

## Research Article

# Asymptotic Stability and Exponential Stability of Impulsive Delayed Hopfield Neural Networks

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A criterion for the uniform asymptotic stability of the equilibrium point of impulsive delayed Hopfield neural networks is presented by using Lyapunov functions and linear matrix inequality approach. The criterion is a less restrictive version of a recent result. By means of constructing the extended impulsive Halanay inequality, we also analyze the exponential stability of impulsive delayed Hopfield neural networks. Some new sufficient conditions ensuring exponential stability of the equilibrium point of impulsive delayed Hopfield neural networks are obtained. An example showing the effectiveness of the present criterion is given.

## 1. Introduction

In the last several years, Hopfield neural networks (HNN) have received especially considerable attention due to their extensive applications in solving optimization problem, traveling salesman problem, and many other subjects in recent years [1–9]. In hardware implementation of neural networks, time delays are inevitably present due to the finite switching speeds of the amplifiers. Hence, it is vital to investigate the stability of delayed HNN. Recently, various results for the stability of delayed HNN are obtained via different approaches. In [3], Rakkiyappan and Balasubramaniam studied the exponential stability for fuzzy impulsive neural networks by utilizing the Lyapunov-Krasovskii functional and the linear matrix inequality approach. In [8], Li studied the global robust stability for stochastic interval neural networks with continuously distributed delays of neutral type based on the similar methods. In [9], Xia et al. derived some sufficient conditions for the synchronization problem of coupled identical Yang-Yang type fuzzy cellular neural networks with time-varying delays based on using the invariance principle of functional differential equations.

On the other hand, impulsive differential equations have attracted a great deal of attention due to its potential applications in biological systems, chemical reactions, and various results are obtained; for instance, see [10–14]. Impulses can make unstable systems stable, and stable systems can become unstable after impulse effects. Hence, the stability properties of impulsive HNN with time delays have become an important topic of theoretical studies and have been investigated by many researchers; see [5, 6, 15–22]. In [5], Zhang and Sun obtained a result for the uniform stability of the equilibrium point of the impulsive HNN systems with time delays by using Lyapunov functions and analysis technique. In [6], global exponential stability of impulsive delay HNN is investigated by applying the piecewise continuous vector Lyapunov function.

The purpose of this paper is to present some sufficient conditions for uniform asymptotic stability and global exponential stability of impulsive HNN with time delays by means of constructing the extended impulsive Halanay inequality which is different from that given in [23], Lyapunov functional methods, and linear matrix inequality approach.

The results here are also discussed from the point of view of their comparison with the earlier results. Our results improve and generalize the earlier results. At last, we discuss an example to illustrate the advantage of the results we obtained.

### 2. Systems Description and Preliminaries

Let  $\mathbb{R}$  denote the set of real numbers, let  $\mathbb{R}_+$  denote the set of nonnegative real numbers, and let  $\mathbb{R}^n$  denote the  $n$ -dimensional real space equipped with the Euclidean norm  $\|\cdot\|$ .

Consider the following impulsive and delayed HNN model:

$$\begin{aligned} x'_i(t) &= -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) \\ &\quad + \sum_{j=1}^n b_{ij} g_j(x_j(t - \tau_j(t))) + I_i, \quad t \neq t_k, t \geq t_0, \quad (1) \\ \Delta x_i|_{t=t_k} &= x_i(t_k) - x_i(t_k^-), \quad i \in \Lambda, k = 1, 2, \dots, \end{aligned}$$

where  $\Lambda = \{1, 2, \dots, n\}$ .  $n \geq 2$  corresponds to the number of units in a neural network; the impulse times  $t_k$  satisfy  $0 \leq t_0 < t_1 < \dots < t_k < \dots, \lim_{k \rightarrow +\infty} t_k = +\infty$ ;  $x_i$  corresponds to the membrane potential of the unit  $i$  at time  $t$ ;  $c_i$  is positive constant;  $f_j, g_j$  denote, respectively, the measures of response or activation to its incoming potentials of the unit  $j$  at time  $t$  and  $t - \tau_j(t)$ ; constant  $a_{ij}$  denotes the synaptic connection weight of the unit  $j$  on the unit  $i$  at time  $t$ ; constant  $b_{ij}$  denotes the synaptic connection weight of the unit  $j$  on the unit  $i$  at time  $t - \tau_j(t)$ ;  $I_i$  is the input of the unit  $i$ ;  $\tau_j(t)$  is the transmission delay of the  $j$ th neuron such that  $0 < \tau_j(t) \leq \tau, t \geq t_0$ , and  $j \in \Lambda$ .

Assume that system (1) is supplemented with initial conditions of the form

$$x(s) = \phi(s), \quad s \in [t_0 - \tau, t_0], \quad (2)$$

where  $x(s) = (x_1(s), x_2(s), \dots, x_n(s))^T$ ,  $\phi(s) = (\phi_1(s), \phi_2(s), \dots, \phi_n(s))^T \in PC([t_0 - \tau, t_0], \mathbb{R}^n)$ , and  $PC(I, \mathbb{R}^n) = \{\psi : I \rightarrow \mathbb{R}^n, \text{ which is continuous everywhere except at finite number of points } t_k, \text{ at which } \psi(t_k^+) \text{ and } \psi(t_k^-) \text{ exist and } \psi(t_k^+) = \psi(t_k^-)\}$ . For any given  $t \geq t_0$ ,  $\psi \in PC([t - \tau, t], \mathbb{R}^n)$ , the norm of  $\psi$  is defined by  $\|\psi\|_\tau = \sup_{t-\tau \leq \theta \leq t} |\psi(\theta)|$ . For any  $\sigma \geq 0$ , let  $PC_\delta(\sigma) = \{\psi \in PC([\sigma - \tau, \sigma], \mathbb{R}^n) : \|\psi\| < \delta\}$ .

In this paper, we assume that some conditions are satisfied, so that the equilibrium point of (1) without impulse does exist denoted by  $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ ; see [2, 5]. Impulsive operator is viewed as perturbation of the equilibrium point of system (1) without impulsive effects. We assume that  $\Delta x_i|_{t=t_k} = x_i(t_k) - x_i(t_k^-) = d_k^{(i)}(x_i(t_k^-) - x^*)$ ,  $d_k^{(i)} \in \mathbb{R}$ , and  $i \in \Lambda, k = 1, 2, \dots$

Since  $x^*$  is an equilibrium point of (1), one can derive from (1) that the transformation  $y_i = x_i - x_i^*, i \in \Lambda$ , transforms system (1) into the following system:

$$\begin{aligned} y'_i(t) &= -c_i y_i(t) + \sum_{j=1}^n a_{ij} \Omega_j(y_j(t)) \\ &\quad + \sum_{j=1}^n b_{ij} \Gamma_j(y_j(t - \tau_j(t))), \quad t \neq t_k, t \geq t_0, \quad (3) \\ y_i(t_k) &= (1 + d_k^{(i)}) y_i(t_k^-), \quad i \in \Lambda, k = 1, 2, \dots, \end{aligned}$$

where  $\Omega_j(y_j(t)) = f_j(x_j^* + y_j(t)) - f_j(x_j^*)$ ,  $\Gamma_j(y_j(t)) = g_j(x_j^* + y_j(t - \tau_j(t))) - g_j(x_j^*)$ .

