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Research Article

New Braided T-Categories over Weak Crossed Hopf Group Coalgebras

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Let H be a weak crossed Hopf group coalgebra over group π; we first introduce a kind of new α-Yetter-Drinfel'd module categories $\mathscr{WYD}_\alpha(H)$ for $\alpha \in \pi$ and use it to construct a braided T-category $\mathscr{WYD}(H)$. As an application, we give the concept of a Long dimodule category ${}_H\mathscr{WL}^H$ for a weak crossed Hopf group coalgebra H with quasitriangular and coquasitriangular structures and obtain that ${}_H\mathscr{WL}^H$ is a braided T-category by translating it into a weak Yetter-Drinfel'd module subcategory $\mathscr{WYD}(H \otimes H)$.

1. Introduction

Braided crossed categories over a group π (i.e., braided T-categories), introduced by Turaev [1] in the study of 3-dimensional homotopy quantum field theories, are braided monoidal categories in Freyd-Yetter categories of crossed π-sets [2]. Such categories play an important role in the construction of homotopy invariants. By using braided T-categories, Virelizier [3, 4] constructed Hennings-type invariants of flat group bundles over complements of links in the 3-sphere. Braided T-categories also provide suitable mathematical formalism to describe the orbifold models of rational conformal field theory (see [5]).

The methods of constructing braided T-categories can be found in [5–8]. Especially, in [8], Zunino gave the definition of α-Yetter-Drinfel'd modules over Hopf group coalgebras and constructed a braided T-category, then proved that both the category of Yetter-Drinfel'd modules $\mathscr{YD}(H)$ and the center of the category of representations of H as well as the category of representations of the quantum double of H are isomorphic as braided T-categories. Furthermore, in [6], Wang considered the dual setting of Zunino's partial results, formed the category of Long dimodules over Hopf group algebras, and proved that the category is a braided T-subcategory of Yetter-Drinfel'd category $\mathscr{YD}(H \otimes B)$.

Weak multiplier Hopf algebras, as a further development of the notion of the well-known multiplier Hopf algebras [9], were introduced by Van Daele and Wang [10]. Examples of such weak multiplier Hopf algebras can be constructed from weak Hopf group coalgebras [10, 11]. Furthermore, the concepts of weak Hopf group coalgebras are also regard as a natural generalization of weak Hopf algebras [12, 13] and Hopf group coalgebras [14].

In this paper, we mainly generalize the above constructions shown in [6, 8], replacing their Hopf group coalgebras (or Hopf group algebras) by weak crossed Hopf group coalgebras [11] and provide new examples of braided T-categories.

This paper is organized as follows. In Section 1, we recall definitions and properties related to braided T-categories and weak crossed Hopf group coalgebras.

In Section 2, let H be a weak crossed Hopf group coalgebra over group π; α is a fixed element in π. We first introduce the concept of a (left-right) weak α-Yetter-Drinfel'd module and define the category $\mathscr{WYD}(H) = \coprod_{\alpha\in\pi}\mathscr{WYD}_\alpha(H)$, where $\mathscr{WYD}_\alpha(H)$ is the category of (left-right) weak α-Yetter-Drinfel'd modules. Then, we show that the category $\mathscr{WYD}(H)$ is a braided T-category.

In Section 3, we introduce a (left-right) weak α-Long dimodule category ${}_H\mathscr{WL}^H_\alpha$ for a weak crossed Hopf group coalgebra H. Then, we obtain a new category ${}_H\mathscr{W}\mathscr{L}^H = \coprod_{\alpha\in\pi} {}_H\mathscr{W}\mathscr{L}_\alpha^H$ and show that as H is a quasitriangular and coquasitriangular weak crossed Hopf group coalgebra, then ${}_H\mathscr{W}\mathscr{L}^H$ is a braided T-subcategory of Yetter-Drinfel'd category $\mathscr{W}\mathscr{Y}\mathscr{D}(H\otimes H)$.

2. Preliminary

Throughout the paper, let π be a group with the unit 1 and let k be a field. All algebras, vector spaces, and so forth are supposed to be over k. We use the Sweedler-type notation [15] for the comultiplication and coaction, t for the flip map, and id for the identity map. In the section, we will recall some basic definitions and results related to our paper.

2.1. Weak Crossed Hopf Group Coalgebras. Recall from Turaev and Virelizier (see [1, 14]) that a group coalgebra over π is a family of k-spaces $C=\{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta=\{\Delta_{\alpha,\beta}: C_{\alpha\beta}\to C_\alpha\otimes C_\beta\}_{\alpha,\beta\in\pi}$ (called a comultiplication) and a k-linear map $\varepsilon: C_1\to k$ (called a counit), such that Δ is coassociative in the sense that

$$\left(\Delta_{\alpha,\beta}\otimes \mathrm{id}_{C_\gamma}\right)\Delta_{\alpha\beta,\gamma}=\left(\mathrm{id}_{C_\alpha}\otimes\Delta_{\beta,\gamma}\right)\Delta_{\alpha,\beta\gamma},\quad \forall\alpha,\beta,\gamma\in\pi.$$
$$\left(\mathrm{id}_{C_\alpha}\otimes\varepsilon\right)\Delta_{\alpha,1}=\mathrm{id}_{C_\alpha}=\left(\varepsilon\otimes\mathrm{id}_{C_\alpha}\right)\Delta_{1,\alpha},\quad\forall\alpha\in\pi. \tag{1}$$

We use the Sweedler-type notation (see [14]) for a comultiplication; that is, we write

$$\Delta_{\alpha,\beta}(c)=c_{(1,\alpha)}\otimes c_{(2,\beta)},\quad \text{for any } \alpha,\beta\in\pi,\ c\in C_{\alpha\beta}. \tag{2}$$

Recall from Van Daele and Wang (see [11]) that a weak semi-Hopf group coalgebra $H=\{H_\alpha,m_\alpha,1_\alpha,\Delta,\varepsilon\}_{\alpha\in\pi}$ is a family of algebras $\{H_\alpha,m_\alpha,1_\alpha\}_{\alpha\in\pi}$ and at the same time a group coalgebra $\{H_\alpha,\Delta=\{\Delta_{\alpha,\beta}\},\varepsilon\}_{\alpha,\beta\in\pi}$, such that the following conditions hold.

(i) The comultiplication $\Delta_{\alpha,\beta}: C_{\alpha\beta}\to C_\alpha\otimes C_\beta$ is a homomorphism of algebras (not necessary unit preserving) such that

$$\begin{aligned}&\left(\Delta_{\alpha,\beta}\otimes\mathrm{id}_{H_\gamma}\right)\Delta_{\alpha\beta,\gamma}\left(1_{\alpha\beta\gamma}\right)\\ &\quad=\left(\Delta_{\alpha,\beta}\left(1_{\alpha\beta}\right)\otimes 1_\gamma\right)\left(1_\alpha\otimes\Delta_{\beta,\gamma}\left(1_{\beta\gamma}\right)\right),\\ &\left(\Delta_{\alpha,\beta}\otimes\mathrm{id}_{H_\gamma}\right)\Delta_{\alpha\beta,\gamma}\left(1_{\alpha\beta\gamma}\right)\\ &\quad=\left(1_\alpha\otimes\Delta_{\beta,\gamma}\left(1_{\beta\gamma}\right)\right)\left(\Delta_{\alpha,\beta}\left(1_{\alpha\beta}\right)\otimes 1_\gamma\right),\end{aligned} \tag{3}$$

for all $\alpha,\beta,\gamma\in\pi$.

(ii) The counit $\varepsilon: H_1\to k$ is a k-linear map satisfying the identity

$$\varepsilon(gxh)=\varepsilon\left(gx_{(2,1)}\right)\varepsilon\left(x_{(1,1)}h\right)=\varepsilon\left(gx_{(1,1)}\right)\varepsilon\left(x_{(2,1)}h\right), \tag{4}$$

for all $g,h,x\in H_1$.

A weak Hopf group coalgebra over π is a weak semi-Hopf group coalgebra $H=\{H_\alpha,m_\alpha,1_\alpha,\Delta,\varepsilon\}_{\alpha\in\pi}$ endowed with a family of k-linear maps $S=\{S_\alpha: H_\alpha\to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ (called an antipode) satisfying the following equations:

$$\begin{aligned}m_\alpha\left(S_{\alpha^{-1}}\otimes\mathrm{id}_{H_\alpha}\right)\Delta_{\alpha^{-1},\alpha}(h)&=1_{(1,\alpha)}\varepsilon\left(h1_{(2,1)}\right),\\ m_\alpha\left(\mathrm{id}_{H_\alpha}\otimes S_{\alpha^{-1}}\right)\Delta_{\alpha,\alpha^{-1}}(h)&=\varepsilon\left(1_{(1,1)}h\right)1_{(2,\alpha)},\\ S_\alpha\left(g_{(1,\alpha)}\right)g_{(2,\alpha^{-1})}S_\alpha\left(g_{(3,\alpha)}\right)&=S_\alpha(g),\end{aligned} \tag{5}$$

for all $h\in H_1$, $g\in H_\alpha$, and $\alpha\in\pi$.

Let H be a weak Hopf group coalgebra. Define a family of linear maps $\varepsilon_t=\{\varepsilon_\alpha^t: H_1\to H_\alpha\}_{\alpha\in\pi}$ and $\varepsilon_s=\{\varepsilon_\alpha^s: H_1\to H_\alpha\}_{\alpha\in\pi}$ by the formulae

$$\begin{aligned}\varepsilon_\alpha^t(h)&=\varepsilon\left(1_{(1,1)}h\right)1_{(2,\alpha)}=m_\alpha\left(\mathrm{id}_{H_\alpha}\otimes S_{\alpha^{-1}}\right)\Delta_{\alpha,\alpha^{-1}}(h),\\ \varepsilon_\alpha^s(h)&=1_{(1,\alpha)}\varepsilon\left(h1_{(2,1)}\right)=m_\alpha\left(S_{\alpha^{-1}}\otimes\mathrm{id}_{H_\alpha}\right)\Delta_{\alpha^{-1},\alpha}(h),\end{aligned} \tag{6}$$

for any $h\in H_1$, where ε^t and ε^s are called the π-target and π-source counital maps.

