points and for $j = 0, 1$

$$f_i(j) = g_i(j) = \delta_{i-1,j}, \quad (10)$$

in order to actually interpolate $\mathbf{x}(0,0), \mathbf{x}(0,1), \mathbf{x}(1,0)$, and $\mathbf{x}(1,1)$ the following equation defines Coons patch:

$$\begin{aligned} \mathbf{x}(u, v) = & \begin{bmatrix} f_1(u) & f_2(u) \end{bmatrix} \begin{bmatrix} \mathbf{x}(0, v) \\ \mathbf{x}(1, v) \end{bmatrix} \\ & + \begin{bmatrix} \mathbf{x}(u, 0) & \mathbf{x}(u, 1) \end{bmatrix} \begin{bmatrix} g_1(v) \\ g_2(v) \end{bmatrix} \\ & - \begin{bmatrix} f_1(u) & f_2(u) \end{bmatrix} \begin{bmatrix} \mathbf{x}(0,0) & \mathbf{x}(0,1) \\ \mathbf{x}(1,0) & \mathbf{x}(1,1) \end{bmatrix} \begin{bmatrix} g_1(v) \\ g_2(v) \end{bmatrix}. \end{aligned} \quad (11)$$

For instance, linear blending functions satisfying the previous conditions may be given by

$$\begin{aligned} f_1(u) &= 1 - u, \qquad f_2(u) = u, \\ g_1(v) &= 1 - v, \qquad g_2(v) = v. \end{aligned} \quad (12)$$

Using these choices of blending functions in (11) we get our initial surface $\mathbf{x}_0(u, v)$. In the present form this prescription is *apparently* limited to a boundary composed of four straight lines or at most four continuous curves. For a more general boundary, we have to reduce, as mentioned previously, it to four continuous curves. The next section describes the algorithm we suggest for achieving this aim.

3. Construction of the Four Continuous Boundary Curves

As said previously, an arbitrary boundary can be written as a limit of a collection of finite number of arbitrary curves. For reducing it to four bounding curves, first we need to group a finite number of curves into four sets (in the notation introduced in the previous section) $\mathbf{c}_1(u) = \mathbf{x}(u, 0)$, $\mathbf{c}_2(u) = \mathbf{x}(u, 1)$, $\mathbf{d}_1(v) = \mathbf{x}(0, v)$, and $\mathbf{d}_2(v) = \mathbf{x}(1, v)$ with $0 \le u, v \le 1$. Let us call the N bounding curves as $\mathbf{L}_i$ for $i = 1, 2, \ldots, N$, the greatest integer $\le N/4$ as m and the residue $s = N - 4m$ which can take values as 0, 1, 2, or 3. We put the first $m + c_1$ curves in the first group $\mathfrak{C}_{i=1}^{m+c_1}$ that eventually becomes $\mathbf{c}_1(u)$ when we join all its elements into one continuous curve by using step functions of the next section. Here $c_1 = \min(s, 1)$, meaning that for four equal groups the first group is just the first quarter and for nonequal groups we put one of the s extra curves in the first group. The next group, to become $\mathbf{d}_1(v)$ after the joining(s), starts with the very next $m + c_1 + 1$st curve and continues till $i = 2m + d_1$ with $d_1 = \min(s, 2)$. The number of curves in this second group is $m + \min(2, s) - \min(1, s)$ meaning that for s larger than one this group gets one of the s curves. We call this group $\mathfrak{C}_{i=m+c_1+1}^{2m+d_1}$. The third group, to become $\mathbf{c}_2(u)$ after the joining(s), starts with the $2m + d_1 + 1$st curve and continues till $i = 3m + s$. The number of curves in this second group is $m + s - \min(2, s)$ meaning that only for $s = 3$ this group gets one of the s curves. We call it $\mathfrak{C}_{i=2m+d_1+1}^{3m+s}$. The fourth and last group, to become $\mathbf{d}_2(v)$, starts with the next $i = 3m + s + 1$ till at the end when $i = 4m + s = N$. The number of curves in this last group is always m, and thus it never gets any of the extra s curves. The label we use for this group is $\mathfrak{C}_{i=3m+s+1}^{N}$. For example for a boundary composed of 9 curves, $N = 9, m = 2, s = 9 - 8 = 1, c_1 = \min(s, 1) = 1, d_1 = \min(s, 2) = 1$ correspond to four sets of curves, namely, $\mathfrak{C}_1^3, \mathfrak{C}_4^5, \mathfrak{C}_6^7$ and $\mathfrak{C}_8^9$ as the bounding curves $\mathbf{c}_1(u), \mathbf{d}_1(v), \mathbf{c}_2(u)$, and $\mathbf{d}_2(v)$, respectively, where $\mathbf{c}_1(u)$ comprises 3 curves and each of $\mathbf{d}_1(v)$, $\mathbf{c}_2(u)$, and $\mathbf{d}_2(v)$ has two curves. For another example note that $N = 14, m = 3, s = 14 - 12 = 2$ correspond to four sets of curves, namely, $\mathfrak{C}_1^4, \mathfrak{C}_5^8, \mathfrak{C}_9^{11}$ and $\mathfrak{C}_{12}^{14}$ as the bounding curves $\mathbf{c}_1(u)$, $\mathbf{d}_1(v)$, $\mathbf{c}_2(u)$ and $\mathbf{d}_2(v)$, respectively, for Coons patch in this case. Each of $\mathfrak{C}_1^4$ and $\mathfrak{C}_5^8$ includes 4 curves, and each of $\mathfrak{C}_9^{11}$ and $\mathfrak{C}_{12}^{14}$ contains 3 curves.

The next task is to join $m + c_1$ curves in the first group $\mathfrak{C}_{i=1}^{m+c_1}$ into *one* continuous curve $\mathbf{c}_1(u)$; the same process is to be done later for other three groups. Let us consider two consecutive curves $\mathbf{L}_i(u)$ and $\mathbf{L}_{i+1}(u)$, for $i = 1, \ldots, m + c_1 - 1$, in this group. These can be combined into a continuous curve

$$\mathbf{L}_i^1(u) = \mathbf{L}_i^0(u) + S(u - u_i)\left(\mathbf{L}_{i+1}^0(u) - \mathbf{L}_i^0(u)\right), \quad (13)$$

at their junction u_i for a smooth approximation ((24) to (26)) of the step function

$$S(u - u_i) = \begin{cases} 0 & \text{if } u < u_i, \\ 1 & \text{if } u > u_i, \end{cases} \quad (14)$$

ensuring that

$$\mathbf{L}_i^1 = \begin{cases} \mathbf{L}_i^0(u) & \text{if } u < u_i, \\ \mathbf{L}_{i+1}^0(u) & \text{if } u > u_i. \end{cases} \quad (15)$$

We have added a superscript j to $\mathbf{L}_i^j$. This denotes the number of junctions in the joined curve and hence $\mathbf{L}_i^0 = \mathbf{L}_i$ are not supposed to have junctions in them. In $\mathbf{L}_i$, $i = 1, 2, \ldots, m + c_1$ of the first group, similarly for the other three groups. To complete the task we have to be able to increase the superscript to the number of junctions of the respective group. We suggest a recursion relation to increase the superscript to any value, namely,

$$\mathbf{L}_i^j(u) = \mathbf{L}_i^0(u) + S(u - u_i)\left[\mathbf{L}_{i+1}^{j-1}(u) - \mathbf{L}_i^0(u)\right]. \quad (16)$$