Clearly,  $x^*$  is uniformly asymptotically stable for system (1) if and only if the trivial solution of system (3) is uniformly asymptotically stable. Hence, we only need to prove the stability of the trivial solution of system (3).

*Remark 1.* If  $x_i(t_k) = d_k^{(i)} x_i(t_k^-)$ , then we cannot get  $y_i(t_k) = d_k^{(i)} y_i(t_k^-)$  through the transformation  $y_i = x_i - x_i^*$ . So some of the results [5] are incorrect.

The following notations will be used throughout the paper. The notation  $A^T$  and  $A^{-1}$  means the transpose of and the inverse of a square matrix  $A$ . Let  $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$ ,  $y_\tau = (y_1(t - \tau_1(t)), y_2(t - \tau_2(t)), \dots, y_n(t - \tau_n(t)))^T$ ;  $C = \text{diag}[-c_1, -c_2, \dots, -c_n]^T$ ,  $A = (a_{ij})_{n \times n}$ ,  $B = (b_{ij})_{n \times n}$ ,  $D_k = \text{diag}[1 + d_k^{(1)}, 1 + d_k^{(2)}, \dots, 1 + d_k^{(n)}]^T$ ;  $\Omega(y) = [\Omega_1(y_1), \Omega_2(y_2), \dots, \Omega_n(y_n)]^T$ ,  $\Gamma(y_\tau) = [\Gamma_1(y_1(t - \tau_1(t))), \Gamma_2(y_2(t - \tau_2(t))), \dots, \Gamma_n(y_n(t - \tau_n(t)))]^T$ . Then system (3) with initial condition becomes

$$\begin{aligned} y'(t) &= Cy(t) + A\Omega(y(t)) + B\Gamma(y_\tau), \quad t \neq t_k, t \geq t_0, \\ y(t_k) &= D_k y(t_k^-), \quad k = 1, 2, \dots, \quad (4) \\ y(t_0 + \theta) &= \varphi(\theta), \quad \theta \in [-\tau, 0], \end{aligned}$$

where  $\varphi(\theta) = x(t_0 + \theta) - x^*$ .

We introduce some definitions as follows.

*Definition 2* (see [10]). The function  $V(t, s) : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}_+$  belongs to class  $v_0$  if

- (A<sub>1</sub>)  $V$  is continuous on each of the sets  $[t_{k-1}, t_k) \times \mathbb{R}^n$  and  $\lim_{(t,z) \rightarrow (t_k^-, s)} V(t, z) = V(t_k^-, s)$  exists;
- (A<sub>2</sub>)  $V(t, s)$  is locally Lipschitzian in  $s$  and  $V(t, 0) \equiv 0$ .

*Definition 3* (see [10]). Let  $V \in v_0$ , for any  $(t, s) \in [t_{k-1}, t_k) \times \mathbb{R}^n$ ; the upper right-hand Dini derivative of  $V(t, s)$  along the solution of (4) is defined by

$$\begin{aligned} D^+V(t, \psi(0)) &= \frac{\limsup_{h \rightarrow 0^+} \{V(t+h, y(t, \psi)(t+h)) - V(t, \psi(0))\}}{h}. \quad (5) \end{aligned}$$

*Definition 4* (see [11]). Assume that  $y(t) = y(\sigma, \varphi)(t)$  is the solution of (4) through  $(\sigma, \varphi)$ . Then the zero solution of (4) is said to be

- (1) uniformly stable, if for any  $\sigma \geq t_0$  and  $\varepsilon > 0$ , there exists a  $\delta = \delta(\varepsilon) > 0$  such that  $\varphi \in PC_\delta(\sigma)$  implies that  $\|y(\sigma, \varphi)(t)\| < \varepsilon, t \geq \sigma$ ;
- (2) uniformly asymptotically stable, if it is uniformly stable, and there exists a  $\delta > 0$  such that for any  $\varepsilon > 0, \sigma \geq t_0$ , there is a  $T = T(\varepsilon) > 0$  such that  $\varphi \in PC_\delta(\sigma)$  implies that  $\|y(\sigma, \varphi)(t)\| < \varepsilon, t \geq \sigma + T$ ;
- (3) globally exponentially stable, if for any  $\varphi \in PC([-\tau, 0], \mathbb{R}^n)$ , there exist constants  $M, \mu > 0$  such that

$$\|y(t)\| < M\|\varphi\|_T e^{-\mu(t-\sigma)}, \quad t \geq \sigma. \quad (6)$$

In this paper, we always assume that the following assumption holds:

(H<sub>0</sub>) there exist constants  $M, N > 0$  such that

$$\Omega^T(y)\Omega(y) \leq My^T y, \quad \Gamma^T(y_\tau)\Gamma(y_\tau) \leq Ny_\tau^T y_\tau. \quad (7)$$

In addition, we have the following basic lemmas.

**Lemma 5** (see [24]). For any vectors  $a, b \in \mathbb{R}^n$ , the inequality

$$\pm 2a^T b \leq a^T X a + b^T X^{-1} b \quad (8)$$

holds, in which  $X$  is any  $n \times n$  matrix with  $X > 0$ .

**Lemma 6** (see [25]). Assume that there exist constants  $P, Q > 0$  and  $m(t) \in PC([t_0 - \tau, \infty), \mathbb{R}_+)$  such that

- (i) for  $t = t_k, m(t_k) \leq \gamma_k m(t_k^-), \gamma_k > 0$  are constants and satisfy  $\max_{k \in \mathbb{Z}_+} \{1/\gamma_k, 1\} < P/Q$ ;
- (ii) for  $t \geq t_0, t \neq t_k$ ,

$$D^+ m(t) \leq -Pm(t) + Q\bar{m}(t), \quad (9)$$

where  $\bar{m}(t) = \sup_{t-\tau \leq s \leq t} m(s)$ .

Then for  $t \geq t_0$ ,

$$m(t) \leq \bar{m}(t_0) \left( \prod_{t_0 < t_k \leq t} \gamma_k \right) e^{-\lambda(t-t_0)}, \quad (10)$$

where  $\lambda$  satisfies the following inequality:

$$0 < \lambda \leq P - Q \max_{k \in \mathbb{Z}_+} \left\{ \frac{1}{\gamma_k}, 1 \right\} \cdot e^{\lambda\tau}. \quad (11)$$

### 3. Main Results

In this section, we will establish some theorems which provide sufficient conditions for uniformly asymptotically stable and global exponential stability of system (1).

**Theorem 7.** *The equilibrium point the system (1) is uniformly asymptotically stable, if there exists  $n \times n$  symmetric, and positive definite matrix  $P$  satisfies the following conditions:*

(H<sub>1</sub>)  $\bar{\eta} \doteq \prod_{k=1}^{\infty} \max\{\eta_k, 1\} < \infty$ , where  $\eta_k$  is the largest eigenvalue of  $P^{-1}D_k P D_k$ ;

(H<sub>2</sub>)  $\lambda_3 < (-M - N)/\lambda_1$ , where  $\lambda_1$  is the smallest eigenvalue of  $P$  and  $\lambda_3$  is the largest eigenvalue of  $P^{-1}(CP + PC + PAA^T P + PBB^T P)$ .

