

Research Article

Stability Analysis and Control of a New Smooth Chua's System

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This paper is concerned with the stability analysis and control of a new smooth Chua's system. Firstly, the chaotic characteristic of the system is confirmed with the aid of the Lyapunov exponents. Secondly, it is proved that the system has globally exponential attractive set and positive invariant set. For the three unstable equilibrium points of the system, a linear controller is designed to globally exponentially stabilize the equilibrium points. Then, a linear controller and an adaptive controller are, respectively, proposed so that two similar types of smooth Chua's systems are globally synchronized, and the estimation errors of the uncertain parameters converge to zero as t tends to infinity. Finally, the numerical simulations are also presented.

1. Introduction

It is well known that Chua's system is the first analog circuit to realize chaos in experiments. The original Chua's system is described by the following ordinary differential equations [1]:

$$\begin{aligned}\dot{x} &= p(y - x - g(x)), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -qy,\end{aligned}\tag{1}$$

where $x, y, z \in R$ are state variables and $g(x) = G_b x + (1/2)(G_a - G_b)(|x - E| - |x + E|)$, $p > 0$, $q > 0$, and G_a, G_b are constants. Due to the form of a simple circuit, there are a large literature on the dynamical behavior of Chua's system [2–8]. By changing the parameters or the corresponding functions of Chua's system, the chaotic phenomenon is very rich, and

it is more convenient to study the chaotic mechanism and characteristics [6–8].

For the chaotic systems, Lagrange stability, stability of equilibrium points and, synchronization are three important problems which attracted more and more attention (refer to [9–16] and the reference therein). In [11–13], the authors studied the Lagrange stability by applying the attractive set and positive invariant set of the chaotic systems. Moreover, the researchers examined the stabilization of the unstable equilibrium points and the synchronization control for the chaotic systems with linear controllers [7, 14]. Recently, adaptive controllers are used in synchronous control of chaotic systems when the parameters of the systems are uncertain [15–17].

Motivated by the previous results, the main purpose of this paper is to construct a new smooth Chua's system and

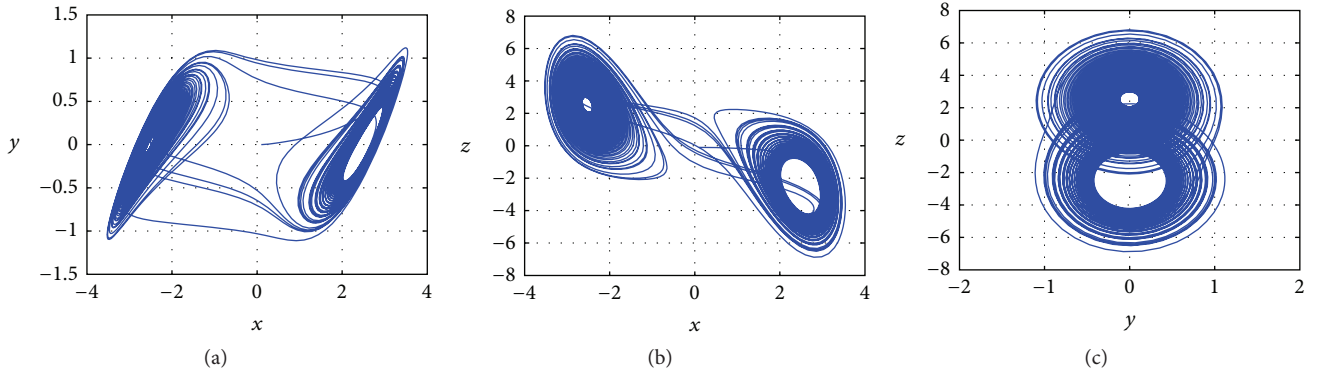


FIGURE 1: The phase portraits of Chua's system (2).

investigate the stability and control problems. More precisely, we will consider the following smooth Chua's system:

$$\begin{aligned}\dot{x} &= p \left(x + y - x \ln \sqrt{1 + x^2} \right), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -qy,\end{aligned}\quad (2)$$

where $x, y, z \in \mathbb{R}$ are state variables and $p > 0, q > 0$ are constants.