For $j = 1$ this equation becomes (13), and this is the least value of the superscript for which (16) should be used. Continuing the iterative process in the previous equation would express $\mathbf{L}_{i+1}^{j-1}(u)$ in terms of the one with superscript further decreased. That is,

$$\mathbf{L}_{i+1}^{j-1}(u) = \mathbf{L}_{i+1}^0(u) + S(u - u_{i+1})\left[\mathbf{L}_{i+2}^{j-2}(u) - \mathbf{L}_{i+1}^0(u)\right], \quad (17)$$

and so on. This iteration allows us to extend our algorithm to merge *any finite number* of curves, that is, (one more than) the value we assign to the superscript j. We illustrated later a merging of three and four curves by assigning the superscript j values of 2 and 3, which are the corresponding number of junctions. In our full scheme these combinations are needed when we partition 9 and 14 total number of curves, respectively, into our usual four groups; when the previously mentioned set of curves $\mathfrak{C}_1^3$ is joined together to make $\mathbf{c}_1(u)$ for a boundary composed of 9 curves, it takes the following form:

$$\mathbf{c}_1(u) = \mathbf{L}_1^2(u) = \mathbf{L}_1^0(u) + S(u - u_1)\left[\mathbf{L}_2^1(u) - \mathbf{L}_1^0(u)\right], \quad (18)$$

where

$$\mathbf{L}_2^1(u) = \mathbf{L}_2^0(u) + S(u - u_2)\left[\mathbf{L}_3^0(u) - \mathbf{L}_2^0(u)\right], \quad (19)$$

as shown in Figure 1.

Likewise, when the previously mentioned set of curves $\mathfrak{C}_1^4$ is joined together to make $\mathbf{c}_1(u)$ for a boundary composed of 14 curves, it takes the following form:

$$\mathbf{c}_1(u) = \mathbf{L}_1^3(u) = \mathbf{L}_1^0(u) + S(u - u_1)\left[\mathbf{L}_2^2(u) - \mathbf{L}_1^0(u)\right], \quad (20)$$

where

$$\mathbf{L}_2^2(u) = \mathbf{L}_2^0(u) + S(u - u_2)\left[\mathbf{L}_3^1(u) - \mathbf{L}_2^0(u)\right], \quad (21)$$

and in turn

$$\mathbf{L}_3^1(u) = \mathbf{L}_3^0(u) + S(u - u_3)\left[\mathbf{L}_4^0(u) - \mathbf{L}_3^0(u)\right]. \quad (22)$$

Substituting (21) and (22) in (20) results in

$$\mathbf{c}_1(u) = \mathbf{L}_1^0(u) + S(u - u_1)\Big[\mathbf{L}_2^0(u) + S(u - u_2)\Big[\mathbf{L}_3^0(u) + S(u - u_3)\left[\mathbf{L}_4^0(u) - \mathbf{L}_3^0(u)\right] - \mathbf{L}_2^0(u)\Big] - \mathbf{L}_1^0(u)\Big], \quad (23)$$

which is shown in Figure 2. In the similar way other constituent parts of Coons patch, namely, $\mathbf{d}_1(v)$, $\mathbf{c}_2(u)$, and $\mathbf{d}_2(v)$ can be constructed using (16).

4. Analytical Representations of the Step Function

In (14), we have used the step function for letting the parameter of our *one* curve map into different given constituent curves as it increases. For this we have to let the step function multiply one curve only till it reaches a value corresponding to a junction; for higher values of the parameter the same step function now multiplies a new curve, and so on till each of the curves gets multiplied by 1 for some range of values of the parameter. In the existing literature [3, 4], a step function is used for another purpose, namely, for effectively deleting through its vanishing value a nondesired piece of a surface and keeping only a desired piece of a surface for a corresponding range of parameters; practically this amounts

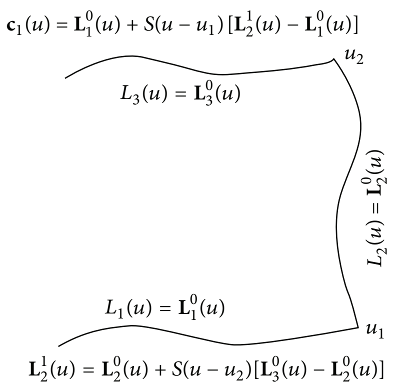

FIGURE 1: $\mathbf{c}_1(u)$ joining $L_1(u)$, $L_2(u)$, and $L_3(u)$ for $N = 9$.

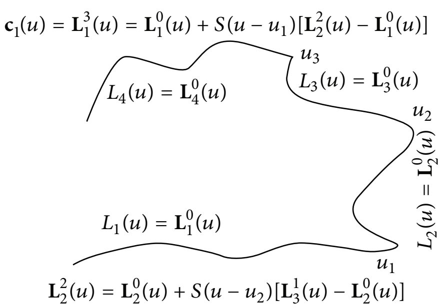

FIGURE 2: $\mathbf{c}_1(u)$ joining $L_1(u)$, $L_2(u)$, $L_3(u)$, and $L_4(u)$ for $N = 14$.

to change the boundary of the piece of the surface that is not deleted. In our present paper we use different forms of the step function to join the curves in each of the four groups into a single analytic curve. (This joining lets us use (11) to write the Coons patch which is used in trying to find a minimal surface for a *fixed boundary* in contrast to an effectively shrinking boundary of [3, 4]; this is called the Plateau problem and has its own importance in the mathematical literature.) We tried three-step function representations $S(u-u_i), S^{\star}(u-u_i)$ and $S^{\star\star}(u-u_i)$ written later. It is to be noted that, in addition to being analytic for $\epsilon \neq 0$ as that in [3], our first and third forms, sketched in Figure 3, are also analytic even for $\epsilon = 0$ provided $u \neq u_i$, and the second form is simply analytic throughout. In writing these, a real scalar l is used to replace unit interval $0 \le u \le 1$ to an interval $0 \le lu \le l$ of arbitrary length l. In the second form another variable k is introduced; larger k takes care of sharp transition at $u = u_i$,

$$S(u-u_i) = \frac{1}{2}\left(1 + \frac{(u-u_i)\,l}{\sqrt{\epsilon + ((u-u_i)\,l)^2}}\right), \qquad (24)$$

$$S^{\star}(u-u_i) = \frac{1}{2}\left(1 + \tanh k\left((u-u_i)\,l\right)\right), \qquad (25)$$

$$S^{\star\star}(u-u_i) = \frac{1}{2}\left(1 + \frac{(u-u_i)\,l}{|(u-u_i)\,l| + \epsilon}\right). \qquad (26)$$

As a special case let us assume that $L_1(u)$ and $L_2(u)$ be two successive analytical smooth curves joined together using a differentiable step-function representation equations (24)–(26) to give us smooth curve $L(u)$ that *exactly interpolates* the constituent curves $L_1(u)$ and $L_2(u)$ such that

$$L(u) = \begin{cases} L_1(u) & \text{if } u < u_1, \\ L_2(u) & \text{if } u > u_1. \end{cases} \qquad (27)$$

In particular for a line $L_1(u)$ obtained by joining $(0,0)$ to $(x_1, y_1) = (lu, y_1)$

$$L_1(u) = \frac{y_1}{lu_1}(lu), \qquad (28)$$

and line $L_2(u)$ obtained by joining the points $(x_1, y_1) = (lu, y_1)$ to $(l, 0)$

$$L_2(u) = \frac{y_1}{lu_1 - l}(lu - l). \qquad (29)$$

Using (13), $L(u)$ may be written as a smooth curve joined through a step function representation given by (24)–(26)

$$L(u) = L_1^1(u) = \frac{y_1}{u_1}u + S(u-u_1)\left(\frac{y_1}{u_1 - 1}(u-1) - \frac{y_1}{u_1}u\right). \qquad (30)$$

The graphs of these combinations of $L_1(u)$ and $L_2(u)$ for the three-step function representations $S(u-u_1), S^{\star}(u-u_1)$ and $S^{\star\star}(u-u_1)$ are sketched in Figure 4. Each of these is an *analytical* function guaranteed to pass through both the curves it combines. One can discretize both the curves and use Splines to generate a smooth curve passing through the resulting points, but that would not assure continuity of third and higher derivatives. If extrapolated, $L(u)$ simply agrees to the respective constituent curves $L_1(u)$ and $L_2(u)$ and thus does not develop any large fluctuations that would result from using an interpolating polynomial of high enough degree in place of Splines.