*Proof.* First, we will prove that the zero solution of system (4) is uniformly stable. For any  $\varepsilon > 0$ , we may choose a  $\delta > 0$  such that  $\delta \leq \sqrt{(\lambda_1/\bar{\eta}\lambda_2)}\varepsilon$ , where  $\lambda_2$  is the largest eigenvalue of  $P$ . For any  $\sigma \geq t_0, \varphi \in PC_\delta(\sigma)$ , let  $y(t) = y(\sigma, \varphi)(t)$  be a solution of (4) through  $(\sigma, \varphi), \sigma \geq t_0$  (for convenience, that we assume  $\sigma = t_0$ ); then we can prove that  $\|y(t)\| < \varepsilon, t \geq t_0$ .

Consider the following Lyapunov function:  $V(t, y(t)) = y^T(t)Py(t)$ ; then we have

$$\lambda_1 \|y(t)\|^2 \leq V(t, y(t)) \leq \lambda_2 \|y(t)\|^2. \quad (12)$$

By virtue of Lemma 5, we obtain for  $t \in [t_k, t_{k+1}), k = 1, 2, \dots$ ,

$$\begin{aligned} D^+ V(t, y(t)) \Big|_{(4)} &= (y^T(t))' Py(t) + y^T(t) Py'(t) \\ &= (Cy(t) + A\Omega(y(t)) + B\Gamma(y_\tau))^T Py(t) \\ &\quad + y^T(t) P(Cy(t) + A\Omega(y(t)) + B\Gamma(y_\tau)) \\ &= y^T(t) CPy(t) + \Omega^T(y(t)) A^T Py(t) \\ &\quad + \Gamma^T(y_\tau) B^T Py(t) + y^T(t) PCy(t) \\ &\quad + y^T(t) PA\Omega(y(t)) + y^T(t) P B\Gamma(y_\tau) \\ &= y^T(t) (CP + PC) y(t) \\ &\quad + 2\Omega^T(y(t)) A^T Py(t) + 2\Gamma^T(y_\tau) B^T Py(t) \\ &\leq y^T(t) (CP + PC) y(t) \\ &\quad + \Omega^T(y(t)) \Omega(y(t)) + y^T(t) PAA^T Py(t) \\ &\quad + \Gamma^T(y_\tau) \Gamma(y_\tau) + y^T(t) PBB^T Py(t) \\ &\leq y^T(t) (CP + PC + PAA^T P + PBB^T P) y(t) \\ &\quad + \Omega^T(y(t)) \Omega(y(t)) + \Gamma^T(y_\tau) \Gamma(y_\tau) \\ &\leq \lambda_3 y^T(t) Py(t) + My^T(t) y(t) + Ny_\tau^T y_\tau \\ &\leq \lambda_3 y^T(t) Py(t) + M\lambda_1^{-1} y^T(t) Py(t) \\ &\quad + N\lambda_1^{-1} y_\tau^T Py_\tau \\ &\leq [\lambda_3 + M\lambda_1^{-1}] y^T(t) Py(t) + N\lambda_1^{-1} y_\tau^T Py_\tau. \end{aligned} \quad (13)$$

First, it is obvious that for  $t_0 - \tau \leq t \leq t_0$ ,

$$\lambda_1 \|y(t)\|^2 \leq V(t, y(t)) \leq \lambda_2 \delta^2 \leq \bar{\eta}^{-1} \lambda_1 \varepsilon^2. \quad (14)$$

Then we can prove that for  $t \in [t_0, t_1)$ ,

$$V(t, y(t)) \leq \bar{\eta}^{-1} \lambda_1 \varepsilon^2. \quad (15)$$

Suppose that this is not true; then there exists  $\hat{t} \in [t_0, t_1)$  such that  $V(\hat{t}, y(\hat{t})) > \bar{\eta}^{-1} \lambda_1 \varepsilon^2$ .

Set

$$\check{t} = \sup \{t \mid s \in [t_0, t), V(s, y(s)) \leq \bar{\eta}^{-1} \lambda_1 \varepsilon^2\}. \quad (16)$$

It is obvious that  $\check{t} < \hat{t}$ . Then it follows that

$$(I_a) \quad V(t, y(t)) \leq \bar{\eta}^{-1} \lambda_1 \varepsilon^2, \quad t \in [t_0, \check{t});$$

$$(II_a) \quad V(\check{t}, y(\check{t})) = \bar{\eta}^{-1} \lambda_1 \varepsilon^2;$$

(III<sub>a</sub>) for any  $\delta > 0$ , there exists  $t_\delta \in (\check{t}, \check{t} + \delta)$  such that  $V(t_\delta, y(t_\delta)) > \bar{\eta}^{-1} \lambda_1 \varepsilon^2$ .

So

$$V(\check{t}, y(\check{t})) = \bar{\eta}^{-1} \lambda_1 \varepsilon^2 \geq V(t, y_\tau), \quad \check{t} - \tau \leq t \leq \check{t}. \quad (17)$$

In view of condition (H<sub>2</sub>), from (13), we obtain

$$\begin{aligned} D^+ V(\check{t}, y(\check{t})) &\leq [\lambda_3 + M\lambda_1^{-1}] y^T(\check{t}) P y(\check{t}) + N\lambda_1^{-1} y_\tau^T P y_\tau \\ &\leq [\lambda_3 + M\lambda_1^{-1} + N\lambda_1^{-1}] V(\check{t}, y(\check{t})) \\ &< 0, \end{aligned} \quad (18)$$

which is a contradiction with (III<sub>a</sub>). Hence, (15) holds.

Considering

$$\begin{aligned} V(t_1, y(t_1)) &= y^T(t_1) P y(t_1) \\ &= y^T(t_1^-) D_1 P D_1 y(t_1^-) \\ &\leq \eta_1 y^T(t_1^-) P y(t_1^-) \\ &= \eta_1 V(t_1^-, y(t_1^-)) \\ &\leq \eta_1 \bar{\eta}^{-1} \lambda_1 \varepsilon^2 \\ &\leq \max\{\eta_1, 1\} \bar{\eta}^{-1} \lambda_1 \varepsilon^2, \end{aligned} \quad (19)$$

we will prove that for  $t \in [t_1, t_2)$ ,

$$V(t, y(t)) \leq \max\{\eta_1, 1\} \bar{\eta}^{-1} \lambda_1 \varepsilon^2. \quad (20)$$

Suppose that this is false; then we can define

$$\check{t} = \sup \{t \mid s \in [t_1, t), V(s, y(s)) \leq \max\{\eta_1, 1\} \bar{\eta}^{-1} \lambda_1 \varepsilon^2\}. \quad (21)$$

Similarly, we can obtain

$$(I_b) \quad V(t, y(t)) \leq \max\{\eta_1, 1\} \bar{\eta}^{-1} \lambda_1 \varepsilon^2, \quad t \in [t_1, \check{t});$$

$$(II_b) \quad V(\check{t}, y(\check{t})) = \max\{\eta_1, 1\} \bar{\eta}^{-1} \lambda_1 \varepsilon^2;$$

(III<sub>b</sub>) for any  $\delta > 0$ , there exists  $t_\delta \in (\check{t}, \check{t} + \delta)$  such that  $V(t_\delta, y(t_\delta)) > \max\{\eta_1, 1\} \bar{\eta}^{-1} \lambda_1 \varepsilon^2$ .