We will show that the chaotic characteristics are depended on the parameters p and q and the initial state values of the system (2). All equilibrium points of system (2) are examined to be unstable when $p = 11$ and $q = 14.87$ (see in Section 3). By computing with MATLAB, the maximum Lyapunov exponent of the system (2) is 0.0021, where the embedding dimension m is 3 and the delay time τ is 5. Since the maximum Lyapunov exponent is greater than 0, the Chua's system is chaotic. It will be of great significance if the solution of (2) is ultimately bounded (Lagrange asymptotically stable). The chaotic phase diagrams of such system is obtained by simulation with MATLAB. Figure 1 shows the phase diagrams of the system (2) with $x(0) = -3, y(0) = 2$, and $z(0) = 1$, the phase diagrams of Chua's system exhibits chaotic.

The remains of this paper are organized as follows. The existence of globally exponential attractive set and positive invariant set for the system (2) will be discussed in Section 2. The asymptotic stability of the equilibrium points will be studied in Section 3, and the synchronization control for two similar types of the Chua's systems will be discussed in Section 4. In Section 5, we will give the numerical simulations to demonstrate the correctness of our results, and finally we will give the conclusions in Section 6.

2. Existence of Globally Exponential Attractive Set and Positive Invariant Set

The Lagrange stability analysis of the system (2) will be studied in this section. To do so, we first give two definitions [7].

Definition 1. If there exists a radially unbounded, positive definite Lyapunov function $V(X(t))$ and positive numbers $l > 0, \alpha > 0$ such that for all $X_0 \in \mathbb{R}^3$, when $V(X_0) > l$, the solution of the system (2), $X(t) = (x(t), y(t), z(t))$ along $V(X(t))$ satisfies $|V(X(t)) - l| \leq |V(x_0) - l|e^{-\alpha(t-t_0)}$, then system (2) has a globally exponential attractive set $\Omega \triangleq \{X \mid V(X(t)) < l\}$.

Definition 2. Let $\Omega \subseteq \mathbb{R}^3$, if $\forall X_0 \in \Omega$ and for all $t \geq t_0$, $X(t, t_0, X_0) \subseteq \Omega$, then Ω is called positive invariant set of the system (2).

It is easy to prove that the globally exponential attractive set is positive invariant. A system with global attractive set is always called Lagrange globally asymptotically stable system or ultimately bounded dissipative system. For the system (2), we will prove the following the Lagrange stability results.

Theorem 3. *The system (2) has the following globally exponential attractive and positive invariant set:*

$$\Omega = \left\{ \begin{aligned} &|x| \leq |x_b| = \min \{|x_{b1}|, |x_{b2}|\} \\ &y^2 + \frac{1}{q}z^2 - \epsilon yz \leq \frac{(5/4\eta) x_b^2}{-\tau_1 \mu_M(G_2^-) / \lambda_M(G_2^+)} \\ &|x_{b1}| > \sqrt{e^{((2-(\epsilon/4))^2 / (2-\epsilon q - (\epsilon/4))) + (\epsilon/4) + 2) - 1}} \\ &|x_{b2}| = \sup_{\bar{\Omega}} |x|, \\ &\bar{\Omega} \triangleq \left\{ x \mid \frac{1}{p}x^2 + y^2 + \frac{1}{q}z^2 - \epsilon yz \right. \\ &\quad \left. = \frac{x_{b1}^2 \ln(1 + x_{b1}^2)}{-\mu_M(G_1^-(x_{b1})) / \lambda_M(G_1^+)} \right\}, \end{aligned} \right. \quad (3)$$