5. A Technique for Variational Improvement

Reducing an arbitrary curve (or a finite collection of many continuous curves) to four continuous curves let us write Coons patch for a surface bounded by it. But that would not tell us anything about its characteristics in the differential geometry. For example, there is no guarantee that such a surface would be a minimal surface spanning its boundary. We can calculate the differential-geometry-related functions of the two parameters u and v of the surface and then do integrations with respect to these parameters to get numbers characterizing the surface. If the rms mean curvature (see Section 2) of the generated surface is nonzero, the surface is not a minimal surface, and a challenge is to modify it so that it either becomes a minimal surface or, if that is not possible, gets closer to being a minimal surface. For this we can write an ansatz for a modification in the surface including a variational parameter (or parameters) and then solve the optimization problem of selecting the value(s) of the parameter(s) so as to minimize either its area directly or its

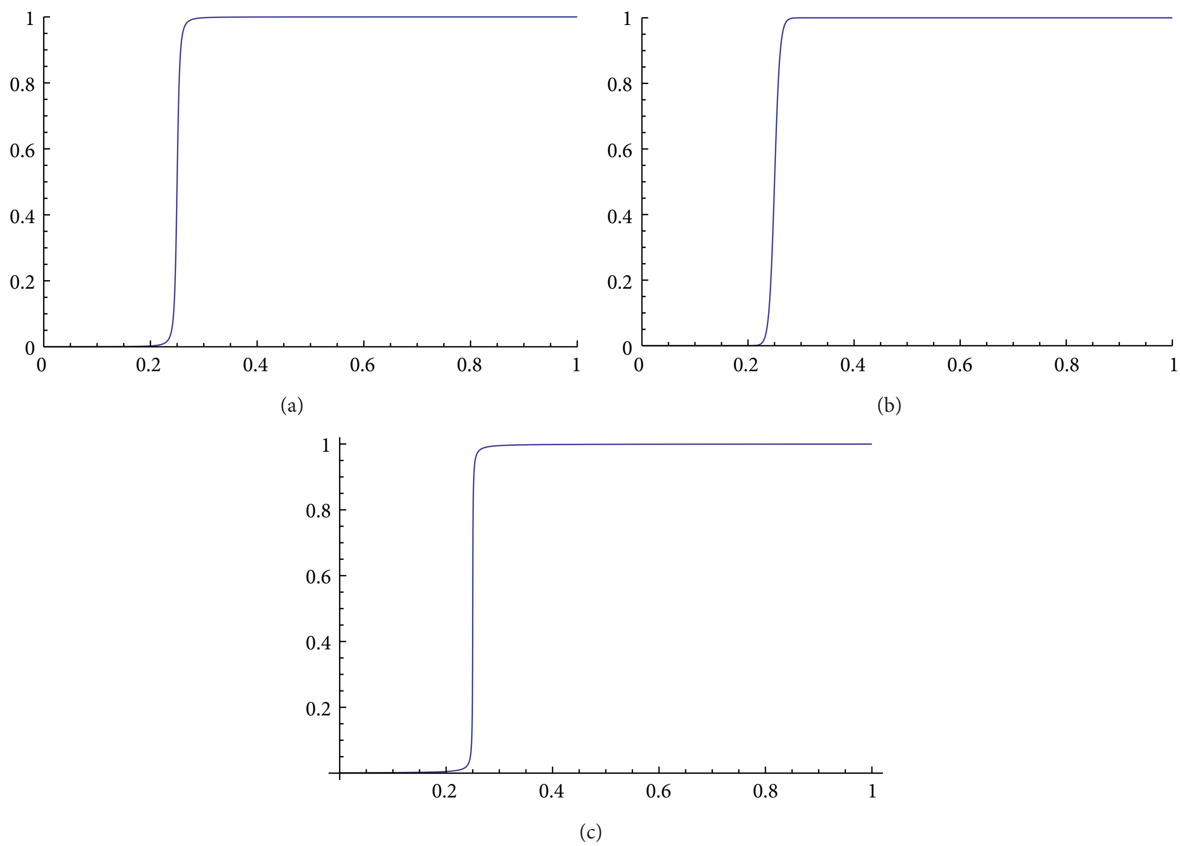

FIGURE 3: Step function representations $S(u-u_i)$, $S^{*}(u-u_i)$ and $S^{**}(u-u_i)$ for $i=1, l=20, u_i=0.25, \epsilon=0.01$ and $k=5$.

rms mean curvature. The ansatz we suggest [9] to minimize the area of a nonminimal surface $\mathbf{x}_0(u,v)$ is

$$\mathbf{x}_1(u,v,t)=\mathbf{x}_0(u,v)+tm(u,v)\,\mathbf{N}_0, \tag{31}$$

where t is our variational parameter; the rest of the modification,

$$m(u,v)=uv(1-u)(1-v)\,H_0, \tag{32}$$

is chosen so that the variation at the boundary curves $u=0, u=1, v=0$, and $v=1$ is zero. (Other choices of $m(u,v)$, for example $m(u,v)=u^2v^2(1-u^2)(1-v^2)H_0$, that vanish at the boundary points are possible, but they would take more CPU time.) In (31), $\mathbf{N}_0$ is unit normal to the nonminimal surface $\mathbf{x}_0(u,v)$, and H_0, given by (6), is numerator of the initial mean curvature of the starting surface $\mathbf{x}_0(u,v)$. Calling the fundamental magnitudes for $\mathbf{x}_1(u,v)$ as $E_1(u,v,t), F_1(u,v,t), G_1(u,v,t)$, $e_1(u,v,t), f_1(u,v,t)$, and $g_1(u,v,t)$, the area $A_1(t)$ of the surface $\mathbf{x}_1(u,v,t)$ for $0\le u\le 1$ and $0\le v\le 1$ is given by

$$A_1(t)=\int_0^1\int_0^1\sqrt{E_1G_1-F_1^2}\,du\,dv. \tag{33}$$

We denote, the numerator of mean curvature for $\mathbf{x}_1(u,v)$ of (31) by $H_1(u,v,t)$. It would have the following familiar expression:

$$H_1(u,v,t)=E_1g_1-2F_1f_1+G_1e_1. \tag{34}$$

As $H_1^2(u,v,t)$ is a polynomial in t, with real coefficients $h_i(u,v)$, we rewrite (34) in the form

$$H_1^2(u,v,t)=\sum_{i=0}^{n}\left(h_i(u,v)\right)t^i. \tag{35}$$

Here n turns out to be 10; there being no higher powers of t in the polynomials as it can be seen from the expression for $E_1(u,v,t)$, $F_1(u,v,t)$, and $G_1(u,v,t)$ which are quadratic in t and $e_1(u,v,t)$, $f_1(u,v,t)$, and $g_1(u,v,t)$ which are cubic in t. Integrating (numerically if needed) these coefficients with respect to u and v in the range $0\le u,v\le 1$ we get rms of mean curvature $H_1(u,v,t)$ of the *new* surface $x_1(u,v,t)$

$$\begin{aligned}\mu_1(t)&=\left(\int_0^1\int_0^1 H_1^2(u,v,t)\,du\,dv\right)^{1/2}\\&=\left(t^i\int_0^1\int_0^1\sum_{i=0}^{n}\left(h_i(u,v)\right)du\,dv\right)^{1/2}.\end{aligned} \tag{36}$$

The expression in the parentheses on right-hand side of previous equation turns out to be a polynomial in t of degree n, which can be minimized with respect to t to find $t_{\min}$. The resulting value of t completely specifies *new* surface $\mathbf{x}_1(u,v)$. New surface $\mathbf{x}_1(u,v)$ is expected to have lesser area than that of original surface $\mathbf{x}_0(u,v)$.