So

$$\begin{aligned} V(\check{t}, y(\check{t})) &= \max\{\eta_1, 1\} \bar{\eta}^{-1} \lambda_1 \varepsilon^2 \geq V(t, y_\tau), \\ &\check{t} - \tau \leq t \leq \check{t}. \end{aligned} \quad (22)$$

In fact, if  $\check{t} - \tau \geq t_1$ , then it is obvious that inequality (22) holds. If  $\check{t} - \tau < t_1$ , then  $V(t, y_\tau) \leq \bar{\eta}^{-1} \lambda_1 \varepsilon^2 \leq \max\{\eta_1, 1\} \bar{\eta}^{-1} \lambda_1 \varepsilon^2 = V(\check{t}, y(\check{t}))$ . So, inequality (22) still holds.

Considering condition (H<sub>2</sub>), from (13), we obtain

$$\begin{aligned} D^+ V(\check{t}, y(\check{t})) &\leq [\lambda_3 + M\lambda_1^{-1}] y^T(\check{t}) P y(\check{t}) + N\lambda_1^{-1} y_\tau^T P y_\tau \\ &\leq [\lambda_3 + M\lambda_1^{-1} + N\lambda_1^{-1}] V(\check{t}, y(\check{t})) \\ &< 0, \end{aligned} \quad (23)$$

which contradicts (III<sub>b</sub>). Hence, (20) holds.

By induction hypothesis, we may prove, in general, that for  $t \in [t_m, t_{m+1})$ ,

$$V(t, y(t)) \leq \prod_{k=1}^m \max\{\eta_k, 1\} \bar{\eta}^{-1} \lambda_1 \varepsilon^2; \quad (24)$$

that is,

$$V(t, y(t)) \leq \prod_{t_0 < t_k \leq t} \max\{\eta_k, 1\} \bar{\eta}^{-1} \lambda_1 \varepsilon^2. \quad (25)$$

Finally, we arrive at

$$\lambda_1 \|y(t)\|^2 \leq V(t, y(t)) \leq \lambda_1 \varepsilon^2, \quad t \geq t_0. \quad (26)$$

Therefore, we obtain  $\|y(t)\| < \varepsilon, t \geq t_0$ . In view of the choice of  $\delta$ , the zero solution of (4) is uniformly stable; that is, the equilibrium point of (1) is uniformly stable.

Next we show the uniformly asymptotical stability. For any given  $G > 0$ , we find a corresponding  $\delta > 0$  such that for any  $\varphi \in PC_\delta(\sigma)$  implies that  $\|y(t)\| = \|y(t_0, \varphi)(t)\| \leq G, t \geq \sigma = t_0$ ; that is,  $V(t, y(t)) \leq \lambda_2 G^2$ .

For any small  $\varepsilon \in (0, G)$ , we choose  $\vec{N} = \vec{N}(\varepsilon) \in \mathbb{Z}_+$  such that

$$\sqrt{\frac{\vec{N} \lambda_1}{2\lambda_2}} \varepsilon < G \leq \sqrt{\frac{(\vec{N} + 1) \lambda_1}{2\lambda_2}} \varepsilon, \quad (27)$$

$$\vec{N} > \frac{-N}{\lambda_3 \lambda_1 + M + N}. \quad (28)$$

In fact, it is feasible to choose small enough  $\varepsilon \in (0, G)$  such that  $\vec{N}$  in (27) is large enough to satisfy (28).

Since  $\bar{\eta} = \prod_{k=1}^{\infty} \max\{\eta_k, 1\} < \infty$  implies that  $\sum_{k=1}^{\infty} (\max\{\eta_k, 1\} - 1) < \infty$ , there exists sufficient large  $N^* \in \mathbb{Z}_+$  such that

$$\begin{aligned} \sum_{i=N^*}^{\infty} (\max\{\eta_i, 1\} - 1) &\leq \frac{\lambda_1 \varepsilon^2}{6\lambda_2 G^2}, \\ \max\{\eta_k, 1\} &< \frac{1}{6\vec{N}} + 1, \quad k \geq N^*. \end{aligned} \quad (29)$$

Let

$$\omega_1 = \frac{[1 + \sum_{k=1}^{\infty} (\max\{\eta_k, 1\} - 1)] \lambda_2 G^2}{|[\lambda_1 \lambda_3 + M + N] \vec{N} + N|} \cdot \frac{2}{\varepsilon^2} + 1. \quad (30)$$

Next we show that there exists  $T_1 \in [N^*, N^* + w_1]$  such that

$$V(T_1, y(T_1)) < \frac{\lambda_1 \varepsilon^2}{2} \vec{N}. \tag{31}$$

Or else, for all  $t \in [N^*, N^* + w_1]$ ,

$$V(t, y(t)) \geq \frac{\lambda_1 \varepsilon^2}{2} \vec{N}. \tag{32}$$

Thus, we get

$$\begin{aligned} V(t, y(t)) + \frac{\lambda_1 \varepsilon^2}{2} &\geq \frac{\lambda_1 \varepsilon^2}{2} (\vec{N} + 1) \geq \lambda_2 G^2 > V(\xi, y_\tau), \\ t - \tau &\leq \xi \leq t. \end{aligned} \tag{33}$$

From (13), we have

$$\begin{aligned} D^+V(t, y(t))|_{(4)} &\leq [\lambda_3 + M\lambda_1^{-1}] y^T(t) P y(t) \\ &\quad + N\lambda_1^{-1} y_\tau^T P y_\tau \\ &\leq [\lambda_3 + M\lambda_1^{-1}] V(t, y(t)) \\ &\quad + N\lambda_1^{-1} \left( V(t, y(t)) + \frac{\lambda_1 \varepsilon^2}{2} \right) \\ &\leq [\lambda_3 + M\lambda_1^{-1} + N\lambda_1^{-1}] V(t, y(t)) + \frac{\varepsilon^2}{2} N \\ &\leq [\lambda_3 + M\lambda_1^{-1} + N\lambda_1^{-1}] \frac{\lambda_1 \varepsilon^2}{2} \vec{N} + \frac{\varepsilon^2}{2} N \\ &\leq \{[\lambda_3 \lambda_1 + M + N] \vec{N} + N\} \frac{\varepsilon^2}{2}. \end{aligned} \tag{34}$$