where $0 < \epsilon \ll 1$,

$$G_1^+ \triangleq \begin{bmatrix} \frac{1}{p} & 0 & 0 \\ 0 & 1 & -\frac{\epsilon}{2} \\ 0 & -\frac{\epsilon}{2} & \frac{1}{q} \end{bmatrix}, \quad G_2^+ \triangleq \begin{bmatrix} 1 & -\frac{\epsilon}{2} \\ -\frac{\epsilon}{2} & \frac{1}{q} \end{bmatrix}, \quad (4)$$

are symmetric positive definite matrices, and

$$G_1^- \triangleq \begin{bmatrix} 2(1 - \ln \sqrt{1 + x_{b1}^2}) & 2 & -\frac{\epsilon}{2} \\ 2 & \epsilon q - 2 & \frac{\epsilon}{2} \\ -\frac{\epsilon}{2} & \frac{\epsilon}{2} & -\epsilon \end{bmatrix}, \quad (5)$$

$$G_2^- \triangleq \begin{bmatrix} \epsilon q - 2 & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & -\epsilon \end{bmatrix},$$

are symmetric negative definite matrices. $\lambda_M(G_1^+)$, $\mu_M(G_1^-)$, $\lambda_M(G_2^+)$, and $\mu_M(G_2^-)$ are the maximum eigenvalues of G_1^+ , G_1^- , G_2^+ , and G_2^- , respectively. ϵ , x_{b1} , and x_{b2} in (3) are chosen to guarantee that G_1^+ , G_2^+ are positive definite and G_1^- , G_2^- are negative definite. τ_1 , τ_2 , and η are chosen such that $\tau_1 > 0$, $\tau_2 > 0$, $\eta > 0$, $\tau_1 + \tau_2 = 1$, and $\tau_2 \mu_M(G_2^-) + \eta \leq 0$.

Proof. The proof is divided into three steps.

Step I. The existence of ϵ , x_{b1} such that G_1^+ is positive definite and G_1^- is negative definite.

It is well known that G_1^+ is positive definite if and only if all the order principal minors of G_1^+ are positive [18]. That is,

$$\frac{1}{p} > 0, \quad \begin{vmatrix} \frac{1}{p} & 0 \\ 0 & 1 \end{vmatrix} > 0,$$

$$\begin{vmatrix} \frac{1}{p} & 0 & 0 \\ 0 & 1 & -\frac{\epsilon}{2} \\ 0 & -\frac{\epsilon}{2} & \frac{1}{q} \end{vmatrix} = \frac{1}{p} \left(\frac{1}{q} - \frac{\epsilon^2}{4} \right) > 0. \quad (6)$$

Let $|\epsilon| < 2/\sqrt{q}$; it is easy to see that G_1^+ is positive definite since $p > 0$, $q > 0$.

Next, let

$$S_{11} = \begin{bmatrix} 2(1 - \ln \sqrt{1 + x_{b1}^2}) & 2 \\ 2 & \epsilon q - 2 \end{bmatrix}, \quad (7)$$

$$S_{12} = \begin{bmatrix} -\frac{\epsilon}{2} \\ \frac{\epsilon}{2} \end{bmatrix}, \quad S_{21} = \begin{bmatrix} -\frac{\epsilon}{2} & \frac{\epsilon}{2} \end{bmatrix}, \quad S_{22} = -\epsilon.$$

Then,

$$G_1^- = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}. \quad (8)$$

From Schur theorem [18], G_1^- is negative definite if and only if $S_{11} < 0$, $S_{22} - S_{21}S_{11}^{-1}S_{12} < 0$. Let $|\epsilon|$, x_{b1} satisfy

$$0 < \epsilon \ll 1, \quad (9)$$

$$|x_{b1}| > \sqrt{e^{((2-\epsilon/4)^2/(2-\epsilon q - (\epsilon/4)))+(\epsilon/4)+2)} - 1}.$$

It is easy to verify that G_1^- is negative definite.

