Research Article

Pricing Options with Credit Risk in Markovian Regime-Switching Markets

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This paper investigates the valuation of European option with credit risk in a reduced form model when the stock price is driven by the so-called Markov-modulated jump-diffusion process, in which the arrival rate of rare events and the volatility rate of stock are controlled by a continuous-time Markov chain. We also assume that the interest rate and the default intensity follow the Vasicek models whose parameters are governed by the same Markov chain. We study the pricing of European option and present numerical illustrations.

1. Introduction

Pricing options with credit risk is an important topic in finance from both theoretical and practical perspectives. Credit risk refers to an investor's risk that a borrower will default on making payments as promised. There are basically two kinds of models to describe the default: structural models and reduced form models. The structural approach was firstly introduced by Merton [1] who investigated European option pricing for modeling single corporate default. The approach is further extended by recent literature: see Ammann [2] and Klein and Inglis [3]. Another tractable approach is called reduced form model, which models the intensity of arrival of default events directly. The reduced form models can be seen in Artzner and Freddy [4], Duffie and Singleton [5], Duffie and Gârleanu [6], and Leung and Kwok [7] and are developed extensively by Su and Wang [8].

Recently, Markovian regime-switching models have attracted attention among researchers and practitioners in economics and mathematics. Elliott et al. [9] introduce a self-calibrating model for short-term interest rate by assuming that the short rate is governed by a finite state space Markov process. Elliott et al. [10] and Elliott and Osakwe [11] use Markov-modulated market parameters to capture the time inhomogeneity generated by the financial market.

Elliott et al. [12] perform the valuation of option under a generalized Markov-modulated jump-diffusion model. Siu et al. [13] consider the pricing currency options under two-factor Markov-modulated stochastic volatility models. Bo et al. [14] derive the valuation of currency option when the spot foreign exchange rates follow Markov-modulated jump-diffusion model.

In this paper, we investigate the valuation of European option with credit risk in a reduced form model in Markovian regime-switching markets. We assume that the recovery rate is constant; that is, when the writer of the option defaults, a specified constant fraction times the payoff will be paid at maturity. In order to incorporate both rare events and time-inhomogeneity in finance market, we model the stock price by the so-called Markov-modulated jump-diffusion process, in which rare events are described as a compound Poisson process and the arrival rate of Poisson process and the volatility rate of stock are governed by a continuous-time Markov chain. The states of Markov chain can be interpreted as the states of the market. The transitions of the states of the market may describe changes of economy, finance, business cycles, and other conditions. In addition, we assume that the interest rate and the default intensity both follow the Vasicek models and the parameters of models are correlated with the same Markov chain. By the method of changing measures,

we obtain the closed form formula for the valuation of the European option.

The rest of this paper is organized as follows. Section 2 presents the model description. In Section 3, by applying Girsanov's measure changing theorem, we derive the formula of the pricing of European-style call option. We provide numerical analysis in Section 4. The concluding remarks are contained in Section 5.

2. The Model Description

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where P is a neutral-risk probability measure. Define $\xi = \{\xi_t, t \geq 0\}$ on $(\Omega, \mathcal{F}, P)$ as a continuous-time, finite state Markov chain with n-state space E. We interpret the state of ξ as the states of the economy as follows (Elliott et al. [10] and Elliott and Osakwe [11]). Without loss of generality, we take the state space of ξ to be a finite set of unite vectors $\{e_1, e_2, \ldots, e_n\}$ with $e_i = (0, \ldots, 1, \ldots, 0) \in R^n$. And ξ has the following semimartingale representation:

$$d\xi_t = Q\xi_t \, dt + dM_t, \tag{1}$$

where $Q = (q_{ij})_{i,j=1,2,\ldots,n}$ is Q-matrix of ξ and $M = \{M_t\}_{0 \leq t \leq T}$ is an R^n-valued martingale with respect to the filtration generated by $\{\xi_t, 0 \leq t \leq T\}$ under P. Suppose that the stock price S_t and interest rate r_t satisfy the following stochastic differential equations (SDE):

$$\frac{dS_t}{S_{t-}} = (r_t - k\nu_t) \, dt + \sigma_{1t} dW_{1t} + \left(e^{Y_{t-}} - 1\right) dN(t), \tag{2}$$

$$dr_t = (a_t - b_t r_t) \, dt + \sigma_{2t} \, dW_{2t},$$

where W_{1t}, W_{2t} are standard Brownian motions and σ_{1t}, σ_{2t} are the stochastic volatility of the stock and the interest rate, respectively. $\{N_t\}_{0 \leq t \leq T}$ is Poisson process with the stochastic jump intensity $\{\nu_t\}_{0 \leq t \leq T}$, and the jump amplitude is controlled by $\{Y_t\}$. Y_s and Y_t for $s \neq t$ independently identify distribution, and write $k = Ee^{Y_i} - 1$. Moreover, Y_t, N_t are assumed to be mutually independent. $\sigma_{1t}, \sigma_{2t}, \nu_t, a_t, b_t$ are controlled by ξ_t, that is,