Integrating the above inequality from  $N^*$  to  $N^* + w_1$ , we have

$$\begin{aligned} &V(N^* + w_1, y(N^* + w_1)) \\ &\leq V(N^*, y(N^*)) + \{[\lambda_3 \lambda_1 + M + N] \vec{N} + N\} \frac{\varepsilon^2}{2} w_1 \\ &\quad + \sum_{N^* < t_k < N^* + w_1} [V(t_k) - V(t_k^-)] \\ &\leq V(N^*, y(N^*)) + \{[\lambda_3 \lambda_1 + M + N] \vec{N} + N\} \frac{\varepsilon^2}{2} w_1 \\ &\quad + \sum_{N^* < t_k < N^* + w_1} [\max\{\eta_k, 1\} - 1] V(t_k^-) \\ &\leq \lambda_2 G^2 + \{[\lambda_3 \lambda_1 + M + N] \vec{N} + N\} \frac{\varepsilon^2}{2} w_1 \\ &\quad + \sum_{N^* < t_k < N^* + w_1} [\max\{\eta_k, 1\} - 1] \lambda_2 G^2 \end{aligned} \tag{40}$$

$$\begin{aligned} &\leq \{[\lambda_3 \lambda_1 + M + N] \vec{N} + N\} \frac{\varepsilon^2}{2} w_1 \\ &\quad + \left\{ 1 + \sum_{N^* < t_k < N^* + w_1} [\max\{\eta_k, 1\} - 1] \right\} \lambda_2 G^2 \\ &< 0, \end{aligned} \tag{35}$$

which is a contradiction. So (31) holds. We may choose  $T_1 = N^* + w_1$ .

We next claim that for all  $t > T_1$ ,

$$V(t, y(t)) < \left( \frac{\vec{N}}{2} + \frac{1}{4} \right) \lambda_1 \varepsilon^2. \tag{36}$$

Suppose that this is not true; then there exists a  $\tau_2 > T_1$  such that

$$V(\tau_2, y(\tau_2)) \geq \left( \frac{\vec{N}}{2} + \frac{1}{4} \right) \lambda_1 \varepsilon^2, \tag{37}$$

and for  $T_1 < t < \tau_2$ ,

$$V(t, y(t)) < \left( \frac{\vec{N}}{2} + \frac{1}{4} \right) \lambda_1 \varepsilon^2. \tag{38}$$

Suppose that  $T_1 \in [t_m, t_{m+1})$ ,  $m \in \mathbb{Z}_+$ . We claim that  $\tau_2 \geq t_{m+1}$ . Otherwise,  $\tau_2 \in [T_1, t_{m+1})$ . Since (31) holds, it is clear that there exists a  $\tau_1 \in [T_1, \tau_2)$  such that

$$\begin{aligned} V(\tau_1, y(\tau_1)) &= \frac{\vec{N}}{2} \lambda_1 \varepsilon^2, \\ V(\tau_1, y(\tau_1)) &\leq V(t, x(t)) \leq V(\tau_2, x(\tau_2)), \\ \tau_1 &\leq t \leq \tau_2. \end{aligned} \tag{39}$$

Furthermore, we note that

$$V(t, y(t)) + \frac{\lambda_1 \varepsilon^2}{2} \geq \frac{\lambda_1 \varepsilon^2}{2} (\vec{N} + 1) \geq \lambda_2 G^2 > V(\xi, y_\tau),$$

$$t - \tau \leq \xi \leq t. \tag{40}$$

From (13), we have

$$\begin{aligned} D^+V(t, y(t)) &\leq \{[\lambda_3 \lambda_1 + M + N] \vec{N} + N\} \frac{\varepsilon^2}{2} < 0, \\ \tau_1 &\leq t \leq \tau_2, \end{aligned} \tag{41}$$

which implies that

$$V(\tau_2, y(\tau_2)) \leq V(\tau_1, y(\tau_1)). \tag{42}$$

This is a contradiction.

Hence, we obtain  $\tau_2 \geq t_{m+1}$ ; without loss of generality, we may suppose that  $\tau_2 \in [t_{m+q}, t_{m+q+1})$ ,  $q \geq 1$ . Next we first claim that there exists  $\tau_1 > 0$  satisfying  $\tau_2 > \tau_1 > T_1$  such that

$$V(\tau_1, y(\tau_1)) \geq \frac{\vec{N}}{2} \lambda_1 \varepsilon^2. \tag{43}$$

Suppose that this is false; then for all  $t \in (T_1, \tau_2)$ ,

$$V(t, y(t)) < \frac{\vec{N}}{2} \lambda_1 \varepsilon^2, \tag{44}$$

which implies that  $\tau_2 = t_{m+q}$  in view of (37). Consequently, we have

$$\begin{aligned} V(t_{m+q}, y(t_{m+q})) &\geq \left(\frac{\vec{N}}{2} + \frac{1}{4}\right) \lambda_1 \varepsilon^2, \\ V(t_{m+q}^-, y(t_{m+q}^-)) &< \frac{\vec{N}}{2} \lambda_1 \varepsilon^2, \end{aligned} \tag{45}$$

which implies that

$$\begin{aligned} &(\max\{\eta_{m+q}, 1\} - 1) \lambda_2 G^2 \\ &\geq (\max\{\eta_{m+q}, 1\} - 1) V(t_{m+q}^-, y(t_{m+q}^-)) > \frac{1}{4} \lambda_1 \varepsilon^2. \end{aligned} \tag{46}$$

Hence, we get  $\max\{\eta_{m+q}, 1\} > 1 + (\lambda_1 \varepsilon^2 / 4 \lambda_2 G^2)$ , which contradicts (29). So (43) holds.

Therefore, there are two situations  $\tau_2 > t_{m+q}$  and  $\tau_2 = t_{m+q}$ . Next we discuss them, respectively.

(b<sub>1</sub>) If  $\tau_2 > t_{m+q}$ , let

$$\begin{aligned} \bar{\tau}_1 = \inf_{\tau_1 \leq t < \tau_2} \left\{ t \mid s \in [t, \tau_2], \frac{\vec{N}}{2} \lambda_1 \varepsilon^2 \leq V(s, y(s)) \right. \\ \left. < V(\tau_2, y(\tau_2)) \right\}. \end{aligned} \tag{47}$$

We first show that  $\bar{\tau}_1 < t_{m+q}$ . Suppose on the contrary that  $\bar{\tau}_1 \in [t_{m+q}, \tau_2)$ ; then

$$\begin{aligned} V(t, y(t)) + \frac{\lambda_1 \varepsilon^2}{2} &\geq \frac{\lambda_1 \varepsilon^2}{2} (\vec{N} + 1) \\ &\geq \lambda_2 G^2 > V(\xi, y_\tau), \\ t - \tau \leq \xi \leq t, \quad t &\in [\bar{\tau}_1, \tau_2]. \end{aligned} \tag{48}$$

From (13), we have

$$\begin{aligned} D^+V(t, y(t)) &\leq \{[\lambda_3 \lambda_1 + M + N] \vec{N} + N\} \frac{\varepsilon^2}{2} < 0, \\ \bar{\tau}_1 &\leq t \leq \tau_2, \end{aligned} \tag{49}$$

which implies that

$$V(\tau_2, y(\tau_2)) \leq V(\bar{\tau}_1, y(\bar{\tau}_1)); \tag{50}$$

which is a contradiction with the definition of  $\bar{\tau}_1$ . Thus we obtain that  $\bar{\tau}_1 < t_{m+q}$ . Suppose that  $\bar{\tau}_1 \in [t_{m+k}, t_{m+k+1})$ ,  $1 \leq k < q$ .