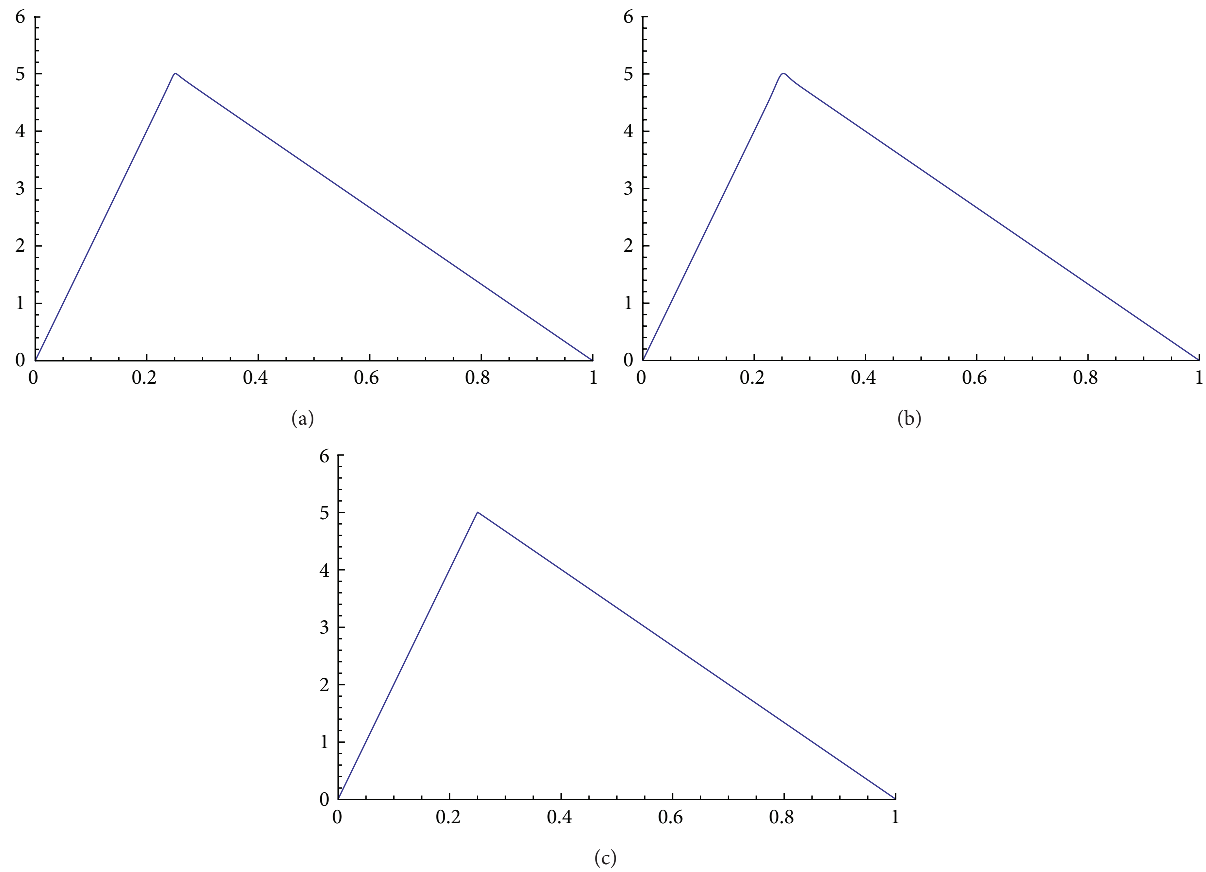

Figure 4: Straight lines $L_i(u)$ and $L_{i+1}(u)$ joined by first-, second, and third-step function representations, respectively, for $i = 1$, $l = 20$, $u_i = 0.25$, $\epsilon = 0.01$, and $k = 5$.

6. The Technique Applied to a Surface Spanned by Five Arbitrary Lines

In this section we use the ansatz of (31) to reduce the area of a surface spanned by five arbitrary lines lying in different planes. For $0 \leq u_i \leq 1$ with l any real scalar, the step function $S(u - u_i)$ satisfying (14) represented by (24) is used to join the curves L_i and L_{i+1}, using the technique discussed in Section 4; other two step function representations $S^{\star}(u - u_i)$ and $S^{\star\star}(u - u_i)$ demand more CPU time, involve complicated trigonometric expressions, and pose issues related to programming. As a special case let us assume that $L_1(u)$ equation (28) and $L_2(u)$ equation (29) be two successive analytical smooth curves joined together to give us smooth curve equation (30) through the step function $S(u - u_i)$ of (24) so that (30) takes the following form:

$$L(u) = L_1^1(u) = \frac{y_1}{u_1}u + \frac{1}{2}\left(1 + \frac{l(u-u_1)}{\sqrt{\varepsilon + (l(u-u_1))^2}}\right) \times \left(\frac{y_1}{u_1 - 1}(u-1) - \frac{y_1}{u_1}u\right), \tag{37}$$

which is continuous and differentiable at every point in the domain $0 \leq u \leq 1$. For using this work for modeling a string breaking, one can take $c_2(u) = L_4(u)$ as the initial string, $d_1(v) = L_3(v)$, and $d_2(v) = L_5(v)$ modeling the time evolution of its ends, the later $L(u)$ as the combination that contains the two final strings as shown below in Figure 5.

For the four curves required in a general Coons patch equation (11), we construct them from the boundary composed of *five* straight lines $L_1(u)$, $L_2(u)$, $L_3(u)$, $L_4(u)$, and $L_5(u)$ connecting five arbitrary corner points. For this, two lines joining three corners are joined into one curve, namely, $c_1(u) = L_1^1(u)$ equation (37) and the remaining three boundary lines $d_1(v) = L_3(v), c_2(u) = L_4(u)$ and $d_2(v) = L_5(v)$. For linear blending functions $f_1 = 1 - u$, $f_2 = u$, $g_1 = 1 - v$, and $g_2 = v$, we have been able to reduce the area spanning pentagons. In case of a pentagon, when we convert it to a Coons patch, for the corners we choose

$$\mathbf{r}_1 = (0, 0, 0), \quad \mathbf{r}_2 = (l, a, 0), \quad \mathbf{r}_3 = (0, a, 0), \quad \mathbf{r}_4 = (l, 0, 0). \tag{38}$$

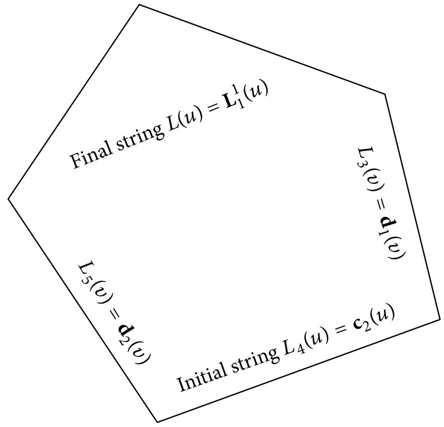

Figure 5: An initial string $\mathbf{c}_2(u) = L_4(u)$ with $\mathbf{d}_1(v) = L_3(v)$ and $\mathbf{d}_2(v) = L_5(v)$ modeling the time evolution of its ends and the combination $L(u)$ contains the two final strings $L_1(u)$ and $L_2(u)$.