$$\sigma_{1t} = \langle \sigma_1, \xi_t \rangle, \quad \sigma_1 = (\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1n}) \in (0, \infty)^n,$$

$$\sigma_{2t} = \langle \sigma_2, \xi_t \rangle, \quad \sigma_2 = (\sigma_{21}, \sigma_{22}, \ldots, \sigma_{2n}) \in (0, \infty)^n,$$

$$\nu_t = \langle \nu, \xi_t \rangle, \quad \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in (0, \infty)^n, \tag{3}$$

$$a_t = \langle a, \xi_t \rangle, \quad a = (a_1, a_2, \ldots, a_n) \in (0, \infty)^n,$$

$$b_t = \langle b, \xi_t \rangle, \quad b = (b_1, b_2, \ldots, b_n) \in (0, \infty)^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n. Let τ denote the default time of the writer of the option with default intensity process λ_t, and λ_t is given by

$$d\lambda_t = (\alpha_t - \beta_t \lambda_t) \, dt + \sigma_{3t} \, dW_{3t}, \tag{4}$$

where W_{3t} is standard Brownian motion and σ_{3t} is the stochastic volatility of default intensity. σ_{3t}, α_t, and β_t are also controlled by ξ_t and satisfy

$$\sigma_{3t} = \langle \sigma_3, \xi_t \rangle, \quad \sigma_3 = (\sigma_{31}, \sigma_{32}, \ldots, \sigma_{3n}) \in (0, \infty)^n,$$

$$\alpha_t = \langle \alpha, \xi_t \rangle, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (0, \infty)^n, \tag{5}$$

$$\beta_t = \langle \beta, \xi_t \rangle, \quad \beta = (\beta_1, \beta_2, \ldots, \beta_n) \in (0, \infty)^n.$$

Moreover, we assume that N_t, Y_t are independent of W_{1t}, W_{2t}, and W_{3t} and the covariance matrix of the Brownian motion (W_{1t}, W_{2t}, W_{3t}) is

$$\begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix} t. \tag{6}$$

The filtration $\mathcal{F}_t$ is generated by $\mathcal{F}_t = \mathcal{F}_t^S \vee \mathcal{F}_t^r \vee \mathcal{F}_t^\lambda \vee \mathcal{F}_T^\xi \vee \mathcal{H}_t$, where $\mathcal{F}_t^S = \sigma(S_s, \ 0 \leq s \leq t)$, $\mathcal{F}_t^r = \sigma(r_s, \ 0 \leq s \leq t)$, $\mathcal{F}_t^\lambda = \sigma(\lambda_s, \ 0 \leq s \leq t)$, $\mathcal{F}_t^\xi = \sigma(\xi_s, \ 0 \leq s \leq t)$, and $\mathcal{H}_t = \sigma(1_{(\tau \leq s)}, s \leq t)$.

3. Pricing Options with Credit Risk

We consider the case of a European call option with credit risk. Assume that the recovery rate is a constant w. When the seller of option defaults, the payoff is given by w times the payoff of the default-free option at maturity. Therefore, by the risk neutral pricing theorem, the valuation of the European call option at time t, with strike price K and maturity T, is given by

$$C(t, T)$$
$$= E\left[e^{-\int_t^T r_s ds} \left(w(S_T - K)^+ 1_{(\tau \leq T)} + (S_T - K)^+ 1_{(\tau > T)}\right) \mid \mathcal{F}_t\right]$$
$$= E\left[e^{-\int_t^T r_s ds} \left(w(S_T - K)^+ + (1-w)(S_T - K)^+ 1_{(\tau > T)}\right) \mid \mathcal{F}_t\right]. \tag{7}$$

Following Lando [15],

$$E\left[e^{-\int_t^T r_s ds}(S_T - K)^+ 1_{(\tau > T)} \mid \mathcal{F}_t\right]$$
$$= 1_{(\tau > t)} E\left[e^{-\int_t^T (r_s + \lambda_s) ds}(S_T - K)^+ \mid \mathcal{F}_t\right]. \tag{8}$$

We can obtain the following expression:

$$C(t, T) = wE\left[e^{-\int_t^T r_s \, ds}(S_T - K)^+ \mid \mathcal{F}_t\right]$$
$$+ (1-w) 1_{(\tau > t)} E\left[e^{-\int_t^T (r_s + \lambda_s) ds}(S_T - K)^+ \mid \mathcal{F}_t\right]$$
$$= I_1 + I_2. \tag{9}$$

Next, we calculate I_1 and I_2, respectively. For $s > t$, we have

$$r_s = e^{-\int_t^s b_u \, du} r_t + \int_t^s e^{-\int_v^s b_u du} a_v \, dv + \int_t^s e^{-\int_v^s b_u du} \sigma_{2v} \, dW_{2v}. \tag{10}$$