We also have two cases.

(b<sub>1a</sub>) If  $\bar{\tau}_1$  is not impulsive point, that is to say  $\bar{\tau}_1 > t_{m+k}$ , then considering the definition of  $\bar{\tau}_1$ , we have

$$V(\bar{\tau}_1, y(\bar{\tau}_1)) = \frac{\vec{N}}{2} \lambda_1 \varepsilon^2. \tag{51}$$

By the same argument as the above mentioned, we obtain that (48) still holds.

Hence, from (13), we get

$$\begin{aligned} \left(\frac{\vec{N}}{2} + \frac{1}{4}\right) \lambda_1 \varepsilon^2 &\leq V(\tau_2, y(\tau_2)) \\ &\leq V(\bar{\tau}_1, y(\bar{\tau}_1)) + \sum_{i=m+k+1}^{m+q} [V(t_i) - V(t_i^-)] \\ &\leq \frac{\vec{N}}{2} \lambda_1 \varepsilon^2 + \sum_{i=m+k+1}^{m+q} (\max\{\eta_i, 1\} - 1) V(t_i^-) \\ &\leq \frac{\vec{N}}{2} \lambda_1 \varepsilon^2 + \sum_{i=m+k+1}^{m+q} (\max\{\eta_i, 1\} - 1) \lambda_2 G^2, \end{aligned} \tag{52}$$

which implies that

$$\frac{1}{4} \lambda_1 \varepsilon^2 \leq \sum_{i=m+k+1}^{m+q} (\max\{\eta_i, 1\} - 1) \lambda_2 G^2, \tag{53}$$

which is a contradiction with (29). So  $\bar{\tau}_1$  is some impulsive point.

(b<sub>1b</sub>) If  $\bar{\tau}_1$  is some impulsive point, that is to see  $\bar{\tau}_1 = t_{m+k}$ , then from the definition of  $\bar{\tau}_1$ , it is clear that

$$V(\bar{\tau}_1^-, y(\bar{\tau}_1^-)) = V(t_{m+k}^-, y(t_{m+k}^-)) < \frac{\vec{N}}{2} \lambda_1 \varepsilon^2, \tag{54}$$

which implies that

$$\begin{aligned} V(\bar{\tau}_1, y(\bar{\tau}_1)) &= V(t_{m+k}, y(t_{m+k})) \\ &\leq \eta_{m+k} V(t_{m+k}^-, y(t_{m+k}^-)) \\ &< \eta_{m+k} \frac{\vec{N}}{2} \lambda_1 \varepsilon^2 \\ &\leq \max\{\eta_{m+k}, 1\} \frac{\vec{N}}{2} \lambda_1 \varepsilon^2. \end{aligned} \tag{55}$$

On the other hand, note that inequality (48) still holds; from (13) and (29), we have

$$\begin{aligned} \left(\frac{\vec{N}}{2} + \frac{1}{4}\right) \lambda_1 \varepsilon^2 &\leq V(\tau_2, y(\tau_2)) \\ &\leq V(\bar{\tau}_1, y(\bar{\tau}_1)) \\ &\quad + \sum_{i=m+k+1}^{m+q} [V(t_i) - V(t_i^-)] \\ &< \max\{\eta_{m+k}, 1\} \frac{\vec{N}}{2} \lambda_1 \varepsilon^2 \\ &\quad + \sum_{i=m+k+1}^{m+q} (\max\{\eta_i, 1\} - 1) V(t_i^-), \end{aligned} \tag{56}$$

which implies that

$$\begin{aligned} \frac{1}{4} \lambda_1 \varepsilon^2 &\leq (\max\{\eta_{m+k}, 1\} - 1) \frac{\vec{N}}{2} \lambda_1 \varepsilon^2 \\ &\quad + \sum_{i=m+k+1}^{m+q} (\max\{\eta_i, 1\} - 1) \lambda_2 G^2 \\ &\leq \frac{1}{6\vec{N}} \frac{\vec{N}}{2} \lambda_1 \varepsilon^2 \\ &\quad + \sum_{i=m+k+1}^{m+q} (\max\{\eta_i, 1\} - 1) \lambda_2 G^2. \end{aligned} \tag{57}$$

That means

$$\frac{1}{6} \lambda_1 \varepsilon^2 < \sum_{i=m+k+1}^{m+q} (\max\{\eta_i, 1\} - 1) \lambda_2 G^2, \tag{58}$$

which contradicts (29).

Hence, the first situation  $\tau_2 > t_{m+q}$  is impossible.

(b<sub>2</sub>) If  $\tau_2 = t_{m+q}$ , then by the same arguments as in the proof in (b<sub>1</sub>) and (43), we have

$$V(\tau_2^-, y(\tau_2^-)) > \frac{\vec{N}}{2} \lambda_1 \varepsilon^2. \tag{59}$$

Then let

$$\begin{aligned} \bar{\tau}_1 &= \inf_{\tau_1 \leq t < \tau_2} \left\{ t \mid s \in [t, \tau_2], \frac{\vec{N}}{2} \lambda_1 \varepsilon^2 \right. \\ &\quad \left. \leq V(s, y(s)) < V(\tau_2, y(\tau_2)) \right\}. \end{aligned} \tag{60}$$

The rest of the arguments are omitted. Finally we can find our desirable contradiction. Hence, (36) holds.

With above mentioned, the same arguments as before, if we replace  $T_1$  with  $N^*$ , then there exists a  $T_2 = T_1 + w_1 = N^* + 2w_1$  such that for  $t > T_2$ ,

$$V(t, y(t)) < \frac{\vec{N}}{2} \lambda_1 \varepsilon^2. \tag{61}$$

Let  $T_2$  replace  $T_1$ ; then there exists a  $T_3 = N^* + 3w_1$  such that for  $t > T_3$ ,

$$V(t, y(t)) < \left(\frac{\vec{N}}{2} - \frac{1}{4}\right) \lambda_1 \varepsilon^2. \tag{62}$$

By induction hypothesis, we may prove, in general, that there exists a  $T_{2\vec{N}} = N^* + 2\vec{N}w_1$  such that  $t > T_{2\vec{N}}$ ,

$$V(t, y(t)) < \left(\frac{\vec{N}}{2} - \frac{\vec{N} - 2}{2}\right) \lambda_1 \varepsilon^2 = \lambda_1 \varepsilon^2. \tag{63}$$

Therefore, we obtain that  $\|y(t)\| < \varepsilon, t > N^* + 2Nw_1$ . In view of the choice of  $N^*, \vec{N}$ , and  $w_1$ , the zero solution of (4) is uniformly asymptotically stable; that is, the equilibrium point of (1) is uniformly asymptotically stable. The proof of Theorem 7 is therefore complete.  $\square$

Let  $|1 + d_k^{(i)}| \neq |1 + d_k^{(j)}|, c_i \neq c_j, i \neq j, i, j \in \Lambda, k = 1, 2, \dots$ , in Theorem 7; then we can have the following result.