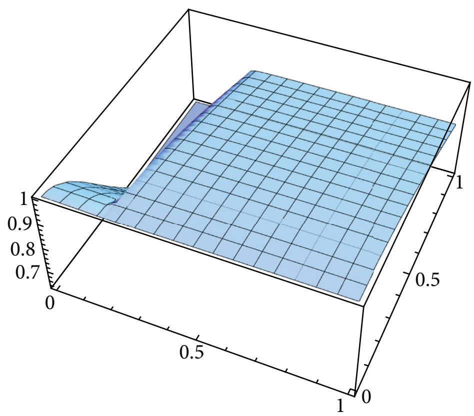

Figure 6: Cosine of the angle between $\mathbf{N}$ and $\mathbf{k}$ is positive shown for Coons patch in particular for $x_i = 4$. Thus angle between $\mathbf{N}$ and $\mathbf{k}$ remains within the range $0 \leq \vartheta \leq \pi/2$.

We use a selection of integer values of l and a. The four corners are labeled by the following Coons convention:

$$\begin{aligned} \mathbf{x}(0,0) &= \mathbf{r}_1, & \mathbf{x}(1,0) &= \mathbf{r}_4, \\ \mathbf{x}(0,1) &= \mathbf{r}_3, & \mathbf{x}(1,1) &= \mathbf{r}_2. \end{aligned} \tag{39}$$

The z component of our surface variable vector $\mathbf{x}_0(u,v)$ is a single valued function for all values of its x and y components, and hence we can replace complicated $\mathbf{N}_0$ in (31) by a unit vector $\mathbf{k}$ along the z-axis to facilitate computations. We checked that this makes a small enough angle with the original normal $\mathbf{N}_0$; it can be seen in Figure 6 that component of the unit normal $\mathbf{N_0}$ along $\mathbf{k}$ remains positive for $0 \leq u, v \leq 1$. Thus angle ϑ between $\mathbf{N}$ and $\mathbf{k}$ remains within the interval $0 \leq \vartheta \leq \pi/2$; this guarantees that signs of changes along $\mathbf{N_0}$ and $\mathbf{k}$ are the same, so a local increase (decrease) in the area integrand while moving along $\mathbf{N}$ means an increase (decrease) in moving along $\mathbf{k}$. But moving along $\mathbf{k}$ is much easier computationally. We get a net decrease in area with an optimal numerical value of the coefficient t even when it multiplies $\mathbf{k}$ not $\mathbf{N_0}$ as in (31); this indicates that the choice $\mathbf{k}$ instead of $\mathbf{N_0}$ has been useful. This choice of unit vector is graphically depicted in the Figure 7. Inserting values of blending functions and boundary points in (11), we find

$$\mathbf{x}(u,v) = \{lu, av, vL(u)\}, \tag{40}$$

whereas fundamental magnitudes have the following expressions:

$$\begin{gathered} E(u,v) = l^2 + v^2\left(L'(u)\right)^2, \\ F(u,v) = vL(u)L'(u), \\ G(u,v) = a^2L^2(u), \\ e(u,v) = alvL''(u), \qquad f(u,v) = alL'(u), \\ g(u,v) = 0, \end{gathered} \tag{41}$$

and (6) gives

$$\begin{aligned} H_0 = &-2alvL(u)L'(u)^2 \\ &+ alv\left(a^2 + L(u)^2\right)L''(u). \end{aligned} \tag{42}$$

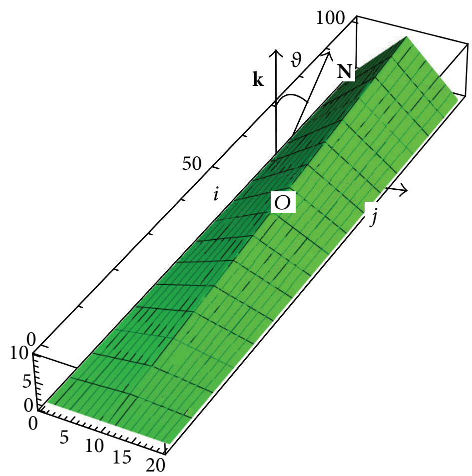

FIGURE 7: A representative graph for Coons patch for $x_i = 10$ that angle ϑ between $\mathbf{N}$ and $\mathbf{k}$ on Coons patch for $x_i = 10$ is smaller.

The root mean square (rms) of beginning curvature given by (7) takes the form:

$$\mu_0 = \left(\int_0^1 \int_0^1 \left(-2alvL(u)L'(u)^2 + alv\left(a^2 + L(u)^2\right)L''(u) \right)^2 du\, dv \right)^{1/2}. \tag{43}$$

The beginning or initial area of the Coons patch given by (8) takes the form in this case

$$A_0 = \int_0^1 \int_0^1 \sqrt{-v^2L(u)^2L'(u)^2 + \left(a^2 + L(u)^2\right)\left(l^2 + v^2L'(u)^2\right)}\, du\, dv. \tag{44}$$

Substituting H_0 from (42) in (32), we have

$$m(u,v) = (1-u)u(1-v)v \times \left(-2alvL(u)L'^2(u) + alv\left(a^2 + L^2(u)\right)L''(u)\right). \tag{45}$$

Variationally improved surface (31) takes following form

$$\mathbf{x}_1(u,v,t) = \Big\{lu, av, vL(u) + t(1-u)u(1-v)v \times \left(-2alvL(u)L'^2(u) + alv\left(a^2 + L^2(u)\right)L''(u)\right)\Big\}. \tag{46}$$

Fundamental magnitudes for this var-iationally improved surface are $E_1(u,v,t)$, $F_1(u,v,t)$, $G_1(u,v,t)$, $e_1(u,v,t)$, $f_1(u,v,t)$, and $g_1(u,v,t)$ included in Appendix A. Inserting these values of fundamental magnitudes in (34) we find the expression for $H_1(u,v,t)$ as

$$\begin{aligned}
H_1(u,v,t) = al\Big(&-2v\Big(L(u) + alt(-1+u)uv(-2+3v) \\
&\quad \times \left(-2L(u)L'^2(u) + a^2L''(u) + L^2(u)L''(u)\right)\Big) \\
&\times \Big(2altv(-2+u(4-6v)+3v)L(u)L'^2(u) \\
&\quad - 2alt(-1+u)uv(-2+3v)L'(u)^3 \\
&\quad + L'(u)\big(1 - 2alt(-1+u) \\
&\qquad \times uv(-2+3v)L(u)L''(u)\big) \\
&\quad + altv(-2+3v)\left(a^2 + L^2(u)\right) \\
&\quad \times \left((-1+2u)L''(u)(-1+u)u\, L^{(3)}(u)\right)\Big) \\
&\times \Big(L'(u) + alt(1-v)v \\
&\quad \times \Big(2(-1+u)uL'(u)^3 \\
&\qquad + 2L(u)L'(u)\left((-1+2u)L'(u) + (-1+u)uL''(u)\right) \\
&\qquad + a^2\left((1-2u)L''(u) - (-1+u)uL^{(3)}(u)\right) \\
&\qquad + L^2(u)\left((1-2u)L''(u) - (-1+u)uL^{(3)}(u)\right)\Big)\Big) \\
&+ 2alt(-1+u)u(-1+3v) \\
&\times \Big(-2L(u)L'^2(u)
\end{aligned}$$