Integrated from t to T in both sides of (10),

$$\int_t^T r_s \, ds = \int_t^T e^{-\int_t^s b_u du} \, ds r_t + \int_t^T a_v \, dv \int_v^T e^{-\int_v^s b_u du} \, ds$$

$$+ \int_t^T \sigma_{2v} \, dW_{2v} \int_v^T e^{-\int_v^s b_u du} \, ds. \tag{11}$$

Let $M(x, y, T) = \int_y^T e^{-\int_y^s x_u du} \, ds$; then

$$\int_t^T r_s \, ds = M(b, t, T) r_t + \int_t^T a_v M(b, v, T) \, dv$$

$$+ \int_t^T \sigma_{2v} M(b, v, T) \, dW_{2v}. \tag{12}$$

Similarly,

$$\int_t^T \lambda_s ds = M(\beta, t, T) \lambda_t + \int_t^T \alpha_v M(\beta, v, T) \, dv$$

$$+ \int_t^T \sigma_{3v} M(\beta, v, T) \, dW_{3v}. \tag{13}$$

From (12) and (13), we have that

$$Z(t, T) := E\left[\exp\left\{ -\int_t^T (r_s + \lambda_s) \, ds \right\} \mid \mathscr{F}_t \right]$$

$$= \exp\left\{ -M(b, t, T) r_t - M(\beta, t, T) \lambda_t \right.$$

$$- \int_t^T a_u M(b, u, T) \, du$$

$$\left. - \int_t^T \alpha_u M(\beta, u, T) \, du \right\}$$

$$\times \exp\left\{ \frac{1}{2} \int_t^T \sigma_{2u}^2 M^2(b, u, T) \, du \right.$$

$$\left. + \frac{1}{2} \int_t^T \sigma_{3u}^2 M^2(\beta, u, T) \, du \right\}$$

$$\times \exp\left\{ \rho_{23} \int_t^T \sigma_{2u} \sigma_{3u} M(b, u, T) M(\beta, u, T) \, du \right\}. \tag{14}$$

Thus, we can define the probability measure Q by

$$\left. \frac{dQ}{dP} \right|_{\mathscr{F}_t} = \frac{\exp\left\{ -\int_t^T (r_s + \lambda_s) \, ds \right\}}{E\left[\exp\left\{ -\int_t^T (r_s + \lambda_s) \, ds \right\} \mid \mathscr{F}_t \right]}$$

$$= \exp\left\{ -\frac{1}{2} \int_t^T \sigma_{2u}^2 M^2(b, u, T) \, du \right.$$

$$\left. - \frac{1}{2} \int_t^T \sigma_{3u}^2 M^2(\beta, u, T) \, du \right\}$$

$$\times \exp\left\{ -\rho_{23} \int_t^T \sigma_{2u} \sigma_{3u} M(b, u, T) M(\beta, u, T) \, du \right\}$$

$$\times \exp\left\{ -\int_t^T \sigma_{2v} M(b, v, T) \, dW_{2v} \right.$$

$$\left. - \int_t^T \sigma_{3v} M(\beta, v, T) \, dW_{3v} \right\}. \tag{15}$$

By Girsanov's theorem,

$$dW_{2t}^Q = dW_{2t} + \sigma_{2t} M(b, t, T) \, dt + \rho_{23} \sigma_{3t} M(\beta, t, T) \, dt,$$

$$dW_{3t}^Q = dW_{3t} + \sigma_{3t} M(\beta, t, T) \, dt + \rho_{23} \sigma_{2t} M(b, t, T) \, dt,$$

$$dW_{1t}^Q = dW_{1t} + \rho_{12} M_1(t) \, dt + \rho_{13} M_2(t) \, dt, \tag{16}$$

where

$$M_1(t) = \sigma_{2t} M(b, t, T) + \rho_{23} \sigma_{3t} M(\beta, t, T),$$

$$M_2(t) = \sigma_{3t} M(\beta, t, T) + \rho_{23} \sigma_{2t} M(b, t, T). \tag{17}$$

W_{1t}^Q, W_{2t}^Q, and W_{3t}^Q are standard Brownian motions under probability measure Q, and $(W_{1t}^Q, W_{2t}^Q, W_{3t}^Q)$ has the same covariance matrix as (W_{1t}, W_{2t}, W_{3t}). Therefore,

$$E\left[e^{-\int_t^T (r_s + \lambda_s) ds} (S_T - K)^+ \mid \mathscr{F}_t \right]$$

$$= Z(t, T) E^Q\left((S_T - K)^+ \mid \mathscr{F}_t \right). \tag{18}$$

In addition,

$$E^Q\left((S_T - K)^+ \mid \mathscr{F}_t \right)$$

$$= E^Q\left(S_T 1_{(S_T \geq K)} \mid \mathscr{F}_t \right) - K E^Q\left(1_{(S_T \geq K)} \mid \mathscr{F}_t \right). \tag{19}$$