By Van Daele and Wang (see [11]), let H be a weak semi-Hopf group coalgebra. Then, we have the following equations:

(1) $\varepsilon(gh)=\varepsilon(g\varepsilon_1^t(h))$, $\varepsilon(gh)=\varepsilon(\varepsilon_1^s(g)h)$, for all $g,h\in H_1$,

(2) $x_{(1,\alpha)}\otimes\varepsilon_\beta^t(x_{(2,1)})=1_{(1,\alpha)}x\otimes 1_{(2,\beta)}$, for all $x\in H_\alpha$, $\alpha,\beta\in\pi$,

(3) $\varepsilon_\beta^s(x_{(1,1)})\otimes x_{(2,\alpha)}=1_{(1,\beta)}\otimes x1_{(2,\alpha)}$, for all $x\in H_\alpha$, $\alpha,\beta\in\pi$,

(4) $\varepsilon_\alpha^t(\varepsilon_1^t(x)y)=\varepsilon_\alpha^t(x)\varepsilon_\alpha^t(y)$, $\varepsilon_\alpha^s(x\varepsilon_1^s(y))=\varepsilon_\alpha^s(x)\varepsilon_\alpha^s(y)$, for all $x,y\in H_1$.

Similarly, for any $\alpha\in\pi$ and $h\in H_1$, define $\overline{\varepsilon}_\alpha^t(h)=\varepsilon(h1_{(1,1)})1_{(2,\alpha)}$, $\overline{\varepsilon}_\alpha^s(h)=1_{(1,\alpha)}\varepsilon(1_{(2,1)}h)$. Then, we have

(1) $\overline{\varepsilon}_\alpha^s(h_{(1,1)})\otimes h_{(2,\beta)}=1_{(1,a)}\otimes 1_{(2,\beta)}h$, for all $h\in H_\beta$, $\alpha,\beta\in\pi$,

(2) $x_{(1,\alpha)}\otimes\overline{\varepsilon}_\alpha^t(x_{(2,1)})=x1_{(1,\alpha)}\otimes 1_{(2,\beta)}$, for all $x\in H_\alpha$, $\alpha,\beta\in\pi$.

A weak Hopf group coalgebra $H=\{H_\alpha,m_\alpha,1_\alpha,\Delta,\varepsilon,S\}_{\alpha\in\pi}$ is called a weak crossed Hopf group coalgebra if it is endowed with a family of algebra isomorphisms $\varphi=\{\varphi_\alpha: H_\beta\to H_{\alpha\beta\alpha^{-1}}\}_{\alpha,\beta\in\pi}$ (called a crossing) such that $(\varphi_\alpha\otimes\varphi_\alpha)\circ\Delta_{\beta,\gamma}=\Delta_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}}\circ\varphi_\alpha$, $\varepsilon\circ\varphi_\alpha=\varepsilon$, and $\varphi_{\alpha\beta}=\varphi_\alpha\circ\varphi_\beta$ for all $\alpha,\beta,\gamma\in\pi$.

If H is crossed with the crossing $\varphi=\{\varphi_\alpha\}_{\alpha\in\pi}$, then we have

$$\begin{aligned}\varphi_\beta\circ\varepsilon_\alpha^s&=\varepsilon_{\beta\alpha\beta^{-1}}^s\circ\varphi_\beta,\ \varphi_\beta\circ\varepsilon_\alpha^t\\ &=\varepsilon_{\beta\alpha\beta^{-1}}^t\circ\varphi_\beta,\quad\forall\alpha,\beta\in\pi.\end{aligned} \tag{7}$$

A quasitriangular weak crossed Hopf group coalgebra over π is a pair (H,R) where H is a weak crossed Hopf group coalgebra together with a family of maps $R=\{R_{\alpha,\beta}\in\overline{\Delta}^{\mathrm{cop}}_{\beta^{-1},\alpha^{-1}}(1_{\alpha\beta})(H_\alpha\otimes H_\beta)\Delta_{\alpha,\beta}(1_{\alpha\beta})\}$ satisfying the following conditions:

(1) $R_{\alpha,\beta}\Delta_{\alpha,\beta}(h) = \overline{\Delta}^{\mathrm{cop}}_{\beta^{-1},\alpha^{-1}}(h)R_{\alpha,\beta}$, for all $h \in H_{\alpha\beta}, \alpha, \beta \in \pi$,

(2) $(\mathrm{id}_{H_\alpha} \otimes \Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}) = (R_{\alpha,\gamma})_{1\beta 3}(R_{\alpha,\beta})_{12\gamma}$, for all $\alpha, \beta, \gamma \in \pi$,

(3) $(\overline{\Delta}_{\alpha,\beta}\otimes \mathrm{id}_{H_\gamma})\ (R_{\beta^{-1}\alpha^{-1},\gamma}) = (R_{\alpha^{-1},\gamma})_{1\beta^{-1}3}(R_{\beta^{-1},\gamma})_{\alpha^{-1}23}$, for all $\alpha, \beta, \gamma \in \pi$,

where $\overline{\Delta}_{\alpha,\beta} = (\varphi_\beta \otimes \mathrm{id}_{H_{\beta^{-1}}})\circ\Delta_{\beta^{-1}\alpha^{-1}\beta,\beta^{-1}}$, $\overline{\Delta}^{\mathrm{cop}}_{\alpha,\beta} = t_{H_{\alpha^{-1}},H_{\beta^{-1}}}\circ(\varphi_\beta \otimes \mathrm{id}_{H_{\beta^{-1}}})\circ\Delta_{\beta^{-1}\alpha^{-1}\beta,\beta^{-1}}$ for all $\alpha, \beta \in \pi$, and such that there exists a family of $\overline{R} = \{\overline{R}_{\alpha,\beta} \in \Delta_{\alpha,\beta}(1_{\alpha\beta})(H_\alpha \otimes H_\beta)\overline{\Delta}^{\mathrm{cop}}_{\beta^{-1},\alpha^{-1}}(1_{\alpha\beta})\}$ with

$$R_{\alpha,\beta}\overline{R}_{\alpha,\beta} = \Delta^{\mathrm{cop}}_{\beta^{-1},\alpha^{-1}}\left(1_{\alpha\beta}\right), \qquad \overline{R}_{\alpha,\beta}R_{\alpha,\beta} = \Delta_{\alpha,\beta}\left(1_{\alpha\beta}\right),$$
$$\left(\varphi_\beta \otimes \varphi_\beta\right)\left(R_{\alpha,\gamma}\right) = R_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}, \tag{8}$$

for all $\alpha, \beta, \gamma \in \pi$. In this paper, we denote $R_{\alpha,\beta} = a_\alpha \otimes b_\beta$.

Recall from [16] that a coquasitriangular weak Hopf group coalgebra (H, σ) consists of a weak Hopf group coalgebra $H = \{H_\alpha, m_\alpha, 1_\alpha, \Delta, \varepsilon, S\}_{\alpha\in\pi}$ and a map $\sigma : H_1 \otimes H_1 \to k$ satisfying

$$\sigma\left(h_{(1,1)}, g_{(1,1)}\right) h_{(2,\alpha)} g_{(2,\alpha)} = g_{(1,\alpha)} h_{(1,\alpha)} \sigma\left(h_{(2,1)}, g_{(2,1)}\right),$$
$$\sigma\left(a, bc\right) = \sigma\left(a_{(1,1)}, c\right) \sigma\left(a_{(2,1)}, b\right),$$
$$\sigma\left(ab, c\right) = \sigma\left(a, c_{(1,1)}\right) \sigma\left(b, c_{(2,1)}\right), \tag{9}$$
$$\varepsilon\left(a_{(1,1)} b_{(1,1)}\right) \sigma\left(b_{(2,1)}, a_{(2,1)}\right) \varepsilon\left(b_{(3,1)} a_{(3,1)}\right) = \sigma\left(b, a\right),$$

and there exists $\sigma^{-1} : H_1 \otimes H_1 \to k$ such that

$$\sigma\left(a_{(1,1)}, b_{(1,1)}\right) \sigma^{-1}\left(a_{(2,1)}, b_{(2,1)}\right) = \varepsilon\left(ba\right),$$
$$\sigma^{-1}\left(a_{(1,1)}, b_{(1,1)}\right) \sigma\left(a_{(2,1)}, b_{(2,1)}\right) = \varepsilon\left(ab\right),$$
$$\varepsilon\left(a_{(1,1)} b_{(1,1)}\right) \sigma^{-1}\left(a_{(2,1)}, b_{(2,1)}\right) \varepsilon\left(b_{(3,1)} a_{(3,1)}\right) = \sigma^{-1}\left(a, b\right), \tag{10}$$

for all $h, g \in H_\alpha$, $a, b, c \in H_1$, where σ^{-1} is called a weak inverse of σ.

2.2. Braided T-Categories. We recall that a monoidal category $\mathscr{C}$ is called a crossed category over group π if it consists of the following data.

(1) A family of subcategories $\{\mathscr{C}_\alpha\}_{\alpha\in\pi}$ such that $\mathscr{C}$ is a disjoint union of this family and such that for any $U \in \mathscr{C}_\alpha$ and $V \in \mathscr{C}_\beta$, $U \otimes V \in \mathscr{C}_{\alpha\beta}$. Here, the subcategory $\mathscr{C}_\alpha$ is called the αth component of $\mathscr{C}$.

(2) A group homomorphism $\psi : \pi \to \mathrm{aut}(\mathscr{C}) : \beta \mapsto \psi_\beta$, the conjugation, (where $\mathrm{aut}(\mathscr{C})$ is the group of invertible strict tensor functors from $\mathscr{C}$ to itself) such that $\psi_\beta(\mathscr{C}_\alpha) = \mathscr{C}_{\beta\alpha\beta^{-1}}$ for any $\alpha, \beta \in \pi$. Here, the functors ψ_β are called conjugation isomorphisms.

We will use the Turaev's left index notation in [1]: for any object $U \in \mathscr{C}_\alpha$, $V, W \in \mathscr{C}_\beta$ and any morphism $f : V \to W$ in $\mathscr{C}_\beta$, we set

$${}^{U}V = \psi_\alpha\left(V\right) \in \mathscr{C}_{\alpha\beta\alpha^{-1}}, \qquad {}^{U}f = \psi_\alpha\left(f\right) : {}^{U}V \longrightarrow {}^{U}W. \tag{11}$$

Recall form [1] that a braided T-category is a crossed category $\mathscr{C}$ endowed with braiding, that is a family of isomorphisms,

$$c = \left\{c_{U,V} \in \mathscr{C}\left(U \otimes V, \left({}^{U}V\right) \otimes U\right)\right\}_{U,V\in\mathscr{C}} \tag{12}$$

satisfying the following conditions:

(1) for any morphism $f \in \mathscr{C}_\alpha(U, U')$ with $\alpha \in \pi$, $g \in \mathscr{C}(V, V')$, we have

$$\left(\left({}^{\alpha}g\right) \otimes f\right) \circ c_{U,V} = c_{U',V'} \circ \left(f \otimes g\right); \tag{13}$$

(2) for all $U, V, W \in \mathscr{C}$, we have

$$c_{U\otimes V,W} = a_{{}^{U\otimes V}W,U,V} \circ \left(c_{U,{}^{V}W} \otimes \mathrm{id}_V\right) \circ a^{-1}_{U,{}^{V}W,V} \circ \left(\mathrm{id}_U \otimes c_{V,W}\right) \circ a_{U,V,W},$$
$$c_{U,V\otimes W} = a^{-1}_{{}^{U}V,{}^{U}W,U} \circ \left(\mathrm{id}_{{}^{U}V} \otimes c_{U,W}\right) \circ a_{{}^{U}V,U,W} \circ \left(c_{U,V} \otimes \mathrm{id}_W\right) \circ a^{-1}_{U,V,W}; \tag{14}$$

(3) for any $U, V \in \mathscr{C}$, $\alpha \in \pi$,

$$\psi_\alpha\left(c_{U,V}\right) = c_{\psi_\alpha(U),\psi_\alpha(V)}. \tag{15}$$

3. Yetter-Drinfel'd Categories for Weak Crossed Hopf Group Coalgebras

In this section, we first introduce the definition of weak α-Yetter-Drinfel'd modules over a weak crossed Hopf group coalgebra H and then use it to construct a class of braided T-categories.