$$
\begin{aligned}
&+a^2L''(u)+L^2(u)L''(u)\Big)\\
&\times\Big(l^2+v^2\Big(L'(u)\\
&\quad+alt(1-v)v\\
&\quad\times\Big(2(-1+u)uL'(u)^3\\
&\quad+2L(u)L'(u)\\
&\quad\times\big((-1+2u)L'(u)\\
&\quad+(-1+u)uL''(u)\big)\\
&\quad+a^2\big((1-2u)L''(u)\\
&\quad-(-1+u)uL^{(3)}(u)\big)\\
&\quad+L^2(u)\\
&\times\ \big((1-2u)L''(u)-(-1+u)uL^{(3)}(u)\big)\Big)\Big)^2\Big)\\
&+v\Big(a^2+\Big(L(u)+alt(-1+u)uv(-2+3v)\\
&\quad\times\big(-2L(u)L'^2(u)\\
&\quad+a^2L''(u)+L^2(u)L''(u)\big)\Big)^2\Big)\\
&\Big(L''(u)+alt(1-v)v\\
&\Big((-4+8u)L'(u)^3\\
&\quad+8(-1+u)uL'^2(u)L''(u)\\
&\quad+2L(u)\Big(2L'^2(u)\\
&\quad+2(-1+2u)L'(u)L''(u)\\
&\quad+(-1+u)uL''^2(u)\Big)\\
&\quad+a^2\Big(-2L''(u)\\
&\quad+(2-4u)L^{(3)}(u)\\
&\quad-(-1+u)uL^{(4)}(u)\Big)\\
&+L^2(u)\Big(-2L''(u)+(2-4u)L^{(3)}(u)\\
&\quad-(-1+u)uL^{(4)}(u)\Big)\Big)\Big)\Big).
\end{aligned}
\tag{47}
$$

Note that $H_1^2(u,v,t)$ is a polynomial in t, with real coefficients. For this purpose we rewrite the previous expression using (35), which reduces to the following form (the coefficients $h_7(u,v)$, $h_8(u,v)$, $h_9(u,v)$, and $h_{10}(u,v)$ of t^7,t^9, t^{10} do not exist as it can be seen from the previous expression for $H_1(u,v,t)$ in which the surviving coefficients are that of t^0, t^1,t^2, and t^3 respectively:

$$
\begin{aligned}
H_1^2(u,v,t)=&\ h_0(u,v)+h_1(u,v)t\\
&+h_2(u,v)t^2+h_3(u,v)t^3\\
&+h_4(u,v)t^4\\
&+h_5(u,v)t^5+h_6(u,v)t^6,
\end{aligned}
\tag{48}
$$

where

$$
\begin{aligned}
h_0(u,v)=&\ \Big(-2alvL(u)L'^2(u)\\
&+a^3lvL''(u)+alvL^2(u)L''(u)\Big)^2,\\
h_1(u,v)=&\ 2\Big(-2alvL(u)L'^2(u)\\
&+a^3lvL''(u)+alvL^2(u)L''(u)\Big)\\
&\Big(-2alvL'^2(u)m_v(u,v)\\
&+2alvL(u)L''(u)m_v(u,v)\\
&+al^3m_{vv}(u,v)\\
&+alv^2L'^2(u)m_{vv}(u,v)\\
&-2alL(u)L'(u)m_u(u,v)\\
&-2alvL(u)L'(u)m_{uv}(u,v)\\
&+a^3lm_{uu}(u,v)+alL^2(u)m_{uu}(u,v)\Big),
\end{aligned}
\tag{49}
$$

Similarly we can find the remaining coefficients $h_2(u,v)$, $h_3(u,v)$, $h_4(u,v)$, $h_5(u,v)$, and $h_6(u,v)$, and they have been included in Appendix B. Thus (36) in this case after performing the integrations mentioned in (36), the mean square curvature $\mu_1^2(t)$ for $\mathbf{x}_1(u,v)$ becomes

$$
\begin{aligned}
\mu_1^2=&\int_0^1\int_0^1 h_0(u,v)du\,dv\\
&+t\int_0^1\int_0^1 h_1(u,v)\,du\,dv\\
&+t^2\int_0^1\int_0^1 h_2(u,v)\,du\,dv+t^3\int_0^1\int_0^1 h_3(u,v)du\,dv\\
&+t^4\int_0^1\int_0^1 h_4(u,v)\,du\,dv+t^5\int_0^1\int_0^1 h_5(u,v)du\,dv\\
&+t^6\int_0^1\int_0^1 h_6(u,v)\,du\,dv,
\end{aligned}
\tag{50}
$$

where the coefficients $h_0(u,v)$ up to $h_6(u,v)$ are given by (49) and (B.1) to (B.5) included in Appendix B. The mean

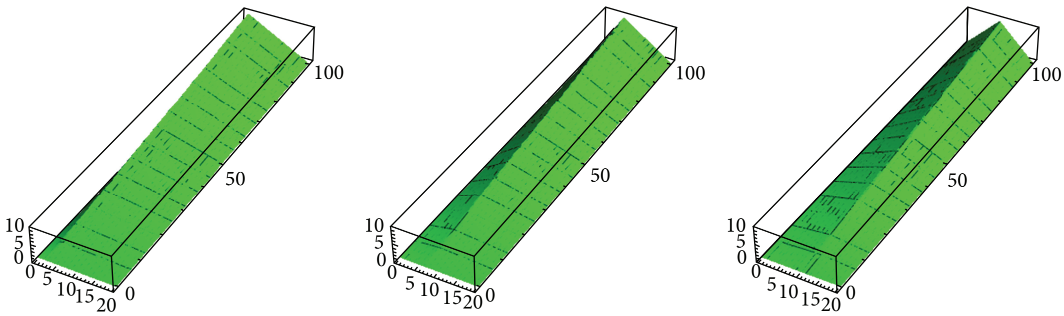

FIGURE 8: Coons patch for $x_i = 2, 6, 10$.

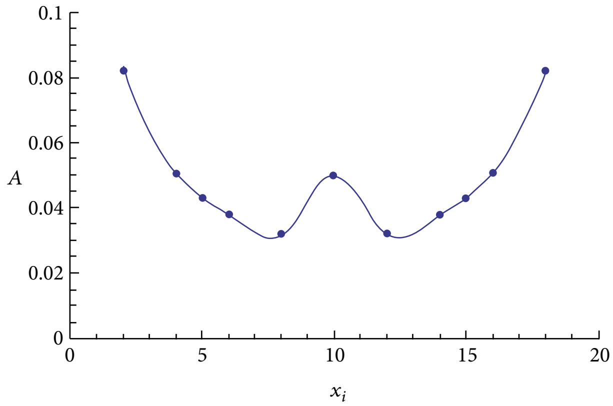

FIGURE 9: Data points and spline curve for percentage decrease in area for $x_i = 2, 4, 5, 6, 8, 10, 12, 14, 16, 18$. Vertical axis A denotes the percentage decrease in area.