By the solution of SDE (2),

$$S_T = S_t \exp\left\{ \int_t^T \left(r_s - k v_s - \frac{1}{2} \sigma_{1s}^2 \right) ds \right.$$

$$\left. + \int_t^T \sigma_{1s} dW_{1s} + \int_t^T Y_{s-} dN_s \right\}. \tag{20}$$

Under Q,

$$S_T = S_t \exp\left\{\Lambda(t,T) + \int_t^T \sigma_{1s} dW_{1s}^Q \right.$$
$$\left. + \int_t^T \sigma_{2s} M(b,s,T) dW_{2s}^Q + \int_t^T Y_{s-} dN_s \right\}, \tag{21}$$

where

$$\Lambda(t,T) = -\int_t^T \left(k\nu_s + \frac{1}{2}\sigma_{1s}^2\right) ds + M(b,t,T) r_t$$
$$+ \int_t^T a_s M(b,s,T) ds - \int_t^T \sigma_{2s}^2 M^2(b,s,T) ds$$
$$- \rho_{12} \int_t^T \sigma_{1s} M_1(s) ds - \rho_{13} \int_t^T \sigma_{1s} M_2(s) ds$$
$$- \rho_{23} \int_t^T \sigma_{2s}\sigma_{3s} M(b,s,T) M(\beta,s,T) ds. \tag{22}$$

So,

$$P^Q\left(S_T \geq K \mid \mathscr{F}_t\right)$$
$$= P^Q\left(\int_t^T \sigma_{1s} dW_{1s}^Q + \int_t^T \sigma_{2s} M(b,s,T) dW_{2s}^Q \right.$$
$$\left. \geq \ln(K/S_t) - \Lambda(t,T) - \int_t^T Y_s dN_s\right)$$
$$= P^Q\left(\frac{\int_t^T \sigma_{1s} dW_{1s}^Q + \int_t^T \sigma_{2s} M(b,s,T) dW_{2s}^Q}{\sqrt{\Delta}} \right. \tag{23}$$
$$\left. \leq \frac{\ln(S_t/K) + \Lambda(t,T) + \int_t^T Y_s dN_s}{\sqrt{\Delta}}\right)$$
$$= \sum_{n=0}^{\infty} \frac{\left(\int_t^T \nu_s ds\right)^n}{n!} e^{-\int_t^T \nu_s ds} E\left(N(d_1)\right),$$

where

$$\Delta = \int_t^T \sigma_{1s}^2 ds + \int_t^T \sigma_{2s}^2 M^2(b,s,T) ds$$
$$+ 2\rho_{12} \int_t^T \sigma_{1s}\sigma_{2s} M(b,s,T) ds, \tag{24}$$
$$d_1 = \frac{\ln(S_t/K) + \Lambda(t,T) + \sum_{j=1}^n Y_j}{\sqrt{\Delta}}.$$

$E(\cdot)$ is the expectation of Y_j under P, and $N(\cdot)$ is distribution function of standard normal distribution. On the other hand, let

$$\mathscr{L}(t,T)$$
$$:= E^Q\left(S_T \mid \mathscr{F}_t\right)$$
$$= S_t e^{\Lambda(t,T)} \exp\left\{\frac{1}{2}\int_t^T \sigma_{1s}^2 ds + \frac{1}{2}\int_t^T \sigma_{2s}^2 M^2(b,s,T) ds\right\}$$
$$\times \exp\left\{\rho_{12} \int_t^T \sigma_{1s}\sigma_{2s} M(b,s,T) ds + k\int_t^T \nu_s ds\right\}. \tag{25}$$

We can define the probability measure R as follows:

$$\left.\frac{dR}{dQ}\right|_{\mathscr{F}_t} = \frac{S_T}{E^Q(S_T \mid \mathscr{F}_t)}$$
$$= \exp\left\{\int_t^T \sigma_{1s} dW_{1s}^Q + \int_t^T \sigma_{2s} M(b,s,T) dW_{2s}^Q \right.$$
$$\left. -\frac{1}{2}\int_t^T \sigma_{1s}^2 ds\right\}$$
$$\times \exp\left\{-\frac{1}{2}\int_t^T \sigma_{2s}^2 M^2(b,s,T) ds \right. \tag{26}$$
$$\left. -\rho_{12} \int_t^T \sigma_{1s}\sigma_{2s} M(b,s,T) ds\right\}$$
$$\times \exp\left\{\int_t^T Y_s dN_s - k\int_t^T \nu_s ds\right\}.$$