Definition 1. Let H be a weak crossed Hopf group coalgebra over group π and let α be a fixed element in π. A (left-right) weak α-Yetter-Drinfel'd module, or simply a $\mathscr{WYD}_\alpha$-module, is a couple $V = (V, \rho^V = \{\rho^V_\lambda\}_{\lambda\in\pi})$, where V is a left H_α-module and, for any $\lambda \in \pi$, $\rho^V_\lambda : V \to V \otimes H_\lambda$ is a k-linear morphism, such that

(1) V is coassociative in the sense that, for any $\lambda_1, \lambda_2 \in \pi$, we have

$$\left(\mathrm{id}_V \otimes \Delta_{\lambda_1,\lambda_2}\right) \circ \rho^V_{\lambda_1\lambda_2} = \left(\rho^V_{\lambda_1} \otimes \mathrm{id}_{H_{\lambda_2}}\right) \circ \rho^V_{\lambda_2}; \tag{16}$$

(2) V is counitary in the sense that

$$\left(\mathrm{id}_V \otimes \varepsilon\right) \circ \rho^V_1 = \mathrm{id}_V; \tag{17}$$

(3) V is crossed in the sense that, for any $\lambda \in \pi$, $h \in H_\alpha$,

$$\rho^V_\lambda\left(h \cdot v\right) = h_{(2,\alpha)} \cdot v_{(0)} \otimes h_{(3,\lambda)} v_{(1,\lambda)} S^{-1} \varphi_{\alpha^{-1}}\left(h_{(1,\alpha\lambda^{-1}\alpha^{-1})}\right), \tag{18}$$

where $\rho^V_\lambda(v) = v_{(0)} \otimes v_{(1,\lambda)}$.

Given two $\mathscr{WYD}_\alpha$-modules (V, ρ^V) and (W, ρ^W), a morphism $f : (V, \rho^V) \to (W, \rho^W)$ of this two $\mathscr{WYD}_\alpha$-modules is an H_α-linear map $f : V \rightharpoonup W$, such that, for any $\lambda \in \pi$,

$$\rho^W_\lambda \circ f = \left(f \otimes \mathrm{id}_{H_\lambda}\right) \circ \rho^V_\lambda. \tag{19}$$

Then, we can form the category $\mathscr{WYD}_\alpha(H)$ of $\mathscr{WYD}_\alpha$-modules where the composition of morphisms of $\mathscr{WYD}_\alpha$-modules is the standard composition of the underlying linear maps.

Proposition 2. *Equation* (18) *is equivalent to the following equations:*

$$\begin{aligned} &h_{(1,\alpha)} \cdot v_{(0)} \otimes h_{(2,\lambda)} v_{(1,\lambda)} \\ &\quad = \left(h_{(2,\alpha)} \cdot v\right)_{(0)} \otimes \left(h_{(2,\alpha)} \cdot v\right)_{(1,\lambda)} \varphi_{\alpha^{-1}}\left(h_{(1,\alpha\lambda\alpha^{-1})}\right), \end{aligned} \tag{20}$$

$$\rho^V_\lambda(v) = v_{(0)} \otimes v_{(1,\lambda)} \in V \otimes_{t_{\alpha\lambda}} H_\lambda := \Delta_{\alpha,\lambda}\left(1_{\alpha\lambda}\right) \cdot \left(V \otimes H_\lambda\right), \tag{21}$$

for any $v \in V$, $h \in H_{\alpha\lambda}$.

Proof. Assume that (20) and (21) hold for all $h \in H_{\alpha\lambda}$, $v \in V$. We compute

$$\begin{aligned} &h_{(2,\alpha)} \cdot v_{(0)} \otimes h_{(3,\lambda)} v_{(1,\lambda)} S^{-1} \varphi_{\alpha^{-1}}\left(h_{(1,\alpha\lambda^{-1}\alpha^{-1})}\right) \\ &\quad = \left(h_{(3,\alpha)} \cdot v\right)_{(0)} \otimes \left(h_{(3,\alpha)} \cdot v\right)_{(1,\lambda)} \\ &\qquad \times \varphi_{\alpha^{-1}}\left(h_{(2,\alpha\lambda\alpha^{-1})}\right) S^{-1} \varphi_{\alpha^{-1}}\left(h_{(1,\alpha\lambda^{-1}\alpha^{-1})}\right) \\ &\quad = \left(h_{(3,\alpha)} \cdot v\right)_{(0)} \otimes \left(h_{(3,\alpha)} \cdot v\right)_{(1,\lambda)} \\ &\qquad \times \varphi_{\alpha^{-1}}\left(h_{(2,\alpha\lambda\alpha^{-1})} S^{-1}\left(h_{(1,\alpha\lambda^{-1}\alpha^{-1})}\right)\right) \\ &\quad = \left(h_{(2,\alpha)} \cdot v\right)_{(0)} \otimes \left(h_{(2,\alpha)} \cdot v\right)_{(1,\lambda)} \\ &\qquad \times \varphi_{\alpha^{-1}} S^{-1}\left(\varepsilon^t_{\alpha\lambda^{-1}\alpha^{-1}}\left(h_{(1,1)}\right)\right) \\ &\quad = \left(1'_{(2,\alpha)} h_{(2,\alpha)} \cdot v\right)_{(0)} \otimes \left(1'_{(2,\alpha)} h_{(2,\alpha)} \cdot v\right)_{(1,\lambda)} \\ &\qquad \times \varphi_{\alpha^{-1}}\left(1_{(1,\alpha\lambda\alpha^{-1})}\right) \times \varepsilon\left(1_{(2,1)} 1'_{(1,1)} h_{(1,1)}\right) \\ &\quad = \left(1_{(2,\alpha)} h \cdot v\right)_{(0)} \otimes \left(1_{(2,\alpha)} h \cdot v\right)_{(1,\lambda)} \varphi_{\alpha^{-1}}\left(1_{(1,\alpha\lambda\alpha^{-1})}\right) \\ &\quad = 1_{(1,\alpha)} \cdot (h \cdot v)_{(0)} \otimes 1_{(2,\lambda)} (h \cdot v)_{(1,\lambda)} \\ &\quad = (h \cdot v)_{(0)} \otimes (h \cdot v)_{(1,\lambda)} \end{aligned} \tag{22}$$

as required.

Conversely, suppose that V is crossed in the sense of (18). We first note that

$$\begin{aligned} v_{(0)} \otimes v_{(1,\lambda)} &= 1_{(2,\alpha)} \cdot v_{(0)} \otimes 1_{(3,\lambda)} v_{(1,\lambda)} S^{-1} \varphi_{\alpha^{-1}}\left(1_{(1,\alpha\lambda^{-1}\alpha^{-1})}\right) \\ &= 1'_{(1,\alpha)} 1_{(2,\alpha)} \cdot v_{(0)} \otimes 1'_{(2,\lambda)} v_{(1,\lambda)} S^{-1} \\ &\qquad \times \varphi_{\alpha^{-1}}\left(1_{(1,\alpha\lambda^{-1}\alpha^{-1})}\right) \\ &= 1'_{(1,\alpha)} \cdot \left(1_{(2,\alpha)} \cdot v_{(0)}\right) \\ &\qquad \otimes 1'_{(2,\lambda)}\left[v_{(1,\lambda)} S^{-1} \varphi_{\alpha^{-1}}\left(1_{(1,\alpha\lambda^{-1}\alpha^{-1})}\right)\right] \\ &\in V \otimes_{t_{\alpha\lambda}} H_\lambda. \end{aligned} \tag{23}$$

To show that (21) is satisfied, for all $h \in H_{\alpha\lambda}$, we do the following calculations:

$$\begin{aligned} &\left(h_{(2,\alpha)} \cdot v\right)_{(0)} \otimes \left(h_{(2,\alpha)} \cdot v\right)_{(1,\lambda)} \varphi_{\alpha^{-1}}\left(h_{(1,\alpha\lambda\alpha^{-1})}\right) \\ &\quad = h_{(3,\alpha)} \cdot v_{(0)} \otimes h_{(4,\lambda)} v_{(1,\lambda)} S^{-1} \varphi_{\alpha^{-1}}\left(h_{(2,\alpha\lambda^{-1}\alpha^{-1})}\right) \\ &\qquad \times \varphi_{\alpha^{-1}}\left(h_{(1,\alpha\lambda\alpha^{-1})}\right) \\ &\quad = h_{(2,\alpha)} \cdot v_{(0)} \otimes h_{(3,\lambda)} v_{(1,\lambda)} \varphi_{\alpha^{-1}} S^{-1}\left(\varepsilon^s_{\alpha\lambda^{-1}\alpha^{-1}}\left(h_{(1,1)}\right)\right) \\ &\quad = h_{(2,\alpha)} 1'_{(2,\alpha)} \cdot v_{(0)} \otimes h_{(3,\lambda)} v_{(1,\lambda)} \varphi_{\alpha^{-1}} S^{-1}\left(1_{(1,\alpha\lambda^{-1}\alpha^{-1})}\right) \\ &\qquad \times \varepsilon\left(h_{(1,1)} 1'_{(1,1)} 1_{(2,1)}\right) \\ &\quad = h_{(1,\alpha)} 1_{(2,\alpha)} \cdot v_{(0)} \otimes h_{(2,\lambda)} v_{(1,\lambda)} \varphi_{\alpha^{-1}} S^{-1}\left(1_{(1,\alpha\lambda^{-1}\alpha^{-1})}\right) \\ &\quad = h_{(1,\alpha)} 1'_{(1,\alpha)} 1_{(2,\alpha)} \cdot v_{(0)} \otimes h_{(2,\lambda)} 1'_{(2,\lambda)} v_{(1,\lambda)} S^{-1} \\ &\qquad \times \varphi_{\alpha^{-1}}\left(1_{(1,\alpha\lambda^{-1}\alpha^{-1})}\right) \\ &\quad = h_{(1,\alpha)} 1_{(2,\alpha)} \cdot v_{(0)} \otimes h_{(2,\lambda)} 1_{(3,\lambda)} v_{(1,\lambda)} S^{-1} \\ &\qquad \times \varphi_{\alpha^{-1}}\left(1_{(1,\alpha\lambda^{-1}\alpha^{-1})}\right) \\ &\quad = h_{(1,\alpha)} \cdot v_{(0)} \otimes h_{(2,\lambda)} v_{(1,\lambda)}. \end{aligned} \tag{24}$$