square curvature μ_1^2 may be minimized for t for fixed values of a, l, and y_k. For this minimum value $t_{\min}$ we find the variationally improved surface $\mathbf{x}_1(u, v)$ and its area using (33). In order to see a geometrically meaningful (relative) change in area we calculate the dimension less area by dividing the difference of the (original) area of the Coons patch and the variationally decreased area by the original area. In particular for $y_i = 10$, $a = 100$, and $l = 20$, following Table 1 provides the percentage decrease in area of the initial surface Coons patch equation (11) for few cases that includes $x_i = lu_i = 2, 4, 5, 6, 8, 10, 12, 14, 16$, and 18. The Table 1, indicates a symmetric behaviour of decrease in area for example for $0 \leq x_i \leq 20$, as it can be seen that the decreases in area for example for $x_i = 2$ and $x_i = 18$, $x_i = 4$ and $x_i = 16$, $x_i = 6$ and $x_i = 14$, $x_i = 8$ and $x_i = 12$, and so forth, agree up to four decimal places. Table 1 and Figure 8 indicate that the relatively flat region staring from $av = 0$ expands inside the surface as the ratio l/a tends to zero for larger values of a ($a \to \infty$). The large a limit of the five line boundary has importance in the mathematical modeling of a string breaking into two, where the $av = 0$ straight line models the original string breaking into two final strings (straight lines) visible at $av = 100$. With our surface as the corresponding string space-time world sheet of relativity, large length a means a large time evolution of the string ends at $u = 0$ and $u = 1$. Combining this with the usual quantum mechanical exponential dependence of the transition amplitudes on both time and energy [35] and Wick's rotation [36] to imaginary time justified by a Contour integration [37] the transition amplitudes for larger energies get damped away for this large time evolution. Thus, in this limit, the string breaking probabilities become specialized to the physically more interesting problem of the ground states of both the initial and final strings. For the gluonic strings connecting a quark and an antiquark, all five lines can be seen for example in [38]; other problems in any bosonic string theory have the same mathematical structure. (In a related application, for any value of time evolution or a the area of string is proportional to its action, and thus reducing area takes us closer to the nonquantum or classical minimal action. This area reduction is what we have done in the second part of our work). Figure 9 represents the data points of Table 1 along with spline curve giving us percentage decrease in area A as numerical function of x_i that shows the outcome of the ansatz used to calculate the decrease in area for the Coons patch for $0 \leq x_i \leq 20$. The dots give computed values of decrease in area, and the smooth graph passing through these points is the spline curve interpolating these points for better predictability that how the decrease in area in the Coons patch is associated with the range of points $0 \leq x_i \leq 20$. Figure 9 indicates that reduction in area is smaller for the string breaking point x_i generally in the middle, and thus such a string breaking world sheet or symmetrical surface may be closer to being a minimal surface than the asymmetrical surfaces for the x_i point significantly away from the middle where we have larger area reductions.

7. Conclusions

We have developed an algorithm (16) to combine N number of curves with the help of step functions equations (24) to (26). We have discussed a technique to reduce the area of a surface $\mathbf{x}(u, v)$ equation (11) obtaining variationally improved surface $\mathbf{x}_1(u, v)$ of (31). The algorithm is applied to reduce the area of a surface of $\mathbf{x}(u, v)$ equation (11) spanned by five nonplanar boundary lines with the help of algorithm (16), extended boundary Coons patch equation (11), satisfying the conditions (9) and (10) along with the linear blending functions equation (12), for a selection of values as given in the previous table. This gave us a much lesser (in the range 0 to 0.82) dimensionless decrease in less area of surface $\mathbf{x}(u, v)$ of equation (11), as seen in the Figure 9. It is to be noted that our variational technique reduces area by 23 percent for a surface mentioned in [9]. A much lesser decrease in this case suggests that $x(u, v)$ equation (11) is already a near minimal surface. The five-line boundary we have worked out has a variety of applications including the string theory one mentioned in the previous paragraph.

Appendices

A. Fundamental Magnitudes of Variationally Improved Surface

The expressions for $E_1(u, v, t)$, $F_1(u, v, t)$, $G_1(u, v, t)$, $e_1(u, v, t)$, $f_1(u, v, t)$, $g_1(u, v, t)$ are given below:

$$
\begin{aligned}
E_1(u, v, t) = {} & l^2 + \Big(vL'(u) \\
& + t\Big((1-u)(1-v)v \\
& \times\Big(-2alvL(u)L'(u)^2 \\
& + alv\left(a^2 + L(u)^2\right)L''(u)\Big) \\
& - u(1-v)v\Big(-2alvL(u)L'(u)^2 \\
& + alv\left(a^2 + L(u)^2\right) \\
& \times L''(u)\Big) \\
& + (1-u)u(1-v)v \\
& \times\Big(-2alvL'(u)^3 \\
& - 2alvL(u)L'(u)L''(u) \\
& + alv\left(a^2 + L(u)^2\right)L^{(3)}(u)\Big)\Big)\Big)^2, \\
F_1(u, v, t) = {} & \Big(L(u) + t\Big((1-u)u(1-v)v \\
& \times\Big(-2alL(u)L'(u)^2 \\
& + al\left(a^2 + L(u)^2\right)L''(u)\Big) \\
& + (1-u)u(1-v) \\
& \times\Big(-2alvL(u)L'(u)^2 \\
& + alv\left(a^2 + L(u)^2\right)L''(u)\Big) \\
& - (1-u)uv \\
& \times\Big(-2alvL(u)L'(u)^2 \\
& + alv\left(a^2 + L(u)^2\right)L''(u)\Big)\Big)\Big) \\
& \times\Big(vL'(u) + \\
& t\Big((1-u)(1-v)v \\
& \times\Big(-2alvL(u)L'(u)^2 \\
& + alv\left(a^2 + L\left(u\right)^2\right)L''(u)\Big) \\
& - u(1-v)v\Big(-2alvL(u)L'(u)^2 \\
& + alv\left(a^2 + L(u)^2\right)L''(u)\Big) \\
& + (1-u)u(1-v)v \\
& \times\Big(-2alvL'(u)^3 - 2alvL(u)L'(u)L''(u) \\
& + alv\left(a^2 + L(u)^2\right)L^{(3)}(u)\Big)\Big)\Big), \\
G_1(u, v, t) = {} & a^2 + \Big(L(u) + t\Big((1-u)u(1-v)v \\
& \times\Big(-2alL(u)L'(u)^2 \\
& + al\left(a^2 + L(u)^2\right)L''(u)\Big) \\
& + (1-u)u(1-v) \\
& \times\Big(-2alvL(u)L'(u)^2 + alv \\
& \times\left(a^2 + L(u)^2\right)L''(u)\Big) \\
& - (1-u)uv \\
& \times\Big(-2alvL(u)L'(u)^2 + alv \\
& \times\left(a^2 + L(u)^2\right)L''(u)\Big)\Big)\Big)^2, \\
e_1(u, v, t) = {} & al\Big(vL''(u) + t\Big(-2(1-v)v \\
& \times\Big(-2alvL(u)L'(u)^2 \\
& + alv\left(a^2 + L(u)^2\right)L''(u)\Big) \\
& + 2(1-u)(1-v)v \\
& \times\Big(-2alvL'(u)^3 \\
& - 2alvL(u)L'(u)L''(u) \\
& + alv\left(a^2 + L(u)^2\right)L^{(3)}(u)\Big) \\
& - 2u(1-v)v \\
& \times\Big(-2alvL'(u)^3 \\
& - 2alvL(u)L'(u)L''(u)
\end{aligned}
$$