By Girsanov's theorem, we have that

$$dW_{1t}^R = dW_{1t}^Q - \sigma_{1t} dt - \rho_{12}\sigma_{2t} M(b,t,T) dt,$$
$$dW_{2t}^R = dW_{2t}^Q - \sigma_{2t} M(b,t,T) dt - \rho_{12}\sigma_{1t} dt, \tag{27}$$

where, under R, W_{1t}^R, W_{2t}^R are standard Brownian motions and their correlation coefficient is ρ_{12}, N_t is Poisson process with intensity $(k+1)\nu_t$, and the density function of Y_1 is $e^y f(y)/(k+1)$, where $f(\cdot)$ is the density function of Y_1 under P. Since, under R,

$$S_T = S_t \exp\left\{\Lambda(t,T) + \Delta + \int_t^T \sigma_{1s} dW_{1s}^R \right.$$
$$\left. + \int_t^T \sigma_{2s} M(b,s,T) dW_{2s}^R + \int_t^T Y_{s-} dN_s\right\}, \tag{28}$$

then

$$E^Q \left(S_T 1_{(S_T \geq K)} \mid \mathcal{F}_t \right)$$

$$= \mathcal{L}(t, T) P^R \left(S_T \geq K \mid \mathcal{F}_t \right)$$

$$= \mathcal{L}(t, T) P^R \left(\int_t^T \sigma_{1s} dW_{1s}^R + \int_t^T \sigma_{2s} M(b, s, T) dW_{2s}^R \right.$$

$$\geq \ln \left(\frac{K}{S_t} \right) - \Lambda(t, T) - \Delta - \int_t^T Y_s dN_s \right)$$

$$= \mathcal{L}(t, T) P^R \left(\frac{\int_t^T \sigma_{1s} dW_{1s}^R + \int_t^T \sigma_{2s} M(b, s, T) dW_{2s}^R}{\sqrt{\Delta}} \right.$$

$$\leq \frac{\ln(S_t/K) + \Lambda(t, T) + \Delta + \int_t^T Y_s dN_s}{\sqrt{\Delta}} \right)$$

$$= \mathcal{L}(t, T) \sum_{n=0}^{\infty} \frac{\left(\int_t^T v_s ds \right)^n}{n!} e^{-\int_t^T v_s ds} E^R \left(N(d_2) \right),$$

$$(29)$$

where

$$d_2 = \frac{\ln(S_t/K) + \Lambda(t, T) + \Delta + \sum_{j=1}^n Y_j}{\sqrt{\Delta}}$$

$$= d_1 + \sqrt{\Delta}$$

$$(30)$$

and $E^R(\cdot)$ is the expectation of Y_j under R. By (23) and (29), we have the following lemma.

Lemma 1. *Consider the Following:*

$$E \left[e^{-\int_t^T (r_s + \lambda_s) ds} (S_T - K)^+ \mid \mathcal{F}_t \right]$$

$$= Z(t, T)$$

$$\times \sum_{n=0}^{\infty} \frac{\left(\int_t^T v_s ds \right)^n}{n!}$$

$$\times e^{-\int_t^T v_s ds} \left[\mathcal{L}(t, T) E^R \left(N(d_2) \right) - K E \left(N(d_1) \right) \right].$$

$$(31)$$

In the following; in order to calculate I_1, we write

$$E \left[e^{-\int_t^T r_s ds} (S_T - K)^+ \mid \mathcal{F}_t \right]$$

$$= E \left[e^{-\int_t^T r_s ds} S_T 1_{(S_T \geq K)} \mid \mathcal{F}_t \right]$$

$$- K E \left[e^{-\int_t^T r_s ds} 1_{(S_T \geq K)} \mid \mathcal{F}_t \right]$$

$$= I_3 - I_4.$$

$$(32)$$

Denote by $P(t, T)$ the value at time t of a T maturity zero coupon bond whose face value is 1. Then

$$P(t, T) = E \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right].$$

$$(33)$$

From (12),

$$P(t, T) = \exp \left\{ -M(b, t, T) r_t - \int_t^T a_v M(b, v, T) dv \right.$$

$$\left. + \frac{1}{2} \int_t^T \sigma_{2v}^2 M^2(b, v, T) dv \right\},$$

$$(34)$$

$$\frac{dP(t, T)}{P(t, T)} = r_t dt - M(b, t, T) \sigma_{2t} dW_{2t}.$$

$$(35)$$

Define the Radon-Nikodym derivative given by

$$\frac{dQ^T}{dP} = \frac{e^{-\int_t^T r_s ds}}{E \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right]}$$

$$= \exp \left\{ -\int_t^T M(b, v, T) \sigma_{2v} dW_{2v} \right.$$

$$\left. - \frac{1}{2} \int_t^T \sigma_{2v}^2 M^2(b, v, T) dv \right\},$$

$$(36)$$

and by Girsanov's theorem, under Q^T,

$$dW_{2t}^{Q^T} = dW_{2t} + \sigma_{2t} M(b, t, T) dt$$

$$dW_{1t}^{Q^T} = dW_{1t} + \rho_{12} \sigma_{2t} M(b, t, T) dt.$$

$$(37)$$

Thus S_T can be rewritten as

$$S_T = S_t \exp \left\{ \mathcal{A}(t, T) + \int_t^T M(b, v, T) \sigma_{2v} dW_{2v}^{Q^T} \right.$$