This completes the proof. □

Proposition 3. *If $(V, \rho^V) \in \mathscr{WYD}_\alpha(H)$, $(W, \rho^W) \in \mathscr{WYD}_\beta(H)$, then $V \otimes_{t_{\alpha\beta}} W = \Delta_{\alpha,\beta}(1_{\alpha\beta}) \cdot (V \otimes W) \in \mathscr{WYD}_{\alpha\beta}(H)$ with the action and coaction structures as follows:*

$$\begin{aligned} h \cdot (v \otimes w) &= h_{(1,\alpha)} \cdot v \otimes h_{(2,\beta)} \cdot w, \\ \rho_\lambda^{V \otimes_{t_{\alpha\beta}} W}(v \otimes w) &= v_{(0)} \otimes w_{(0)} \\ &\qquad \otimes w_{(1,\lambda)} \varphi_{\beta^{-1}}\left(v_{(1,\beta\lambda\beta^{-1})}\right), \end{aligned} \tag{25}$$

for all $h \in H_{\alpha\beta}$, $\lambda \in \pi$, $v \otimes w \in V \otimes_{t_{\alpha\beta}} W$.

Proof. It is easy to prove that $V \otimes_{t_{\alpha\beta}} W$ is a left $H_{\alpha\beta}$-module, and the proof of coassociativity of $V \otimes_{t_{\alpha\beta}} W$ is similar to the Hopf group coalgebra case. For all $v \otimes w \in V \otimes_{t_{\alpha\beta}} W$, we have

$$\begin{aligned} &\left(\mathrm{id}_{V \otimes_{t_{\alpha\beta}} W} \otimes \varepsilon\right) \circ \rho_1^{V \otimes_{t_{\alpha\beta}} W}(v \otimes w) \\ &\quad = \varepsilon\left(w_{(1,1)} \varphi_{\beta^{-1}}\left(1'\right)_{(2,1)}\right) \\ &\qquad \times \varepsilon\left(\varphi_{\beta^{-1}}\left(1'\right)_{(1,1)} \varphi_{\beta^{-1}}\left(v_{(1,1)}\right)\right) v_{(0)} \otimes w_{(0)} \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon\left(\left(1_{(2,\beta)}\cdot w\right)_{(1,1)}\varphi_{\beta^{-1}}\left(1_{(1,1)}\right)\varphi_{\beta^{-1}}\left(1'_{(2,1)}\right)\right)\\
&\quad\times\varepsilon\left(\varphi_{\beta^{-1}}\left(1'_{(1,1)}\right)\varphi_{\beta^{-1}}\left(v_{(1,1)}\right)\right)v_{(0)}\otimes\left(1_{(2,\beta)}\cdot w\right)_{(0)}\\
&= \varepsilon\left(1_{(5,1)}w_{(1,1)}S^{-1}\varphi_{\beta^{-1}}\left(1_{(3,1)}\right)\varphi_{\beta^{-1}}\left(1_{(2,1)}\right)\right)\\
&\quad\times\varepsilon\left(1_{(1,1)}v_{(1,1)}\right)v_{(0)}\otimes 1_{(4,\beta)}\cdot w_{(0)}\\
&= \varepsilon\left(1_{(4,1)}w_{(1,1)}\varphi_{\beta^{-1}}S^{-1}\left(\varepsilon_1^s\left(1_{(2,1)}\right)\right)\right)\varepsilon\left(1_{(1,1)}v_{(1,1)}\right)v_{(0)}\\
&\quad\otimes 1_{(3,\beta)}\cdot w_{(0)}\\
&= \varepsilon\left(1_{(4,1)}w_{(1,1)}\varepsilon_1^t S^{-1}\varphi_{\beta^{-1}}\left(1_{(2,1)}\right)\right)\varepsilon\left(1_{(1,1)}v_{(1,1)}\right)v_{(0)}\\
&\quad\otimes 1_{(3,\beta)}\cdot w_{(0)}\\
&= \varepsilon\left(\left(1_{(2,\beta)}\cdot w\right)_{(1,1)}\right)\varepsilon\left(1_{(1,1)}v_{(1,1)}\right)v_{(0)}\otimes\left(1_{(2,\beta)}\cdot w\right)_{(0)}\\
&= \varepsilon\left(\left(1_{(2,\alpha)}\cdot v\right)_{(1,1)}\varphi_{\alpha^{-1}}\left(1_{(1,1)}\right)\right)\left(1_{(2,\alpha)}\cdot v\right)_{(0)}\otimes 1_{(3,\beta)}\cdot w\\
&= \varepsilon\left(1_{(3,1)}v_{(1,1)}\varphi_{\alpha^{-1}}S^{-1}\varepsilon_1^s\left(1_{(1,1)}\right)\right)1_{(2,\alpha)}\cdot v_{(0)}\otimes 1_{(4,\beta)}\cdot w\\
&= \varepsilon\left(\left(1_{(1,\alpha)}\cdot v\right)_{(1,1)}\right)\left(1_{(1,\alpha)}\cdot v\right)_{(0)}\otimes 1_{(2,\beta)}\cdot w\\
&= v\otimes w.
\end{aligned}
\tag{26}
$$

This shows that $V\otimes_{t_{\alpha\beta}}W$ is satisfing counitary condition (17).

Then, we check the equivalent form of crossed conditions (20) and (21). In fact, for all $h\in H_{\alpha\beta\lambda}$, $v\otimes w\in V\otimes_{t_{\alpha\beta}}W$, we have

$$
\begin{aligned}
&\left(h_{(2,\alpha\beta)}\cdot(v\otimes w)\right)_{(0)}\\
&\quad\otimes\left(h_{(2,\alpha\beta)}\cdot(v\otimes w)\right)_{(1,\lambda)}\varphi_{(\alpha\beta)^{-1}}\left(h_{(1,\alpha\beta\lambda\beta^{-1}\alpha^{-1})}\right)\\
&= \left(h_{(2,\alpha)}\cdot v\right)_{(0)}\otimes\left(h_{(3,\beta)}\cdot w\right)_{(0)}\\
&\quad\otimes\left(h_{(3,\beta)}\cdot w\right)_{(1,\lambda)}\varphi_{\beta^{-1}}\left(\left(h_{(2,\alpha)}\cdot v\right)_{(1,\beta\lambda\beta^{-1})}\right)\\
&\varphi_{\beta^{-1}\alpha^{-1}}\left(h_{(1,\alpha\beta\lambda\beta^{-1}\alpha^{-1})}\right)\\
&= h_{(3,\alpha)}\cdot v_{(0)}\otimes h_{(6,\beta)}\cdot w_{(0)}\\
&\quad\otimes h_{(7,\lambda)}w_{(1,\lambda)}S^{-1}\varphi_{\beta^{-1}}\left(h_{(5,\beta\lambda^{-1}\beta^{-1})}\right)\\
&\varphi_{\beta^{-1}}\left(h_{(4,\beta\lambda\beta^{-1})}v_{(1,\beta\lambda\beta^{-1})}\right)\varphi_{\beta^{-1}}\varphi_{\alpha^{-1}}S^{-1}\\
&\quad\times\left(h_{(2,\alpha\beta\lambda^{-1}\beta^{-1}\alpha^{-1})}\right)\varphi_{\beta^{-1}\alpha^{-1}}\left(h_{(1,\alpha\beta\lambda\beta^{-1}\alpha^{-1})}\right)\\
&= h_{(3,\alpha)}\cdot v_{(0)}\otimes h_{(6,\beta)}\cdot w_{(0)}\\
&\quad\otimes h_{(7,\lambda)}w_{(1,\lambda)}S^{-1}\varphi_{\beta^{-1}}\left(h_{(5,\beta\lambda^{-1}\beta^{-1})}\right)\\
&\varphi_{\beta^{-1}}\left(h_{(4,\beta\lambda\beta^{-1})}v_{(1,\beta\lambda\beta^{-1})}\right)\varphi_{\beta^{-1}\alpha^{-1}}S^{-1}\\
&\quad\times\left(S\left(h_{(1,\alpha\beta\lambda\beta^{-1}\alpha^{-1})}\right)h_{(2,\alpha\beta\lambda^{-1}\beta^{-1}\alpha^{-1})}\right)\\
&= h_{(1,\alpha)}1_{(2,\alpha)}\cdot v_{(0)}\otimes h_{(4,\beta)}\cdot w_{(0)}\\
&\quad\otimes h_{(5,\lambda)}w_{(1,\lambda)}S^{-1}\varphi_{\beta^{-1}}\left(h_{(3,\beta\lambda^{-1}\beta^{-1})}\right)\\
&\varphi_{\beta^{-1}}\left(h_{(2,\beta\lambda\beta^{-1})}v_{(1,\beta\lambda\beta^{-1})}\right)\varphi_{\beta^{-1}\alpha^{-1}}S^{-1}\left(1_{(1,\alpha\beta\lambda^{-1}\beta^{-1}\alpha^{-1})}\right)\\
&= h_{(1,\alpha)}1_{(2,\alpha)}\cdot v_{(0)}\otimes h_{(4,\beta)}\cdot w_{(0)}\\
&\quad\otimes h_{(5,\lambda)}w_{(1,\lambda)}S^{-1}\varphi_{\beta^{-1}}\left(h_{(3,\beta\lambda^{-1}\beta^{-1})}\right)\\
&\varphi_{\beta^{-1}}\left(h_{(2,\beta\lambda\beta^{-1})}1_{(3,\beta\lambda\beta^{-1})}v_{(1,\beta\lambda\beta^{-1})}S^{-1}\right.\\
&\quad\left.\times\varphi_{\alpha^{-1}}\left(1_{(1,\alpha\beta\lambda^{-1}\beta^{-1}\alpha^{-1})}\right)\right)\\
&= h_{(1,\alpha)}\cdot v_{(0)}\otimes h_{(3,\beta)}\cdot w_{(0)}\otimes h_{(4,\lambda)}w_{(1,\lambda)}\\
&\quad\times\varphi_{\beta^{-1}}S^{-1}\left(\varepsilon^s_{\beta\lambda^{-1}\beta^{-1}}\left(h_{(2,1)}\right)\right)\varphi_{\beta^{-1}}\left(v_{(1,\beta\lambda\beta^{-1})}\right)\\
&= h_{(1,\alpha)}\cdot v_{(0)}\otimes h_{(2,\beta)}1'_{(1,\beta)}1_{(2,\beta)}\cdot w_{(0)}\\
&\quad\otimes h_{(3,\lambda)}1'_{(2,\lambda)}w_{(1,\lambda)}S^{-1}\varphi_{\beta^{-1}}\left(1_{(1,\beta\lambda^{-1}\beta^{-1})}\right),\\
&\varphi_{\beta^{-1}}\left(v_{(1,\beta\lambda\beta^{-1})}\right)\\
&= h_{(1,\alpha)}\cdot v_{(0)}\otimes h_{(2,\beta)}\cdot w_{(0)}\otimes h_{(3,\lambda)}w_{(1,\lambda)}\varphi_{\beta^{-1}}\left(v_{(1,\beta\lambda\beta^{-1})}\right)\\
&= h_{(1,\alpha\beta)}\cdot(v\otimes w)_{(0)}\otimes h_{(2,\lambda)}(v\otimes w)_{(1,\lambda)},\\
&1_{(1,\alpha\beta)}\cdot(v\otimes w)_{(0)}\otimes 1_{(2,\lambda)}\ (v\otimes w)_{(1,\lambda)}\\
&= 1_{(1,\alpha\beta)}\cdot\left(v_{(0)}\otimes w_{(0)}\right)\otimes 1_{(2,\lambda)}w_{(1,\lambda)}\varphi_{\beta^{-1}}\left(v_{(1,\beta\lambda\beta^{-1})}\right)\\
&= 1_{(1,\alpha)}\cdot v_{(0)}\otimes 1_{(2,\beta)}\cdot w_{(0)}\otimes 1_{(3,\lambda)}w_{(1,\lambda)}\varphi_{\beta^{-1}}\left(v_{(1,\beta\lambda\beta^{-1})}\right)\\
&= 1_{(1,\alpha)}\cdot v_{(0)}\otimes 1_{(2,\beta)}1'_{(1,\beta)}\cdot w_{(0)}\otimes 1'_{(2,\lambda)}w_{(1,\lambda)}\\
&\quad\times\varphi_{\beta^{-1}}\left(v_{(1,\beta\lambda\beta^{-1})}\right)\\
&= 1_{(1,\alpha)}\cdot v_{(0)}\otimes 1_{(2,\beta)}\cdot w_{(0)}\otimes w_{(1,\lambda)}\varphi_{\beta^{-1}}\left(v_{(1,\beta\lambda\beta^{-1})}\right)\\
&= v_{(0)}\otimes w_{(0)}\otimes w_{(1,\lambda)}\varphi_{\beta^{-1}}\left(v_{(1,\beta\lambda\beta^{-1})}\right)\\
&= (v\otimes w)_{(0)}\otimes(v\otimes w)_{(1,\lambda)}.
\end{aligned}
\tag{27}
$$