Step II. Existence of globally exponential attractive set of x .

Let $\lambda_m(G_1^+)$ and $\lambda_m(G_1^-)$ be the minimum eigenvalues of G_1^+ and G_1^- , respectively. By constructing a radially unbounded Lyapunov function as follows:

$$V = \frac{1}{p}x^2 + y^2 + \frac{1}{q}z^2 - \epsilon yz = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T G_1^+ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad (10)$$

then, one can obtain that

$$\lambda_m(G_1^+)(x^2 + y^2 + z^2) \leq V \leq \lambda_M(G_1^+)(x^2 + y^2 + z^2). \quad (11)$$

The time derivative of V along the system (2) is given by

$$\begin{aligned} \frac{dV}{dt} \Big|_{(2)} &= \frac{2}{p}x\dot{x} + 2y\dot{y} + \frac{2}{q}z\dot{z} - \epsilon y\dot{z} - \epsilon y\dot{z} \\ &= \frac{2}{p}xp(x + y - x \ln \sqrt{1 + x^2}) + 2y(x - y + z) \\ &\quad + \frac{2}{q}(-qy) - \epsilon(x - y + z)z - \epsilon y(-qy) \\ &= \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} 2(1 - \ln \sqrt{1 + x^2}) & 2 & -\frac{\epsilon}{2} \\ 2 & \epsilon q - 2 & \frac{\epsilon}{2} \\ -\frac{\epsilon}{2} & \frac{\epsilon}{2} & -\epsilon \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T G_1^- \begin{bmatrix} x \\ y \\ z \end{bmatrix} - 2(\ln \sqrt{1 + x^2} - \ln \sqrt{1 + x_{b1}^2})x^2. \end{aligned} \quad (12)$$

If $|x| > |x_{b1}|$, one has

$$\frac{dV}{dt} \Big|_{(2)} \leq \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T G_1^- \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leq \frac{\mu_M(G_1^-)}{\lambda_M(G_1^+)}V. \quad (13)$$

Then, the following inequality holds

$$V(X(t, t_0, X_0)) \leq V(X_0) e^{(\mu_M(G_1^-)/\lambda_M(G_1^+))(t-t_0)}. \quad (14)$$

Thus, the trajectory of the system (2) will exponentially decay into the area $U = \{x \mid |x| \leq |x_{b1}|\}$ if $|x| > |x_{b1}|$.

When the trajectory of the system (2) is in the area U , it holds that

$$\begin{aligned} \left. \frac{dV}{dt} \right|_{(2)} &= \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T G_1^- \begin{bmatrix} x \\ y \\ z \end{bmatrix} + 2 \left(\ln \sqrt{1+x_{b1}^2} - \ln \sqrt{1+x^2} \right) x^2 \\ &\leq \frac{\mu_M(G_1^-)}{\lambda_M(G_1^+)} V + 2 \left(\ln \sqrt{1+x_{b1}^2} - \ln \sqrt{1+x^2} \right) x^2 \\ &\leq \frac{\mu_M(G_1^-)}{\lambda_M(G_1^+)} V + x_{b1}^2 \ln(1+x_{b1}^2) \\ &= \frac{\mu_M(G_1^-)}{\lambda_M(G_1^+)} \left(\frac{V - x_{b1}^2 \ln(1+x_{b1}^2)}{-\mu_M(G_1^-)/\lambda_M(G_1^+)} \right). \end{aligned} \tag{15}$$

Let $V_l \triangleq (x_{b1}^2 \ln(1+x_{b1}^2))/(-\mu_M(G_1^-)/\lambda_M(G_1^+))$. If $V(X_0) > V_l$ and $V(X(t, t_0, X_0)) > V_l$, it holds that

$$V(X(t, t_0, X_0)) - V_l \leq (V(X_0) - V_l) e^{(\mu_M(G_1^-)/\lambda_M(G_1^+))(t-t_0)}. \tag{16}$$

Let $\bar{\Omega} \triangleq \{X \mid V(X) = V_l\}$, $|x_{b2}| \triangleq \sup_{x \in \bar{\Omega}} |x|$. Then, $V(X(t, t_0, X_0)) - V_l$ exponentially decays when $|x| \geq |x_b| \triangleq \min\{|x_{b1}|, |x_{b2}|\}$. Thus, $\Omega_x \triangleq \{x \mid |x| \leq |x_b|\}$ is the globally exponential attractive set of x .