**Corollary 8.** *The equilibrium point of system (1) is uniformly asymptotically stable, if there exists  $n \times n$  symmetric and positive definite matrix  $P$  satisfying*

- (I<sub>1</sub>)  $\bar{\eta} \doteq \prod_{k=1}^{\infty} \max\{\max_{i \in \Lambda} (1 + d_k^{(i)})^2, 1\} < \infty$ ;
- (I<sub>2</sub>)  $\lambda_3 < (-M - N)/\lambda_1$ , where  $\lambda_1$  is the smallest eigenvalue of  $P$  and  $\lambda_3$  is the largest eigenvalue of  $2C + AA^T P + BB^T P$ .

*Remark 9.* For using the less conservative conditions in Theorem 7, our results obviously improve some results established in the earlier references. In [5], condition  $y(t_k) = Dy(t_k^-)$  holds for all  $k \in \mathbb{Z}_+$ ; here note in our Theorem 7 that we only require that the solutions satisfy the hypothesis ( $H_1$ ) at impulsive points. In addition, our conditions are without requirement of the range of the largest eigenvalues of  $P^{-1}D^T P D$  on  $(0, 1)$ , which are milder than the restrictions in [5].

By utilizing Lemma 6, we will give some sufficient conditions for globally exponential stability of the equilibrium point of system (1).

**Theorem 10.** *Assume that there exists  $n \times n$  symmetric and positive definite matrix  $P$  such that  $\lambda_1 > 0$  is the smallest eigenvalue of  $P$ ,  $\lambda_3$  is the largest eigenvalue of  $P^{-1}(CP + PC + PAA^T P + PBB^T P)$ ,  $\eta_k$  is the largest eigenvalue of  $P^{-1}D_k P D_k$ , and  $\lambda_1, \lambda_3$  and  $\eta_k$  satisfy the following conditions:*

(H<sub>3</sub>) one has

$$\lambda_3 + \frac{M}{\lambda_1} + \frac{N}{\lambda_1} \cdot \max_{k \in \mathbb{Z}_+} \left\{ \frac{1}{\eta_k}, 1 \right\} < 0, \tag{64}$$

(H<sub>4</sub>) there exist constants  $\mathbb{U} (> 0), \delta (\geq 0)$  such that  $\delta < \mu$  and the following inequality

$$\sum_{k=1}^m \ln \eta_k - \delta (t_m - t_0) < \mathbb{U} \quad \forall m \in \mathbb{Z}_+ \text{ holds,} \tag{65}$$

where  $\mu$  satisfies the following inequality:

$$0 < \mu \leq -\lambda_3 - \frac{M}{\lambda_1} - \frac{N}{\lambda_1} \max_{k \in \mathbb{Z}_+} \left\{ \frac{1}{\eta_k}, 1 \right\} \cdot e^{\mu\tau}. \quad (66)$$

Then the equilibrium point of the system (1) is globally exponentially stable, and the exponential convergence rate is equal to  $(\mu - \delta)/2$ .

*Proof.* Let  $y(t) = y(\sigma, \varphi)(t)$  be a solution of (4) through  $(\sigma, \varphi)$ ,  $\sigma \geq t_0$  (for convenience, we assume that  $\sigma = t_0$ ). We next will prove that the zero solution of (4) is globally exponentially stable. Construct the Lyapunov functional as in Theorem 7; that is,  $V(t, y(t)) = y^T(t)Py(t)$ ; then we obtain the following:

- (1)  $\lambda_1 \|y(t)\|^2 \leq V(t, y(t)) \leq \lambda_2 \|y(t)\|^2$ , where  $\lambda_2$  is the largest eigenvalue of  $P$ ;
- (2) for  $t \in [t_k, t_{k+1})$ ,  $k = 1, 2, \dots$ ,

$$D^+V(t, y(t))|_{(4)} \leq [\lambda_3 + M\lambda_1^{-1}] y^T(t)Py(t) + N\lambda_1^{-1} y_\tau^T P y_\tau; \quad (67)$$

- (3)  $V(t_k, y(t_k)) \leq \eta_k V(t_k^-, y(t_k^-))$ .

From (2), we have for  $t \in [t_k, t_{k+1})$ ,  $k = 1, 2, \dots$ ,

$$D^+V(t, y(t))|_{(4)} \leq -PV(t, y(t)) + Q\tilde{V}(t, y(t)), \quad (68)$$

where  $\tilde{V}(t) = \sup_{t-\tau \leq s \leq t} V(s)$ ,  $P = -\lambda_3 - M\lambda_1^{-1}$ , and  $Q = N\lambda_1^{-1}$ .

For any  $t \geq t_0$ , suppose that  $t \in [t_m, t_{m+1})$ ,  $m \geq 0$ . By Lemma 6 and condition  $(H_4)$ , we obtain

$$\begin{aligned} V(t) &\leq \tilde{V}(t_0) \left( \prod_{k=1}^m \eta_k \right) e^{-\mu(t-t_0)} \\ &\leq \tilde{V}(t_0) e^{\cup} \cdot e^{\delta(t_m-t_0)} \cdot e^{-\mu(t-t_0)} \\ &\leq \tilde{V}(t_0) e^{\cup} \cdot e^{\delta(t-t_0)} \cdot e^{-\mu(t-t_0)} \\ &\leq \lambda_2 e^{\cup} \|\varphi\|_\tau^2 e^{-(\mu-\delta)(t-t_0)}, \end{aligned} \quad (69)$$

where  $\mu$  satisfies inequality (66).

Hence, we obtain for any  $t \geq t_0$ ,

$$\|y(t)\| \leq e^{(1/2)\cup} \sqrt{\frac{\lambda_2}{\lambda_1}} \|\varphi\|_\tau e^{-(\mu-\delta)/2(t-t_0)}, \quad t \geq t_0, \quad (70)$$

where  $\mu$  satisfies inequality (66).

Therefore, the zero solution of system (4) is globally exponentially stable; that is, the equilibrium point of system (1) is globally exponentially stable. The proof of Theorem 10 is complete.  $\square$

*Remark 11.* In Theorem 10, if  $\sup_{n \in \mathbb{Z}_+} (\prod_{k=1}^n \eta_k) < \infty$ , then we can choose  $\delta = 0$  in condition  $(H_3)$ .

Let  $\eta_k \in (0, 1]$ ,  $k = 1, 2, \dots$  in Theorem 7; then we can have the following result.

**Corollary 12.** Assume that there exists  $n \times n$  symmetric and positive definite matrix  $P$  such that  $\lambda_1$  is the smallest eigenvalue of  $P$ ,  $\lambda_3$  is the largest eigenvalue of  $P^{-1}(CP + PC + PAA^T P + PBB^T P)$ ,  $\eta_k$  is the largest eigenvalue of  $P^{-1}D_k P D_k$ , and condition

$$\lambda_1 \lambda_3 + M + N < 0 \quad \text{holds.} \quad (71)$$

Then the equilibrium point of the system (1) is globally exponentially stable, and the exponential convergence rate is equal to  $\mu/2$ , where  $\mu$  satisfies the following inequality:

$$0 < \mu \leq -\lambda_3 - \frac{M}{\lambda_1} - \frac{N}{\lambda_1} \max_{k \in \mathbb{Z}_+} \eta_k \cdot e^{\mu\tau}. \quad (72)$$

### 4. Example

In this section, we present a numerical example to illustrate our results.