This finishes the proof. □

Proposition 4. *Let $(V,\rho^V)\in\mathscr{WYD}_\alpha(H)$, and let $\beta\in\pi$. Set ${}^\beta V = V$ as vector space, with action and coaction structures defined by*

$$
\begin{aligned}
h\triangleright{}^\beta v &= {}^\beta\!\left(\varphi_{\beta^{-1}}(h)\cdot v\right),\qquad \forall h\in H_{\beta\alpha\beta^{-1}},\ {}^\beta v\in{}^\beta V,\\
\rho_\lambda^{{}^\beta V}\left({}^\beta v\right) &= {}^\beta\!\left(v_{(0)}\right)\otimes\varphi_\beta\left(v_{(1,\beta^{-1}\lambda\beta)}\right)\\
&:= v_{\langle 0\rangle}\otimes v_{\langle 1,\lambda\rangle},\quad \forall\,{}^\beta v\in{}^\beta V.
\end{aligned}
\tag{28}
$$

Then, ${}^\beta V\in\mathscr{WYD}_{\beta\alpha\beta^{-1}}(H)$.

Proof. Obviously, ${}^{\beta}V$ is a left $H_{\beta\alpha\beta^{-1}}$-module, and conditions (16) and (17) are straightforward. Then, it remains to show that conditions (20) and (21) hold. For all ${}^{\beta}v \in {}^{\beta}V$, we have

$$\begin{aligned} & 1_{(1,\beta\alpha\beta^{-1})} \triangleright v_{\langle 0\rangle} \otimes 1_{(2,\lambda)} v_{\langle 1,\lambda\rangle} \\ &\quad = {}^{\beta}\left(\varphi_{\beta^{-1}}\left(1_{(1,\beta\alpha\beta^{-1})}\right) \cdot v_{(0)}\right) \otimes 1_{(2,\lambda)} \varphi_{\beta}\left(v_{(1,\beta^{-1}\lambda\beta)}\right) \\ &\quad = {}^{\beta}\left(1_{(1,\alpha)} \cdot v_{(0)}\right) \otimes \varphi_{\beta}\left(1_{(2,\lambda)}\right) \varphi_{\beta}\left(v_{(1,\beta^{-1}\lambda\beta)}\right) \\ &\quad = v_{\langle 0\rangle} \otimes v_{\langle 1,\lambda\rangle}. \end{aligned} \tag{29}$$

Next, for all $h \in H_{\beta\alpha\beta^{-1}\lambda}$, ${}^{\beta}v \in {}^{\beta}V$, we get

$$\begin{aligned} & \left(h_{(2,\beta\alpha\beta^{-1})} \triangleright {}^{\beta}v\right)_{\langle 0\rangle} \otimes \left(h_{(2,\beta\alpha\beta^{-1})} \triangleright {}^{\beta}v\right)_{\langle 1,\lambda\rangle} \\ &\qquad \times \varphi_{\beta\alpha^{-1}\beta^{-1}}\left(h_{(1,\beta\alpha\beta^{-1}\lambda\beta\alpha^{-1}\beta^{-1})}\right) \\ &\qquad = {}^{\beta}\left(\left(\varphi_{\beta^{-1}}\left(h_{(2,\beta\alpha\beta^{-1})}\right) \cdot v\right)_{(0)}\right) \\ &\qquad\quad \otimes \varphi_{\beta}\left(\left(\varphi_{\beta^{-1}}\left(h_{(2,\beta\alpha\beta^{-1})}\right) \cdot v\right)_{(1,\beta^{-1}\lambda\beta)}\right), \\ & \varphi_{\beta\alpha^{-1}\beta^{-1}}\left(h_{(1,\beta\alpha\beta^{-1}\lambda\beta\alpha^{-1}\beta^{-1})}\right) \\ &\qquad = {}^{\beta}\left(\left(\varphi_{\beta^{-1}}(h)_{(2,\alpha)} \cdot v\right)_{(0)}\right) \\ &\qquad\quad \otimes \varphi_{\beta}\left(\left(\varphi_{\beta^{-1}}(h)_{(2,\alpha)} \cdot v\right)_{(1,\beta^{-1}\lambda\beta)}\right. \\ &\qquad\qquad \left.\times \varphi_{\alpha^{-1}}\left(\varphi_{\beta^{-1}}(h)_{(1,\alpha\beta^{-1}\lambda\beta\alpha^{-1})}\right)\right) \\ &\qquad = {}^{\beta}\left(\varphi_{\beta^{-1}}(h)_{(1,\alpha)} \cdot v_{(0)}\right) \\ &\qquad\quad \otimes \varphi_{\beta}\left(\varphi_{\beta^{-1}}(h)_{(2,\beta^{-1}\lambda\beta)} v_{(1,\beta^{-1}\lambda\beta)}\right) \\ &\qquad = {}^{\beta}\left(\varphi_{\beta^{-1}}\left(h_{(1,\beta\alpha\beta^{-1})}\right) \cdot v_{(0)}\right) \\ &\qquad\quad \otimes \varphi_{\beta}\left(\varphi_{\beta^{-1}}\left(h_{(2,\lambda)}\right) v_{(1,\beta^{-1}\lambda\beta)}\right) \\ &\qquad = h_{(1,\beta\alpha\beta^{-1})} \triangleright {}^{\beta}\left(v_{(0)}\right) \\ &\qquad\quad \otimes \varphi_{\beta}\left(\varphi_{\beta^{-1}}\left(h_{(2,\lambda)}\right) v_{(1,\beta^{-1}\lambda\beta)}\right) \\ &\qquad = h_{(1,\beta\alpha\beta^{-1})} \triangleright {}^{\beta}v_{\langle 0\rangle} \otimes h_{(2,\lambda)} v_{\langle 1,\lambda\rangle}. \end{aligned} \tag{30}$$

This completes the proof of the proposition. □

Remark 5. Let $(V, \rho^V) \in \mathscr{WYD}_{\alpha}(H)$ and let $(W, \rho^W) \in \mathscr{WYD}_{\beta}(H)$; then we have ${}^{st}V = {}^{s}({}^{t}V)$ as an object in $\mathscr{WYD}_{st\alpha t^{-1}s^{-1}}(H)$ and ${}^{s}(V \otimes_{t_{\alpha\beta}} W) = {}^{s}V \otimes_{t_{s\alpha\beta s^{-1}}} {}^{s}W$ as an object in $\mathscr{WYD}_{s\alpha\beta s^{-1}}(H)$.

Proposition 6. *Let $(V, \rho^V) \in \mathscr{WYD}_{\alpha}(H)$; $(W, \rho^W) \in \mathscr{WYD}_{\beta}(H)$. Set ${}^{V}W = {}^{\alpha}W$ as an object in $\mathscr{WYD}_{\alpha\beta\alpha^{-1}}(H)$. Define the map*

$$\begin{aligned} c_{V,W} &: V \otimes W \longrightarrow {}^{V}W \otimes V, \\ c_{V,W}(v \otimes w) &= {}^{\alpha}\left(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right) \cdot w\right) \otimes v_{(0)}. \end{aligned} \tag{31}$$

Then, $c_{V,W}$ is H-linear, H-colinear and satisfies the following conditions:

$$\begin{aligned} c_{V\otimes W,U} &= \left(c_{V,{}^{W}U} \otimes \mathrm{id}_W\right) \circ \left(\mathrm{id}_V \otimes c_{W,U}\right), \\ c_{V,W\otimes U} &= \left(\mathrm{id}_{{}^{V}W} \otimes c_{V,U}\right) \circ \left(c_{V,W} \otimes \mathrm{id}_U\right). \end{aligned} \tag{32}$$

Furthermore, $c_{{}^{\gamma}V,{}^{\gamma}W} = c_{V,W}$, for all $\gamma \in \pi$.