Step III. Existence of globally exponential attractive set of y and z .

Let

$$G_2^+ \triangleq \begin{bmatrix} 1 & -\frac{\epsilon}{2} \\ -\frac{\epsilon}{2} & 1 \end{bmatrix}, \quad G_2^- \triangleq \begin{bmatrix} \epsilon q - 2 & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & -\epsilon \end{bmatrix}, \tag{17}$$

then, G_2^+ is a positive matrix if $\epsilon \in (0, 2/\sqrt{q})$, and G_2^- is a negative matrix if $\epsilon \in (0, 8/(4q+1))$.

At the same time, it is easy to examine that G_2^+ is positive and G_2^- is negative if let $\epsilon \in (0, 8/(4q+1))$, since $q > 0$ and $8/(4q+1) = 2/(q+1/4) = 2/\sqrt{q}$.

Let $\lambda_m(G_2^+)$, $\lambda_M(G_2^+)$ be, respectively, minimum and maximum eigenvalues of G_2^+ , and let $\mu_m(G_2^-)$ and $\mu_M(G_2^-)$ be the minimum and maximum eigenvalues of G_2^- , respectively. Since $|x| \leq |x_b|$, a radially unbounded and positive definite Lyapunov function about y and z is constructed as follows:

$$W = y^2 + \frac{1}{q} z^2 - \epsilon yz = \begin{bmatrix} y \\ z \end{bmatrix}^T G_2^+ \begin{bmatrix} y \\ z \end{bmatrix}. \tag{18}$$

Then, the time derivative of W along the system (2) yields

$$\begin{aligned} \left. \frac{dW}{dt} \right|_{(2)} &= 2y\dot{y} + \frac{2}{q} z\dot{z} - \epsilon y\dot{z} - \epsilon y\dot{z} \\ &= \begin{bmatrix} y \\ z \end{bmatrix}^T G_2^- \begin{bmatrix} y \\ z \end{bmatrix} + 2xy - \epsilon xz \\ &\leq \mu_M(G_2^-) (y^2 + z^2) + 2|x_b||y| + \epsilon|x_b||z| \\ &\leq \tau_1 \mu_M(G_2^-) (y^2 + z^2) + \tau_2 \mu_M(G_2^-) (y^2 + z^2) \\ &\quad + \eta y^2 + \frac{x_b^2}{\eta} + \eta z^2 + \frac{x_b^2}{4\eta} \\ &\leq \frac{\tau_1 \mu_M(G_2^-)}{\lambda_M(G_2^+)} W + (\tau_2 \mu_M(G_2^-) + \eta) (y^2 + z^2) + \frac{5}{4\eta} x_b^2 \\ &\leq \frac{\tau_1 \mu_M(G_2^-)}{\lambda_M(G_2^+)} \left(W - \frac{(5/4\eta) x_b^2}{-\tau_1 \mu_M(G_2^-)/\lambda_M(G_2^+)} \right), \end{aligned} \tag{19}$$

where τ_1 , τ_2 , and η are chosen such that $\tau_1 > 0$, $\tau_2 > 0$, $\eta > 0$, $\tau_1 + \tau_2 = 1$, and $\tau_2 \mu_M(G_2^-) + \eta \leq 0$.