$$
\begin{aligned}
&+alv\left(a^2+L(u)^2\right)L^{(3)}(u)\Big)\\
&+(1-u)u(1-v)v\\
&\times\Big(-8alvL'(u)^2L''(u)\\
&-2alvL(u)L''(u)^2\\
&+alv\left(a^2+L(u)^2\right)L^{(4)}(u)\Big)\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
f_1(u,v,t)={}&alL'(u)-4a^2l^2tvL(u)L'(u)^2\\
&+8a^2l^2tuvL(u)L'(u)^2\\
&+6a^2l^2tv^2L(u)L'(u)^2\\
&-12a^2l^2tuv^2L(u)L'(u)^2\\
&-4a^2l^2tuvL'(u)^3+4a^2l^2tu^2vL'(u)^3\\
&+6a^2l^2tuv^2L'(u)^3\\
&-6a^2l^2tu^2v^2L'(u)^3+2a^4l^2tvL''(u)\\
&-4a^4l^2tuvL''(u)-3a^4l^2tv^2L''(u)\\
&+6a^4l^2tuv^2L''(u)+2a^2l^2tvL(u)^2L''(u)\\
&-4a^2l^2tuvL(u)^2L''(u)\\
&-3a^2l^2tv^2L(u)^2L''(u)\\
&+6a^2l^2tuv^2L(u)^2L''(u)k\\
&-4a^2l^2tuvL(u)L'(u)L''(u)\\
&+4a^2l^2tu^2vL(u)L'(u)L''(u)\\
&+6a^2l^2tuv^2L(u)L'(u)L''(u)\\
&-6a^2l^2tu^2v^2L(u)L'(u)L''(u)\\
&+2a^4l^2tuvL^{(3)}(u)-2a^4l^2tu^2vL^{(3)}(u)\\
&-3a^4l^2tuv^2L^{(3)}(u)\\
&+3a^4l^2tu^2v^2L^{(3)}(u)\\
&+2a^2l^2tuvL(u)^2L^{(3)}(u)\\
&-2a^2l^2tu^2vL(u)^2L^{(3)}(u)\\
&-3a^2l^2tuv^2L(u)^2L^{(3)}(u)\\
&+3a^2l^2tu^2v^2L(u)^2L^{(3)}(u),
\end{aligned}
$$

$$
\begin{aligned}
g_1(u,v,t)={}&alt\Big(2(1-u)u(1-v)\\
&\times\Big(-2alL(u)L'(u)^2\\
&+al\left(a^2+L(u)^2\right)L''(u)\Big)\\
&-2(1-u)uv\\
&\times\Big(-2alL(u)L'(u)^2\\
&+al\left(a^2+L(u)^2\right)L''(u)\Big)\\
&-2(1-u)u\\
&\times\Big(-2alvL(u)L'(u)^2\\
&+alv\left(a^2+L(u)^2\right)L''(u)\Big)\Big).
\end{aligned}
\tag{A.1}
$$

B. Coefficients of the Expansion of $H_1^2(u,v,t)$ in Powers of t

The expressions for the coefficients $h_2(u,v)$, $h_3(u,v)$, $h_4(u,v)$, $h_5(u,v)$, and $h_6(u,v)$ are given below:

$$
\begin{aligned}
h_2(u,v)={}&\Big(-2alvL'^2(u)m_v(u,v)\\
&+2alvL(u)L''(u)m_v(u,v)+al^3m_{vv}(u,v)\\
&+alv^2L'^2(u)m_{vv}(u,v)\\
&-2alL(u)L'(u)m_u(u,v)\\
&-2alvL(u)L'(u)m_{uv}(u,v)\\
&+a^3lm_{uu}(u,v)+alL^2(u)m_{uu}(u,v)\Big)^2\\
&+2\Big(-2alvL(u)L'^2(u)\\
&+a^3lvL''(u)+alvL^2(u)L''(u)\Big)\\
&\times\Big(alvL''(u)m_v^2(u,v)\\
&-2alL'(u)m_v(u,v)m_u(u,v)\\
&+2alvL'(u)m_{vv}(u,v)m_u(u,v)\\
&-2alvL'(u)m_v(u,v)m_{uv}(u,v)\\
&-2alL(u)m_u(u,v)m_{uv}(u,v)\\
&+2alL(u)m_v(u,v)m_{uu}(u,v)\Big),
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
h_3(u,v)={}&2\Big(-2alvL'^2(u)m_v(u,v)\\
&+2alvL(u)L''(u)m_v(u,v)\\
&+al^3m_{vv}(u,v)+alv^2L'^2(u)m_{vv}(u,v)\\
&-2alL(u)L'(u)m_u(u,v)\\
&-2alvL(u)L'(u)m_{uv}(u,v)+a^3lm_{uu}(u,v)\\
&+alL^2(u)m_{uu}(u,v)\Big)\\
&\times\Big(alvL''(u)m_v^2(u,v)\\
&-2alL'(u)m_v(u,v)m_u(u,v)\\
&+2alvL'(u)m_{vv}(u,v)m_u(u,v)\\
&-2alvL'(u)m_v(u,v)m_{uv}(u,v)\\
&-2alL(u)m_u(u,v)m_{uv}(u,v)\\
&+2alL(u)m_v(u,v)m_{uu}(u,v)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+2\left(-2alvL(u)L'^2(u)+a^3lvL''(u)\right.\\
&\quad\left.+alvL^2(u)L''(u)\right)\\
&\quad\times\left(alm_{vv}(u,v)m_u^2(u,v)\right.\\
&\qquad-2alm_v(u,v)m_u(u,v)m_{uv}(u,v)\\
&\qquad\left.+alm_v^2(u,v)m_{uu}(u,v)\right),
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
h_4(u,v)=&\left(alvL''(u)m_v^2(u,v)\right.\\
&-2alL'(u)m_v(u,v)m_u(u,v)\\
&+2alvL'(u)m_{vv}(u,v)m_u(u,v)\\
&-2alvL'(u)m_v(u,v)m_{uv}(u,v)\\
&-2alL(u)m_u(u,v)m_{uv}(u,v)\\
&\left.+2alL(u)m_v(u,v)m_{uu}(u,v)\right)^2\\
&+2\left(-2alvL'^2(u)m_v(u,v)\right.\\
&\quad+2alvL(u)L''(u)m_v(u,v)\\
&\quad+al^3m_{vv}(u,v)+alv^2L'^2(u)m_{vv}(u,v)\\
&\quad-2alL(u)L'(u)m_u(u,v)\\
&\quad-2alvL(u)L'(u)m_{uv}(u,v)\\
&\quad\left.+a^3lm_{uu}(u,v)+alL^2(u)m_{uu}(u,v)\right)\\
&\times\left(alm_{vv}(u,v)m_u^2(u,v)\right.\\
&\quad-2alm_v(u,v)m_u(u,v)m_{uv}(u,v)\\
&\quad\left.+alm_v^2(u,v)m_{uu}(u,v)\right),
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
h_5(u,v)=&\,2\left(alvL''(u)m_v^2(u,v)\right.\\
&-2alL'(u)m_v(u,v)m_u(u,v)\\
&+2alvL'(u)m_{vv}(u,v)m_u(u,v)\\
&-2alvL'(u)m_v(u,v)m_{uv}(u,v)\\
&-2alL(u)m_u(u,v)m_{uv}(u,v)\\
&\left.+2alL(u)m_v(u,v)m_{uu}(u,v)\right)\\
&\times\left(alm_{vv}(u,v)m_u^2(u,v)\right.\\
&\quad-2alm_v(u,v)m_u(u,v)m_{uv}(u,v)\\
&\quad\left.+alm_v^2(u,v)m_{uu}(u,v)\right),
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
h_6(u,v)=&\left(alm_{vv}(u,v)m_u^2(u,v)\right.\\
&-2alm_v(u,v)m_u(u,v)m_{uv}(u,v)\\
&\left.+alm_v^2(u,v)m_{uu}(u,v)\right)^2.
\end{aligned}
\tag{B.5}
$$

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