Proof. Firstly, we need to show that $c_{V,W}$ is well defined. Indeed, we have

$$\begin{aligned} & c_{V\cdot W}\left(1_{(1,\alpha)} \cdot v \otimes 1_{(2,\beta)} \cdot w\right) \\ &\quad = {}^{\alpha}\left(S_{\beta^{-1}}\left(\left(1_{(1,\alpha)} \cdot v\right)_{(1,\beta^{-1})}\right) 1_{(2,\beta)} \cdot w\right) \otimes \left(1_{(1,\alpha)} \cdot v\right)_{(0)} \\ &\quad = {}^{\alpha}\left(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})} S^{-1}\varphi_{\alpha^{-1}}\left(1_{(1,\alpha\beta\alpha^{-1})}\right)\right)\right. \\ &\qquad \left.\times S_{\beta^{-1}}\left(1_{(3,\beta^{-1})}\right) 1_{(4,\beta)} \cdot w\right) \otimes 1_{(2,\alpha)} \cdot v_{(0)} \\ &\quad = {}^{\alpha}\left(S_{\beta^{-1}} S^{-1}\varphi_{\alpha^{-1}}\left(1_{(1,\alpha\beta\alpha^{-1})}\right)\right. \\ &\qquad \left.\times S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right) \varepsilon_{\beta}^{s}\left(1_{(3,1)}\right) \cdot w\right) \otimes 1_{(2,\alpha)} \cdot v_{(0)} \\ &\quad = {}^{\alpha}\left(S_{\beta^{-1}} S^{-1}\varphi_{\alpha^{-1}}\left(1_{(1,\alpha\beta\alpha^{-1})}\right) S_{\beta^{-1}}\left(1_{(3,\beta^{-1})} v_{(1,\beta^{-1})}\right) \cdot w\right) \\ &\qquad \otimes 1_{(2,\alpha)} \cdot v_{(0)} \\ &\quad = {}^{\alpha}\left(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right) \cdot w\right) \otimes v_{(0)} \\ &\quad = c_{V,W}(v \otimes w). \end{aligned} \tag{33}$$

Secondly, we prove that $c_{V,W}$ is H-linear. For all $h \in H_{\alpha\beta}$, we compute

$$\begin{aligned} & c_{V,W}(h \cdot (v \otimes w)) \\ &\quad = {}^{\alpha}\left(S_{\beta^{-1}}\left(\left(h_{(1,\alpha)} \cdot v\right)_{(1,\beta^{-1})}\right) h_{(2,\beta)} \cdot w\right) \otimes \left(h_{(1,\alpha)} \cdot v\right)_{(0)} \\ &\quad = {}^{\alpha}\left(S_{\beta^{-1}}\left(h_{(3,\beta^{-1})} v_{(1,\beta^{-1})} S^{-1}\varphi_{\alpha^{-1}}\left(h_{(1,\alpha\beta\alpha^{-1})}\right)\right) h_{(4,\beta)} \cdot w\right) \\ &\qquad \otimes h_{(2,\alpha)} \cdot v_{(0)} \\ &\quad = {}^{\alpha}\left(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})} S^{-1}\varphi_{\alpha^{-1}}\left(h_{(1,\alpha\beta\alpha^{-1})}\right)\right) \varepsilon_{\beta}^{s}\left(h_{(3,1)}\right) \cdot w\right) \\ &\qquad \otimes h_{(2,\alpha)} \cdot v_{(0)} \\ &\quad = {}^{\alpha}\left(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})} S^{-1}\varphi_{\alpha^{-1}}\left(h_{(1,\alpha\beta\alpha^{-1})}\right)\right)\right. \\ &\qquad \left.\times S_{\beta^{-1}}\left(1_{(2,\beta^{-1})}\right) \cdot w\right) \otimes h_{(2,\alpha)} 1_{(1,\alpha)} \cdot v_{(0)} \end{aligned}$$

$$
\begin{aligned}
&= {}^{\alpha}\Big(S_{\beta^{-1}}\left(1_{(3,\beta^{-1})}v_{(1,\beta^{-1})}S^{-1}\varphi_{\alpha^{-1}}\left(1_{(1,\alpha\beta\alpha^{-1})}\right)\right.\\
&\qquad \left.\times S^{-1}\varphi_{\alpha^{-1}}\left(h_{(1,\alpha\beta\alpha^{-1})}\right)\right)\cdot w\Big)\otimes h_{(2,\alpha)}1_{(2,\alpha)}\cdot v_{(0)}\\
&= {}^{\alpha}\Big(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}S^{-1}\varphi_{\alpha^{-1}}\left(h_{(1,\alpha\beta\alpha^{-1})}\right)\right)\cdot w\Big)\\
&\quad \otimes h_{(2,\alpha)}\cdot v_{(0)}\\
&= {}^{\alpha}\Big(\varphi_{\alpha^{-1}}\left(h_{(1,\alpha\beta\alpha^{-1})}\right)S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right)\cdot w\Big)\otimes h_{(2,\alpha)}\cdot v_{(0)}\\
&= h_{(1,\alpha\beta\alpha^{-1})}\triangleright {}^{\alpha}\Big(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right)\cdot w\Big)\otimes h_{(2,\alpha)}\cdot v\\
&= h\cdot c_{V,W}(v\otimes w)
\end{aligned}
\tag{34}
$$

as required.

Finally, we check that $c_{V,W}$ is satisfing the H-colinear condition. In fact,

$$
\begin{aligned}
&\rho_{\lambda}^{{}^{V}W\otimes V}\circ c_{V,W}(v\otimes w)\\
&\quad= {}^{\alpha}\Big(\left(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right)\cdot w\right)_{(0)}\Big)\otimes v_{(0)(0)}\\
&\qquad\otimes v_{(0)(1,\lambda)}\left(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right)\cdot w\right)_{(1,\lambda)}\\
&\quad= {}^{\alpha}\Big(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right)_{(2,\beta)}\cdot w_{(0)}\Big)\otimes v_{(0)(0)}\\
&\qquad\otimes v_{(0)(1,\lambda)}S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right)_{(3,\lambda)}w_{(1,\lambda)}S^{-1}\varphi_{\beta^{-1}}\\
&\left(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right)_{(1,\beta\lambda^{-1}\beta^{-1})}\right)\\
&\quad= {}^{\alpha}\Big(S_{\beta^{-1}}\left(v_{(3,\beta^{-1})}\right)\cdot w_{(0)}\Big)\otimes v_{(0)}\\
&\qquad\otimes v_{(1,\lambda)}S_{\lambda^{-1}}\left(v_{(2,\lambda^{-1})}\right)\\
&\qquad\times w_{(1,\lambda)}S^{-1}\varphi_{\beta^{-1}}S_{\beta\lambda\beta^{-1}}\left(v_{(4,\beta\lambda\beta^{-1})}\right)\\
&\quad= {}^{\alpha}\Big(S_{\beta^{-1}}\left(v_{(2,\beta^{-1})}\right)\cdot w_{(0)}\Big)\otimes v_{(0)}\otimes\varepsilon_{\lambda}^{t}\left(v_{(1,1)}\right)\\
&\qquad\times w_{(1,\lambda)}\varphi_{\beta^{-1}}\left(v_{(3,\beta\lambda\beta^{-1})}\right)\\
&\quad= {}^{\alpha}\Big(S_{\beta^{-1}}\left(1_{(2,\beta^{-1})}v_{(1,\beta^{-1})}\right)\cdot w_{(0)}\Big)\otimes v_{(0)}\\
&\qquad\otimes S_{\lambda^{-1}}\left(1_{(1,\lambda^{-1})}\right)w_{(1,\lambda)}\varphi_{\beta^{-1}}\left(v_{(2,\beta\lambda\beta^{-1})}\right)\\
&\quad= {}^{\alpha}\Big(S_{\beta^{-1}}\left(1_{(2,\beta^{-1})}v_{(1,\beta^{-1})}\right)\cdot w_{(0)}\Big)\otimes v_{(0)}\\
&\qquad\otimes S_{\lambda^{-1}}\left(1_{(1,\lambda^{-1})}\right)w_{(1,\lambda)}\varphi_{\beta^{-1}}S^{-1}\\
&S_{\beta\lambda\beta^{-1}}\left(1_{(3,\beta\lambda\beta^{-1})}\right)\varphi_{\beta^{-1}}\left(v_{(2,\beta\lambda\beta^{-1})}\right)\\
&\quad= {}^{\alpha}\Big(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right)S_{\beta^{-1}}(1)_{(2,\beta)}\cdot w_{(0)}\Big)\\
&\qquad\otimes v_{(0)}\otimes S_{\beta^{-1}}(1)_{(3,\lambda)}w_{(1,\lambda)},\\
&\varphi_{\beta^{-1}}S^{-1}\left(S_{\beta^{-1}}(1)_{(1,\beta\lambda^{-1}\beta^{-1})}\right)\varphi_{\beta^{-1}}\left(v_{(2,\beta\lambda\beta^{-1})}\right)\\
&\quad= {}^{\alpha}\Big(S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right)\cdot w_{(0)}\Big)\otimes v_{(0)}\\
&\qquad\otimes w_{(1,\lambda)}\varphi_{\beta^{-1}}\left(v_{(2,\beta\lambda\beta^{-1})}\right)\\
&\quad= \left(c_{V,W}\otimes \mathrm{id}_{H_{\lambda}}\right)\Big(v_{(0)}\otimes w_{(0)}\otimes w_{(1,\lambda)}\varphi_{\beta^{-1}}\\
&\qquad\times\left(v_{(1,\beta\lambda\beta^{-1})}\right)\Big)\\
&\quad= \left(c_{V,W}\otimes \mathrm{id}_{H_{\lambda}}\right)\circ\rho_{\lambda}^{V\otimes W}(v\otimes w).
\end{aligned}
\tag{35}
$$

The rest of proof is easy to get and we omit it. □

Lemma 7. *The map $c_{V,W}$ defined by* (31) *is bijective with inverse*

$$
\begin{aligned}
&c_{V,W}^{-1}: {}^{V}W\otimes V\longrightarrow V\otimes W,\\
&c_{V,W}^{-1}\left({}^{\alpha}w\otimes v\right)=v_{(0)}\otimes v_{(1,\beta)}\cdot w,
\end{aligned}
\tag{36}
$$

for all $v\in V$, ${}^{\alpha}w\in {}^{V}W$.