$$\left. + \int_t^T \sigma_{1v} dW_{1v}^{Q^T} + \int_t^T Y_{s-} dN_s \right\},$$

$$(38)$$

where

$$\mathcal{A}(t, T) = -\int_t^T \left(k v_s + \frac{1}{2} \sigma_{1s}^2 \right) ds + M(b, t, T) r_t$$

$$+ \int_t^T a_s M(b, s, T) ds - \int_t^T \sigma_{2s}^2 M^2(b, s, T) ds$$

$$- \rho_{12} \int_t^T \sigma_{1s} \sigma_{2s} M(b, s, T) ds.$$

$$(39)$$

Then

$$P^{Q^T}\left(S_T \geq K \mid \mathcal{F}_t\right)$$

$$= P^{Q^T}\left(\int_t^T \sigma_{1s} dW_{1s}^{Q^T} + \int_t^T \sigma_{2s} M\left(b, s, T\right) dW_{2s}^{Q^T}\right.$$

$$\geq \ln\left(\frac{K}{S_t}\right) - \mathcal{A}\left(t, T\right) - \int_t^T Y_{s-} dN_s\right)$$

$$= P^{Q^T}\left(\frac{\int_t^T \sigma_{1s} dW_{1s}^{Q^T} + \int_t^T \sigma_{2s} M\left(b, s, T\right) dW_{2s}^{Q^T}}{\sqrt{\Delta}}\right. \qquad (40)$$

$$\left.\leq \frac{\ln\left(S_t/K\right) + \mathcal{A}\left(t, T\right) + \int_t^T Y_{s-} dN_s}{\sqrt{\Delta}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{\left(\int_t^T \nu_s ds\right)^n}{n!} e^{-\int_t^T \nu_s ds} E\left(N\left(d_3\right)\right),$$

where $E(\cdot)$ is the expectation of Y_j under P,

$$d_3 = \frac{\ln\left(S_t/K\right) + \mathcal{A}\left(t, T\right) + \sum_{j=1}^n Y_j}{\sqrt{\Delta}}. \qquad (41)$$

Therefore,

$$I_4 = KP\left(t, T\right) E^{Q^T}\left(1_{(S_T \geq K)} \mid \mathcal{F}_t\right)$$

$$\qquad (42)$$

$$= KP\left(t, T\right) \sum_{n=0}^{\infty} \frac{\left(\int_t^T \nu_s ds\right)^n}{n!} e^{-\int_t^T \nu_s ds} E\left(N\left(d_3\right)\right).$$

Finally, let

$$A\left(t, T\right) := E\left(e^{-\int_t^T r_s ds} S_T \mid \mathcal{F}_t\right)$$

$$= S_t \exp\left\{-\int_t^T \left(k\nu_s + \frac{1}{2}\sigma_{1s}^2\right) ds \right. \qquad (43)$$

$$\left. + \frac{1}{2}\int_t^T \sigma_{1s}^2 ds + k\int_t^T \nu_s ds\right\}.$$

We define

$$\frac{dQ^S}{dP} = \frac{e^{-\int_t^T r_s ds} S_T}{E\left[e^{-\int_t^T r_s ds} S_T \mid \mathcal{F}_t\right]}$$

$$= \exp\left\{\int_t^T \sigma_{1\nu} dW_{1\nu} - \frac{1}{2}\int_t^T \sigma_{2\nu}^2 d\nu \right. \qquad (44)$$

$$\left. + \int_t^T Y_{s-} dN\left(s\right) - k\int_t^T \nu_s ds\right\},$$

and by Girsanov's theorem, under Q^S,

$$dW_{1t}^{Q^S} = dW_{1t} - \sigma_{1t} dt$$

$$\qquad (45)$$

$$dW_{2t}^{Q^S} = dW_{2t} - \rho_{12}\sigma_{1t} dt,$$

and, under Q^S, N_t is Poisson process with intensity $(k+1)\nu_t$, and the density function of Y_1 is $e^y f(y)/(k+1)$, where $f(\cdot)$ is the density function of Y_1 under P. Since, under Q^S,

$$S_T = S_t \exp\left\{\mathcal{A}\left(t, T\right) + \Delta + \int_t^T M\left(b, \nu, T\right) \sigma_{2\nu} dW_{2\nu}^{Q^S}\right.$$

$$\qquad (46)$$

$$\left. + \int_t^T \sigma_{1\nu} dW_{1\nu}^{Q^S} + \int_t^T Y_{s-} dN_s\right\},$$

then

$$P^{Q^S}\left(S_T \geq K \mid \mathcal{F}_t\right)$$

$$= P^{Q^S}\left(\int_t^T \sigma_{1s} dW_{1s}^{Q^S} + \int_t^T \sigma_{2s} M\left(b, s, T\right) dW_{2s}^{Q^S}\right.$$

$$\geq \ln\left(\frac{K}{S_t}\right) - \mathcal{A}\left(t, T\right) - \Delta - \int_t^T Y_{s-} dN_s\right)$$

$$= P^{Q^T}\left(\frac{\int_t^T \sigma_{1s} dW_{1s}^{Q^S} + \int_t^T \sigma_{2s} M\left(b, s, T\right) dW_{2s}^{Q^S}}{\sqrt{\Delta}}\right.$$