Let $W_l \triangleq (5/4\eta)x_b^2/(-\tau_1 \mu_M(G_2^-)/\lambda_M(G_2^+))$; if $W(X_0) > W_l$ and $W(X(t, t_0, X_0)) > W_l$, it holds that

$$\begin{aligned} W(X(t, t_0, X_0)) - W_l &\leq (W(X_0) - W_l) e^{(\tau_1 \mu_M(G_2^-)/\lambda_M(G_2^+))(t-t_0)}. \end{aligned} \tag{20}$$

Similarly, if $W(X_0) \leq W_l$, the state trajectory will stay in the area such that $W(X(t, t_0, X_0)) \leq W_l$ holds. Hence, y and z are exponentially decreased and ultimately enter into the attractive region $W(X(t, t_0, X_0)) \leq W_l$, that is,

$$y^2 + \frac{1}{q} z^2 - \epsilon yz \leq \frac{(5/4\eta) x_b^2}{-\tau_1 \mu_M(G_2^-)/\lambda_M(G_2^+)}. \tag{21}$$

Combining Steps I, II, and III, Theorem 3 is obtained and the proof is completed. \square

Remark 4. In this section, a constructive method is proposed to prove the main results of existence of globally exponential attractive set and positive invariant set for the Chua's systems. By constructing the matrices G_1^+ , G_1^- , G_2^+ , G_2^- , and Lyapunov function candidate V and W , the problem is solved ingeniously.

3. Global Linear Stabilization of the Equilibrium Points

In this section, the stability of the equilibrium points for the system (2) will be discussed with the aid of a linear controller.

Firstly, it is easy to examine that the system (2) has three equilibrium points:

$$\begin{aligned} S_0 &= (0, 0, 0), \quad S_+ = (\sqrt{e^2-1}, 0, -\sqrt{e^2-1}), \\ S_- &= (-\sqrt{e^2-1}, 0, \sqrt{e^2-1}). \end{aligned} \tag{22}$$

Moreover, the equilibrium points of the system is independent of parameters p and q . It should be mentioned that, however, the stability of the equilibrium points is depended on p and q . In the following, we will design a linear controller to stabilize the unstable equilibrium points.

Let $p = 11, q = 14.87$, and (x^*, y^*, z^*) be any equilibrium point of the system (2), the corresponding Jacobian matrix is given by

$$J_0 = \begin{bmatrix} p \left(-\ln \sqrt{1+x^{*2}} + \frac{1}{1+x^{*2}} \right) & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{bmatrix}. \quad (23)$$

Then the characteristic equation of the corresponding local linearization system of system (2) is as follows:

$$a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0, \quad (24)$$

where $a_0 = 1, a_1 = 1 - pw, a_2 = -q - p - pw, a_3 = pqw$, and $w = -\ln \sqrt{1+x^{*2}} + (1/(1+x^{*2}))$.

- (i) If $x^* = 0$, one has $a_0 = 1 > 0, a_1 = -10 < 0$. According to Hurwitz stability criterion [19], the necessary condition of stable equilibrium point is the same sign of the coefficients of the characteristic equation. Consequently, the equilibrium S_0 is unstable.
- (ii) If $x^* = \pm \sqrt{e^2 - 1}$, one has $a_0 = 1, a_1 = 10.5113 > 0$, and $a_2 = -16.3587 < 0$. Similarly, one can obtain that S_+ and S_- are the unstable equilibrium points.