Proof. Firstly, we prove $c_{V,W}^{-1}\circ c_{V,W}=\mathrm{id}_{V\otimes W}$. For all $v\in V$, $w\in W$, we have

$$
\begin{aligned}
&c_{V,W}^{-1}\circ c_{V,W}(v\otimes w)\\
&\quad= v_{(0)(0)}\otimes v_{(0)(1,\beta)}S_{\beta^{-1}}\left(v_{(1,\beta^{-1})}\right)\cdot w\\
&\quad= v_{(0)}\otimes\varepsilon_{\beta}^{t}\left(v_{(1,1)}\right)\cdot w=1_{(1,\alpha)}\cdot v_{(0)}\\
&\qquad\otimes 1_{(2,\beta)}\varepsilon_{\beta}^{t}\left(v_{(1,1)}\right)\cdot w\\
&\quad= 1_{(1,\alpha)}S^{-1}\varepsilon_{\alpha^{-1}}^{t}\left(v_{(1,1)}\right)\cdot v_{(0)}\otimes 1_{(2,\beta)}\cdot w\\
&\quad= \varepsilon\left(1'_{(2,1)}v_{(1,1)}\right)1_{(1,\alpha)}1'_{(1,\alpha)}\cdot v_{(0)}\otimes 1_{(2,\beta)}\cdot w\\
&\quad= 1_{(1,\alpha)}\cdot v\otimes 1_{(2,\beta)}\cdot w=v\otimes w.
\end{aligned}
\tag{37}
$$

Secondly, we check $c_{V,W}\circ c_{V,W}^{-1}=\mathrm{id}_{{}^{V}W\otimes V}$ as follows:

$$
\begin{aligned}
&c_{V,W}\circ c_{V,W}^{-1}\left({}^{\alpha}w\otimes v\right)\\
&\quad= {}^{\alpha}\Big(S_{\beta^{-1}}\left(v_{(0)(1,\beta^{-1})}\right)v_{(1,\beta)}\cdot w\Big)\otimes v_{(0)(0)}\\
&\quad= {}^{\alpha}\Big(\varphi_{\alpha^{-1}}\left(1_{(1,\alpha\beta\alpha^{-1})}\right)S_{\beta^{-1}}\left(v_{(0)(1,\beta^{-1})}\right)v_{(1,\beta)}\cdot w\Big)\\
&\qquad\otimes 1_{(2,\alpha)}\cdot v_{(0)(0)}\\
&\quad= {}^{\alpha}\Big(\varphi_{\alpha^{-1}}\left(1_{(1,\alpha\beta\alpha^{-1})}\right)\varepsilon_{\beta}^{s}\left(v_{(1,1)}\right)\cdot w\Big)\otimes 1_{(2,\alpha)}\cdot v_{(0)}\\
&\quad= {}^{\alpha}\Big(\varphi_{\alpha^{-1}}\left(1_{(1,\alpha\beta\alpha^{-1})}\varepsilon_{\alpha\beta\alpha^{-1}}^{s}\varphi_{\alpha}\left(v_{(1,1)}\right)\right)\cdot w\Big)\otimes 1_{(2,\alpha)}\cdot v_{(0)}\\
&\quad= {}^{\alpha}\Big(\varphi_{\alpha^{-1}}\left(1_{(1,\alpha\beta\alpha^{-1})}\right)\cdot w\Big)\otimes 1_{(2,\alpha)}\\
&\qquad\times S_{\alpha^{-1}}\varepsilon_{\alpha}^{s}\left(\varphi_{\alpha}\left(v_{(1,1)}\right)\right)\cdot v_{(0)}
\end{aligned}
$$

$$= {}^{\alpha}\left(\varphi_{\alpha^{-1}}\left(1_{(1,\alpha\beta\alpha^{-1})}\right)\cdot w\right)\otimes 1_{(2,\alpha)}\varepsilon_{\alpha}^{t}\varphi_{\alpha}S\left(v_{(1,1)}\right)\cdot v_{(0)}$$
$$= {}^{\alpha}\left(\varphi_{\alpha^{-1}}\left(1_{(1,\alpha\beta\alpha^{-1})}\right)\cdot w\right)\otimes 1_{(2,\alpha)}$$
$$\times\varepsilon\left(\varphi_{\alpha^{-1}}\left(1'_{(1,1)}\right)S\left(v_{(1,1)}\right)\right)1'_{(2,\alpha)}\cdot v_{(0)}$$
$$= {}^{\alpha}\left(\varphi_{\alpha^{-1}}\left(1_{(1,\alpha\beta\alpha^{-1})}\right)\cdot w\right)\otimes 1_{(2,\alpha)}\varepsilon S\left(v_{(1,1)}\right)\cdot v_{(0)}$$
$$= 1_{(1,\alpha\beta\alpha^{-1})}\triangleright {}^{\alpha}w\otimes 1_{(2,\alpha)}\cdot v$$
$$= {}^{\alpha}w\otimes v. \tag{38}$$

This completes the proof. □

Define $\mathscr{WYD}(H) = \coprod_{\alpha\in\pi}\mathscr{WYD}_{\alpha}(H)$, the disjoint union of the categories $\mathscr{WYD}_{\alpha}(H)$ for all $\alpha\in\pi$. If we endow $\mathscr{WYD}(H)$ with tensor product as in Proposition 3, then $\mathscr{WYD}(H)$ becomes a monoidal category. The unit is $H^T = \{H_{\alpha}^t := \varepsilon_{\alpha}^t(H_1)\}_{\alpha\in\pi}$.

The group homomorphism $\psi : G \to \operatorname{aut}(\mathscr{WYD}(H))$; $\beta\to\psi_{\beta}$ is given on components as

$$\psi_{\beta}:\mathscr{WYD}_{\alpha}(H)\longrightarrow\mathscr{WYD}_{\beta\alpha\beta^{-1}}(H), \tag{39}$$

where the functor ψ_{β} acts as follows: given a morphism $f : (V,\rho^V)\to(W,\rho^W)$, for any $v\in V$, we set $({}^{\beta}f)({}^{\beta}v) = {}^{\beta}(f(v))$.

The braiding in $\mathscr{WYD}(H)$ is given by the family $\{c_{V,W}\}$ as shown in Proposition 6. Then, we have the following theorem.

Theorem 8. *For a weak crossed Hopf group coalgebra H, $\mathscr{WYD}(H)$ is a braided T-category over group π.*

Example 9. Let H be a weak Hopf algebra, G a finite group, and $k(G)$ the dual Hopf algebra of the group algebra kG.

Then, have the weak Hopf group coalgebra $k(G)\otimes H$; the multiplication in $k(G)\otimes H$ is given by

$$\left(p_{\alpha}\otimes h\right)\left(p_{\beta}\otimes g\right) = p_{\alpha}p_{\beta}\otimes hg, \tag{40}$$

for all $p_{\alpha},p_{\beta}\in k(G)$, $h,g\in H$, and the comultiplication, counit, and antipode are given by

$$\Delta_{u,v}\left(p_{\alpha}\otimes h\right) = \sum_{uv=\alpha}\left(p_u\otimes h_1\right)\otimes\left(p_v\otimes h_2\right),$$
$$\varepsilon\left(p_{\alpha}\otimes h\right) = \delta_{\alpha,1}\varepsilon(h), \tag{41}$$
$$S\left(p_{\alpha}\otimes h\right) = p_{\alpha^{-1}}\otimes S(h).$$

Moreover, $k(G)\otimes H$ is a weak crossed Hopf group coalgebra with the following crossing:

$$\Phi_{\beta}\left(p_{\alpha}\otimes h\right) = p_{\beta^{-1}\alpha\beta}\otimes h. \tag{42}$$

By Theorem 8, $\mathscr{WYD}(k(G)\otimes H)$ is a braided T-category.

4. Braided T-Categories over Weak Long Dimodule Categories

In this section, we introduce the notion of a (left-right) weak α-Long dimodule over a weak crossed Hopf group coalgebra H and prove that the category ${}_H\mathscr{WL}^H$ is a braided T-subcategory of Yetter-Drinfel'd category $\mathscr{WYD}(H\otimes H)$ when H is a quasitriangular and coquasitriangular weak crossed Hopf group coalgebra.

Definition 10. Let H be a weak crossed Hopf group coalgebra over π. For a fixed element $\alpha\in\pi$, a (left-right) weak α-Long dimodule is a couple $V = (V,\rho^V = \{\rho_{\lambda}^V\}_{\lambda\in\pi})$, where V is a left H_{α}-module and, for any $\lambda\in\pi$, $\rho_{\lambda}^V : V\to V\otimes H_{\lambda}$ is a k-linear morphism, such that

(1) V is coassociative in the sense that, for any $\lambda_1,\lambda_2\in\pi$, we have

$$\left(\mathrm{id}_V\otimes\Delta_{\lambda_1,\lambda_2}\right)\circ\rho_{\lambda_1\lambda_2}^V = \left(\rho_{\lambda_1}^V\otimes\mathrm{id}_{H_{\lambda_2}}\right)\circ\rho_{\lambda_2}^V; \tag{43}$$

(2) V is counitary in the sense that

$$\left(\mathrm{id}_V\otimes\varepsilon\right)\circ\rho_1^V = \mathrm{id}_V; \tag{44}$$

(3) V satisfies the following compatible condition:

$$\rho_{\lambda}^V(x\cdot v) = x\cdot v_{(0)}\otimes v_{(1,\lambda)}; \tag{45}$$

where $x\in H_{\alpha}$ and $v\in V$.

Now, we can form the category ${}_H\mathscr{WL}_{\alpha}^H$ of (left-right) weak α-Long dimodules where the composition of morphisms of weak α-Long dimodules is the standard composition of the underlying linear maps.

Let ${}_H\mathscr{WL}^H = \coprod_{\alpha\in\pi}{}_H\mathscr{WL}_{\alpha}^H$, the disjoint union of the categories ${}_H\mathscr{WL}_{\alpha}^H$ for all $\alpha\in\pi$.

Proposition 11. *The category ${}_H\mathscr{WL}^H$ is a monoidal category. Moreover, for any $\alpha,\beta\in G$, let $V\in{}_H\mathscr{WL}_{\alpha}^H$ and let $W\in{}_H\mathscr{WL}_{\beta}^H$. Set*

$$V\widetilde{\otimes}W = \left\{v\otimes w\in V\otimes W\mid v\otimes w\right.$$
$$= 1_{(1,\alpha)}\cdot v\otimes 1_{(2,\beta)}\cdot w \tag{46}$$
$$\left. = \varepsilon\left(w_{(1,1)}\varphi_{\beta^{-1}}\left(v_{(1,1)}\right)\right)v_{(0)}\otimes w_{(0)}\right\}.$$

Then, $V\widetilde{\otimes}W\in{}_H\mathscr{WL}_{\alpha}^H$ with the following structures:

$$x\cdot(v\otimes w) = x_{(1,\alpha)}\cdot v\otimes x_{(2,\beta)}\cdot w,$$
$$\rho_{\lambda}^{V\widetilde{\otimes}W}(v\otimes w) = v_{(0)}\otimes w_{(0)}\otimes w_{(1,\lambda)}\varphi_{\beta^{-1}}\left(v_{(1,\beta\lambda\beta^{-1})}\right), \tag{47}$$

for all $x\in H_{\alpha\beta}$, $v\otimes w\in V\widetilde{\otimes}W$.