$$\left.\leq \frac{\ln\left(S_t/K\right) + \mathcal{A}\left(t, T\right) + \Delta + \int_t^T Y_{s-} dN_s}{\sqrt{\Delta}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{\left(\int_t^T \nu_s ds\right)^n}{n!} e^{-\int_t^T \nu_s ds} E^S\left(N\left(d_4\right)\right),$$

$$\qquad (47)$$

where $E^S(\cdot)$ is the expectation of Y_j under Q^S,

$$d_4 = \frac{\ln\left(S_t/K\right) + \mathcal{A}\left(t, T\right) + \Delta + \sum_{j=1}^n Y_j}{\sqrt{\Delta}}$$

$$\qquad (48)$$

$$= d_3 + \sqrt{\Delta}.$$

From (42) and (33), we can conclude the following lemma.

Lemma 2. *Consider the following:*

$$E\left[e^{-\int_t^T r_s ds}\left(S_T - K\right)^+ \mid \mathcal{F}_t\right]$$

$$= \sum_{n=0}^{\infty} \frac{\left(\int_t^T \nu_s ds\right)^n}{n!}$$

$$\times e^{-\int_t^T \nu_s ds}\left[A\left(t, T\right) E^S\left(N\left(d_4\right)\right) - KP\left(t, T\right) E\left(N\left(d_3\right)\right)\right].$$

$$\qquad (49)$$

Combined with the previous lemmas, the price of European call option at time t is given by the following theorem.

TABLE 1: The parameter values.

Parameter name	Value in state e_1	Value in state e_2
Volatility of S	$\sigma_{11} = 0.2$	$\sigma_{12} = 0.4$
Jump intensity	$\nu_1 = 15$	$\nu_2 = 30$
Speed of reversion of r	$b_1 = 2$	$b_2 = 1$
Long-term average of r	$a_1/b_1 = 0.04$	$a_2/b_2 = 0.02$
Volatility of r	$\sigma_{21} = 0.15$	$\sigma_{22} = 0.3$
Speed of reversion of λ	$\beta_1 = 1.5$	$\beta_2 = 2$
Long-term average of λ	$\alpha_1/\beta_1 = 0.01$	$\alpha_2\beta_2 = 0.02$
Volatility of λ	$\sigma_{31} = 0.25$	$\sigma_{32} = 0.45$
Initial stock price	$S_0 = 100$	
Initial interest rate	$r_0 = 0.04$	
Initial default intensity	$\lambda_0 = 0.5$	
Mean jump size	$\mu_J = 0$	
Standard deviation of jump size	$\sigma_J = 0.1$	

Theorem 3. *The price of European call option with credit risk at time t is*

$$
\begin{aligned}
&C(t,T) \\
&= \sum_{n=0}^{\infty} \frac{\left(\int_t^T \nu_s \, ds\right)^n}{n!} \\
&\quad \times e^{-\int_t^T \nu_s ds} \left[w \left(A(t,T) E^S \left(N(d_4) \right) \right.\right. \\
&\qquad\qquad\qquad \left. - KP(t,T) E\left(N(d_3)\right) \right) \\
&\qquad\qquad + (1-w) 1_{(\tau>t)} Z(t,T) \\
&\qquad\qquad\quad \times \left(\mathcal{L}(t,T) E^R \left(N(d_2) \right) \right. \\
&\qquad\qquad\qquad \left.\left. - KE\left(N(d_1) \right) \right) \right].
\end{aligned}
\tag{50}
$$

4. Numerical Analysis

In this section, we will employ Monte Carlo simulation and perform a numerical analysis for European-style call option with credit risk under the Markov-modulated jump-diffusion model. We consider that the Markov chain ξ has two states, which means that macroeconomic shifts between the two states: e_1 ("boom" or "good") and e_2 ("recession" or "bad"). We assume that the current economy is in boom and that the transition probability matrix of the two-state Markov chain ξ is given by

$$
\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}.
\tag{51}
$$

Assume that Y_t is normally distributed with mean μ_J and standard deviation σ_J. We will adopt the specimen values for the model parameters as Table 1. We consider a range of spot-to-strike ratios S_0/K from 0.8 to 1.2 and assume that one year has 252 trading days. We perform 10 000 simulations for computing the option price.
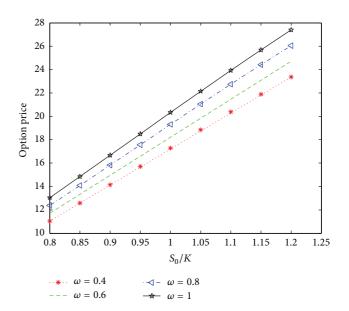

FIGURE 1: The option price with different recovery rate for $T = 1$, $\rho_{12} = 0.7$, $\rho_{13} = 0.5$, and $\rho_{23} = 0.6$.
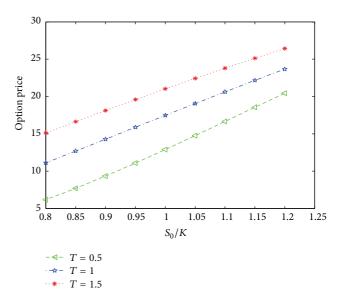

FIGURE 2: The option price with different maturity T for $\omega = 0.4$, $\rho_{12} = 0.7$, $\rho_{13} = 0.5$, and $\rho_{23} = 0.6$.