*Example 1.* We consider Example 1 in [2] as follows:

$$\begin{aligned} u_1'(t) &= -2.5u_1(t) - 0.5f(u_1(t)) \\ &\quad + 0.1f(u_2(t)) - 0.1f(u_1(t-\tau)) \\ &\quad + 0.2f(u_2(t-\tau)) - 1, \\ u_2'(t) &= -2u_1(t) + 0.2f(u_1(t)) \\ &\quad - 0.1f(u_2(t)) + 0.2f(u_1(t-\tau)) \\ &\quad + 0.1f(u_2(t-\tau)) + 4, \end{aligned} \quad (73)$$

with impulses

$$d_k^{(1)} = \sqrt{1 + \frac{1}{2k^2}} - 1, \quad (74)$$

$$d_k^{(2)} = \sqrt{1 + \frac{1}{k^2}} - 1, \quad t_k = k, \quad k \in \mathbb{Z}_+.$$

The delayed feedback matrixes  $A$ ,  $B$ , and  $C$  are

$$\begin{aligned} C &= \begin{pmatrix} -2.5 & 0 \\ 0 & -2 \end{pmatrix}, \quad A = \begin{pmatrix} -0.5 & 0.1 \\ 0.2 & -0.1 \end{pmatrix}, \\ B &= \begin{pmatrix} -0.1 & 0.2 \\ 0.2 & 0.1 \end{pmatrix}. \end{aligned} \quad (75)$$

Since the activation function in [2] is described by  $f = f_i = 0.5(|x + 1| - |x - 1|)$ ,  $i = 1, 2$ , then we have  $M = N = 1$ . We may choose  $P = E$  (unit matrix); then  $\lambda_1 = \lambda_2 = 1$ . Note that

$$\begin{aligned} P^{-1}D_k P D_k &= D_k^2 = \begin{pmatrix} 1 + \frac{1}{2k^2} & 0 \\ 0 & 1 + \frac{1}{k^2} \end{pmatrix}, \\ P^{-1}(CP + PC + PAA^T P + PBB^T P) &= 2C + AA^T + BB^T \end{aligned} \quad (76)$$

$$= \begin{pmatrix} -4.69 & -0.11 \\ -0.11 & -3.9 \end{pmatrix} = \Delta.$$

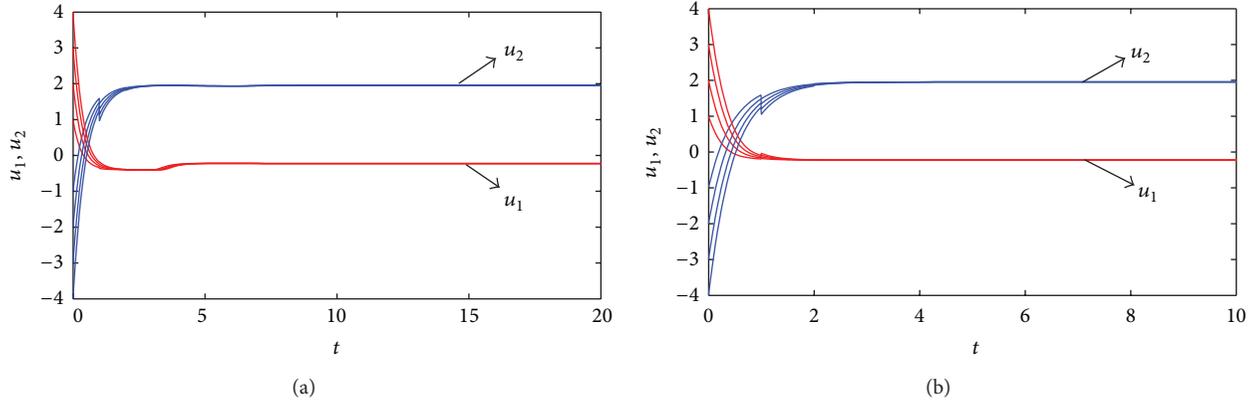


FIGURE 1: (a) State trajectories of system (73) with impulses (74) and  $\tau = 3$ . (b) State trajectories of system (73) with impulses (79) and  $\tau = 0.17$ .

then we get  $\eta_k = 1 + 1/k^2$ ,  $\bar{\eta} = \prod_{k=1}^{\infty} (1 + 1/k^2) < \infty$ . Matrix  $\Delta$ 's characteristic equation is

$$\lambda^2 + 8.59\lambda + 18.2789 = 0. \tag{77}$$

By a straightforward calculation, we obtain that the largest characteristic root  $\lambda_3 \approx -3.885 < -2$ . By Theorem 7, the equilibrium point of system (73) is uniformly asymptotically stable with impulses (74) for any  $\tau > 0$ , which is shown in Figure 1(a).

However, the criteria in [5] are invalid here. In fact, condition  $y(t_k) = Dy(t_k^-)$  is not satisfied here. Moreover, because of the impulsive effect, the criteria in [2] are also invalid here. Therefore, our results are less conservative than those given result in [2, 5].

Furthermore, let  $\tau = 0.17$ . Note that  $\prod_{k=1}^{\infty} (1 + 1/k^2) < \infty$ ,  $\lambda_3 = -3.885 < -2$ ; then one can choose  $\delta = 0$  in Theorem 10. So all conditions of Theorem 10 are satisfied. Therefore, the equilibrium point of system (73) is globally exponentially stable, and the exponential convergence rate is equal to  $\mu$ , where  $\mu > 0$  satisfies

$$\mu + e^{0.17\mu} - 2.885 < 0. \tag{78}$$

If  $d_k^{(1)}, d_k^{(2)}$  in above example are given as follows:

$$d_k^{(1)} = \begin{cases} \sqrt{2.2} - 1, & k = 2n - 1, \\ \sqrt{0.23} - 1, & k = 2n, n \in \mathbb{Z}_+, \end{cases} \tag{79}$$

$$d_k^{(2)} \begin{cases} \sqrt{1.7} - 1, & k = 2n - 1, \\ \sqrt{0.4} - 1, & k = 2n, n \in \mathbb{Z}_+, \end{cases}$$

then we finally get  $\lambda_3 = -3.885 < -3.500, \delta = 0$ . So all conditions of Theorem 10 are still satisfied. Therefore, the equilibrium point of system (73) is globally exponentially stable with impulses (79), and the exponential convergence rate is equal to  $\mu$ , where  $\mu > 0$  satisfies

$$\mu + 2.5e^{0.17\mu} - 2.885 < 0. \tag{80}$$

The simulation is shown in Figure 1(b). However, it is easy to check that the impulsive delayed Halanay inequality in [23, 26] is not feasible here. Thus our results can be applied to the case not covered in [23, 26].

### 5. Conclusion

The uniform asymptotic stability and global exponential stability of impulsive HNN with time delays are considered in this paper. Some new stability conditions are obtained by means of constructing the extended impulsive Halanay inequality, Lyapunov functional methods, and linear matrix inequality approach. Moreover, our results can be applied to the case not covered in some other existing criteria. Hence, the results extend and improve the earlier publications. An example is given to illustrate the feasibility of the results and the effects of impulses.

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