Proof. It is straightforward. □

Let (H,σ,R) be a coquasitriangular and quasitriangular weak crossed Hopf group coalgebra with crossing φ. Define a family of vector spaces $H\otimes H=\{(H\otimes H)_\alpha=H_1\otimes H_\alpha\}_{\alpha\in\pi}$, where, the H on the left we consider its coquasitriangular structure and for the right one we consider its quasitriangular structure. Then, $H\otimes H$ is a weak crossed Hopf group coalgebra with the natural tensor product and the crossing $\Phi=\{\mathrm{id}\otimes\varphi_\alpha\}_{\alpha\in\pi}$.

Theorem 12. *Let (H,σ,R) be a weak crossed Hopf group coalgebra with coquasitriangular structure σ and quasitriangular structure R. Then, the category ${}_H\mathscr{W}\mathscr{L}^H$ is a braided T-subcategory of Yetter-Drinfel'd category $\mathscr{W}\mathscr{Y}\mathscr{D}(H\otimes H)$ under the following action and coaction given by*

$$\delta^V_\lambda(v)=a_\alpha\cdot v_{(0)}\otimes v_{(1,1)}\otimes S^{-1}(b_{\lambda^{-1}})=:v_{[0]}\otimes v_{[1,\lambda]},$$
$$(h\otimes x)\rhd{}_\alpha v=\sigma(h,v_{(1,1)})\,x\cdot v_{(0)}, \tag{48}$$

where $h\otimes x\in(H\otimes H)_\alpha$, $h\in H_1$, $x\in H_\alpha$, $v\in V$, and $V\in{}_H\mathscr{W}\mathscr{L}^H_\alpha$.

The braiding on ${}_H\mathscr{W}\mathscr{L}^H$, $\tau_{V,W}:V\otimes W\to{}^VW\otimes V$ is given by

$$\tau_{V,W}(v\otimes w)=\sigma(S(v_{(1,1)}),w_{(1,1)})\,{}^\alpha(b_\beta\cdot w_{(0)})\otimes a_\alpha\cdot v_{(0)}, \tag{49}$$

for all $V\in{}_H\mathscr{W}\mathscr{L}^H_\alpha$, $W\in{}_H\mathscr{W}\mathscr{L}^H_\beta$.

Proof. Obviously, V is a left $(H\otimes H)_\alpha$-module. Then, we show that V satisfies the conditions in Definition 1. First, we need to check that V is coassociative. In fact, for all $v\in V\in{}_H\mathscr{W}\mathscr{L}^H_\alpha$ and $\lambda_1,\lambda_2\in\pi$

$$\begin{aligned}
&(\mathrm{id}_V\otimes\Delta_{\lambda_1,\lambda_2})\circ\delta^V_{\lambda_1\lambda_2}(v)\\
&=a_\alpha\cdot v_{(0)}\otimes v_{(1,1)}\otimes S^{-1}(b_{\lambda_2^{-1}\lambda_1^{-1}(2,\lambda_1^{-1})})\\
&\quad\otimes v_{(2,1)}\otimes S^{-1}(b_{\lambda_2^{-1}\lambda_1^{-1}(1,\lambda_2^{-1})})\\
&=a_\alpha a'_\alpha\cdot v_{(0)}\otimes v_{(1,1)}\otimes S^{-1}(b_{\lambda_1^{-1}})\\
&\quad\otimes v_{(2,1)}\otimes S^{-1}(b'_{\lambda_2^{-1}})\\
&=a_\alpha\cdot(a'_\alpha\cdot v_{(0)})_{(0)}\otimes(a'_\alpha\cdot v_{(0)})_{(1,1)}\\
&\quad\otimes S^{-1}(b_{\lambda_1^{-1}})\otimes v_{(1,1)}\otimes S^{-1}(b'_{\lambda_2^{-1}})\\
&=(\delta^V_{\lambda_1}\otimes\mathrm{id}_{(H\otimes H)_{\lambda_2}})(a'_\alpha\cdot v_{(0)}\otimes v_{(1,1)}\otimes S^{-1}(b'_{\lambda_2^{-1}}))\\
&=(\delta^V_{\lambda_1}\otimes\mathrm{id}_{(H\otimes H)_{\lambda_2}})\circ\delta^V_{\lambda_2}(v).
\end{aligned} \tag{50}$$

Next, one directly shows that counitary condition (17) holds as follows:

$$\begin{aligned}
(\mathrm{id}_V\otimes\varepsilon)\circ\delta^V_1(v)&=a_\alpha\cdot v_{(0)}\varepsilon(m_{(1,1)})\,\varepsilon S^{-1}(b_1)\\
&=a_\alpha\cdot v\varepsilon(b_1)=1_\alpha\cdot v=v.
\end{aligned} \tag{51}$$

Then, we have to prove that crossed condition (18) is satisfied. For all $h\in H_1$, $x\in H_\alpha$, and $v\in V\in{}_H\mathscr{W}\mathscr{L}^H_\alpha$, we have

$$\begin{aligned}
&(h\otimes x)_{(2,\alpha)}\cdot v_{[0]}\otimes(h\otimes x)_{(3,\lambda)}\\
&\quad\times v_{[1,\lambda]}S^{-1}\Phi_{\alpha^{-1}}((h\otimes x)_{(1,\alpha\lambda^{-1}\alpha^{-1})})\\
&=\sigma(h_{(2,1)},(a_\alpha\cdot v_{(0)})_{(1,1)})\;x_{(2,\alpha)}\cdot(a_\alpha\cdot v_{(0)})_{(0)}\\
&\quad\otimes h_{(3,1)}v_{(1,1)}S^{-1}(h_{(1,1)})\otimes x_{(3,\lambda)},\\
&S^{-1}(b_{\lambda^{-1}})\,S^{-1}\psi_{\alpha^{-1}}(x_{(1,\alpha\lambda^{-1}\alpha^{-1})})\\
&=\sigma(h_{(2,1)},v_{(1,1)})\,x_{(2,\alpha)}a_\alpha\cdot v_{(0)}\\
&\quad\otimes h_{(3,1)}v_{(2,1)}S^{-1}(h_{(1,1)})\otimes x_{(3,\lambda)}S^{-1}(b_{\lambda^{-1}}),\\
&S^{-1}\psi_{\alpha^{-1}}(x_{(1,\alpha\lambda^{-1}\alpha^{-1})})\\
&=\sigma(h_{(3,1)},v_{(2,1)})\,x_{(2,\alpha)}a_\alpha\cdot v_{(0)}\\
&\quad\otimes v_{(1,1)}h_{(2,1)}S^{-1}(h_{(1,1)})\otimes x_{(3,\lambda)},\\
&S^{-1}(\psi_{\alpha^{-1}}(x_{(1,\alpha\lambda^{-1}\alpha^{-1})})\,b_{\lambda^{-1}})\\
&=\sigma(h_{(2,1)},v_{(2,1)})\,a_\alpha x_{(1,\alpha)}\cdot v_{(0)}\otimes v_{(1,1)}S^{-1}\varepsilon^t_1(h_{(1,1)})\\
&\quad\otimes x_{(3,\lambda)}S^{-1}(b_{\lambda^{-1}}(x_{(2,\lambda^{-1})}))\\
&=\sigma(h_{(2,1)},v_{(2,1)})\,a_\alpha x_{(1,\alpha)}\cdot v_{(0)}\otimes v_{(1,1)}S^{-1}\varepsilon^t_1(h_{(1,1)})\\
&\quad\otimes S^{-1}\varepsilon^t_{\lambda^{-1}}(x_{(2,1)})\,S^{-1}(b_{\lambda^{-1}})\\
&=\sigma(1'_{(2,1)}h,v_{(2,1)})\;a_\alpha 1_{(1,\alpha)}x\cdot v_{(0)}\otimes v_{(1,1)}1'_{(1,1)}\\
&\quad\otimes S^{-1}(1_{(2,\lambda^{-1})})\,S^{-1}(b_{\lambda^{-1}})\\
&=\sigma(1'_{(2,1)},v_{(2,1)})\,\sigma(h,v_{(3,1)})\,a_\alpha 1_{(1,\alpha)}x\cdot v_{(0)}\\
&\quad\otimes v_{(1,1)}1'_{(1,1)}\otimes S^{-1}(b_{\lambda^{-1}}1_{(2,\lambda^{-1})})\\
&=\varepsilon(v_{(2,1)}1_{(2,1)})\,\sigma(h,v_{(3,1)})\,a_\alpha x\cdot v_{(0)}\\
&\quad\otimes v_{(1,1)}1_{(1,1)}\otimes S^{-1}(b_{\lambda^{-1}})\\
&=\sigma(h,v_{(2,1)})\,a_\alpha x\cdot v_{(0)}\otimes v_{(1,1)}\otimes S^{-1}(b_{\lambda^{-1}})\\
&=\sigma(h,v_{(1,1)})\,a_\alpha\cdot(x\cdot v_{(0)})_{(0)}\\
&\quad\otimes(x\cdot v_{(0)})_{(1,1)}\otimes S^{-1}(b_{\lambda^{-1}})\\
&=\delta^V_\lambda((h\otimes x)\rhd{}_\alpha v).
\end{aligned} \tag{52}$$

Finally, it follows from Proposition 6, the braiding on $\mathscr{W}\mathscr{Y}\mathscr{D}(H\otimes H)$, that the braiding on ${}_H\mathscr{W}\mathscr{L}^H$ is as the following:

$$\begin{aligned}
\tau_{V,W}(v\otimes w)&={}^\alpha(S_{\beta^{-1}}(v_{[1,\beta^{-1}]}))\rhd{}_\beta w\otimes v_{[0]}\\
&=\sigma(S(v_{(1,1)}),w_{(1,1)})\,{}^\alpha(b_\beta\cdot w_{(0)})\otimes a_\alpha\cdot v_{(0)},
\end{aligned} \tag{53}$$

for all $V\in{}_H\mathscr{W}\mathscr{L}^H_\alpha$, $W\in{}_H\mathscr{W}\mathscr{L}^H_\beta$, $v\in V$, and $w\in W$.

This completes the proof. □

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