For each $\omega = 0.4, 0.6, 0.8, 1$, we consider the impact of the spot-to-strike ratio on the option price. From Figure 1, we observe that the option price increases as the spot-to-strike ratio increases. We can also see that the greater the ω, the greater the option price. When $\omega = 1$, it follows that there is not default risk. It is a result of the fact that the payoff at the maturity will increase as the recovery rate increases. Figure 2 depicts the plot of the option price against the spot-to-strike ratio for each maturity $T = 0.5, 1, 1.5$. From these plots, we find that the longer the maturities, the greater the option price.

In the following, we compare the option price with different correlation coefficients for fixed maturity $T = 1$
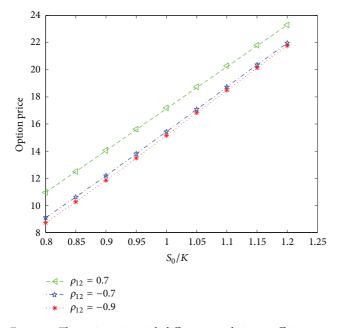
FIGURE 3: The option price with different correlation coefficients ρ_{12} for $T = 1$, $\omega = 0.4$, $\rho_{13} = 0.5$, and $\rho_{23} = 0.6$.
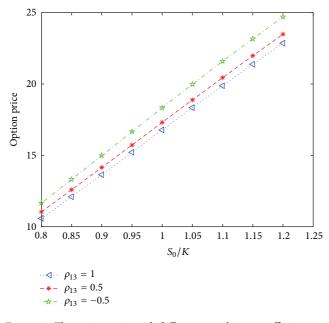

FIGURE 5: The option price with different ν for $\omega = 0.4$, $\rho_{12} = 0.7$, $\rho_{13} = 0.5$, and $\rho_{23} = 0.6$.

FIGURE 4: The option price with different correlation coefficients ρ_{13} for $T = 1$, $\omega = 0.4$, $\rho_{12} = 0.7$, and $\rho_{23} = 0.6$.

FIGURE 6: The option price in Markov-modulated model and non-Markov-modulated models.

and recovery rate $\omega = 0.4$. Figure 3 illustrates that option price increases as correlation coefficient ρ_{12} increases. From Figure 4, we can also see that option price decreases as correlation coefficient ρ_{13} increases.

In Figure 5, we present how the option prices vary with the changes of the annual jump intensity ν. Figure 5 displays a large change in the option price due to the variation of the jump intensity. The option price increases as jump intensity increases.
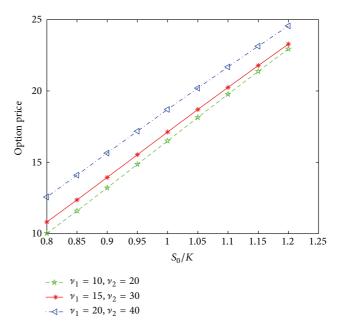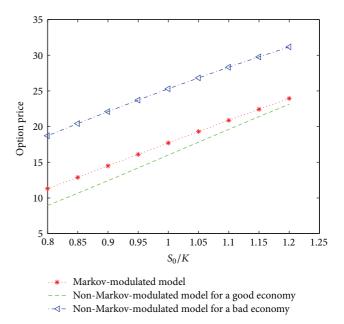
Finally, we investigate the difference of the option price in Markov-modulated model and non-Markov-modulated models. From Figure 6, we can observe that the call option price in good economy is lower than that in bad economy. The reason is that when economy is bad, the volatility of the risky asset is great so that the option price is higher. In our model, we assume that the current economy is good, so Markov-modulated model is close to the model for a good economy, while the two plots and the model for a bad economy are relatively far apart. In Markov-modulated model, we take into

consideration the changes of the state of the economy, so Markov-modulated model is more in accord with the reality for the pricing of the defaultable options.

5. Conclusion

The pricing of European option with credit risk in a reduced form model was studied, while the stock price was driven by Markov-modulated jump-diffusion models. The interest rate and the default intensity followed the Vasicek models, and the parameters were also controlled by the same Markov chain. Compared with most of the credit risk models, the main advantage is that we incorporated Markov-modulated rates into the models. We applied Girsanov's theorem to obtain the equivalent martingale measure and derived the closed form formula for the valuation of the European option. Finally, from the numerical illustrations, we obtain the effects of the recovery rate, the maturity, the correlation coefficients, and the jump intensity of stock on the option price.

Acknowledgments

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