Now, we will discuss how to design a linear feedback controller such that the unstable equilibrium points are exponentially stable. For this purpose, we add the control terms to the system (2):

$$\begin{aligned} \dot{x} &= p \left(x + y - x \ln \sqrt{1+x^2} \right) + u_1, \\ \dot{y} &= x - y + z + u_2, \\ \dot{z} &= -qy + u_3. \end{aligned} \quad (25)$$

Let $X^* = (x^*, y^*, z^*)$ be any of the three unstable equilibrium points, let $X(t) = (x(t), y(t), z(t))$ be the solution of the system (25), and $\tilde{X}(t) = (\tilde{x}, \tilde{y}, \tilde{z}) = X(t) - X^*$, then the error system is given by

$$\begin{aligned} \dot{\tilde{x}} &= p \left(\tilde{x} + \tilde{y} - x \ln \sqrt{1+x^2} + x^* \ln \sqrt{1+x^{*2}} \right) + u_1, \\ \dot{\tilde{y}} &= \tilde{x} - \tilde{y} + \tilde{z} + u_2, \\ \dot{\tilde{z}} &= -q\tilde{y} + u_3. \end{aligned} \quad (26)$$

Definition 5. $\forall X(0) = (x(0), y(0), z(0)) \in R^3$, if u_i ($i = 1, 2, 3$) is appropriately selected such that

$$\left| \tilde{X}(t, t_0, X_0) \right| \leq \left| X(0) - X^* \right| e^{-\alpha(t-t_0)} \quad (27)$$

holds ($\alpha > 0$). Then, the control input u_i ($i = 1, 2, 3$) can globally exponentially stabilize the equilibrium point X^* .

Theorem 6. *If the following linear controller is added to the error system (26),*

$$u_1 = -p\sigma_x \tilde{x}, \quad u_2 = u_3 = 0, \quad (28)$$

where σ_x is any parameter given beforehand such that $\sigma_x > 2$; then the equilibrium point X^* is globally exponentially stable.

Proof. The proof is divided into two steps.

- (1) We will find the existence of $\epsilon > 0$ such that G_3^+ is positive definite and G_3^- is negative definite, where

$$G_3^+ \triangleq \begin{bmatrix} \frac{1}{p} & 0 & 0 \\ 0 & 1 & -\frac{\epsilon}{2} \\ 0 & -\frac{\epsilon}{2} & \frac{1}{q} \end{bmatrix}, \quad G_3^- \triangleq \begin{bmatrix} 2(1-\sigma_x) & 2 & -\frac{\epsilon}{2} \\ 2 & \epsilon q - 2 & \frac{\epsilon}{2} \\ -\frac{\epsilon}{2} & \frac{\epsilon}{2} & -\epsilon \end{bmatrix}. \quad (29)$$

It is easy to obtain that G_3^+ is positive definite if $|\epsilon| < 2/\sqrt{q}$. Now we focus on choosing $\epsilon > 0$ such that G_3^- is negative definite. By the Schur theorem [18], G_3^- is negative definite if and only if

$$\epsilon > 0,$$

$$\epsilon q + \frac{\epsilon}{4} - 2 < 0, \quad (30)$$

$$\left(2(\sigma_x - 1) - \frac{\epsilon}{4} \right) \left(2 - \epsilon q - \frac{\epsilon}{4} \right) > \left(2 - \frac{\epsilon}{4} \right)^2,$$

that is

$$\epsilon > 0,$$

$$\epsilon < \frac{8}{4q+1}, \quad (31)$$

$$\sigma_x > \frac{64 - 16\epsilon + \epsilon^2}{64 - 32\epsilon q - 8\epsilon} + 1 + \frac{\epsilon}{8}.$$

Obviously, G_3^- is negative definite if $0 < \epsilon \ll 1, \sigma_x > 2$.

- (2) We construct a positive definite and radially unbounded Lyapunov function to prove the stability of closed-loop systems (26) with controller (28) which is written as

$$\begin{aligned} V_1(\tilde{X}(t, t_0, \tilde{X}_0)) &= \frac{1}{p} \tilde{x}^2 + \tilde{y}^2 + \frac{1}{q} \tilde{z}^2 - \epsilon \tilde{y} \tilde{z} \\ &= \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix}^T G_3^+ \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix}. \end{aligned} \quad (32)$$

Suppose $\lambda_m(G_3^+)$ and $\lambda_M(G_3^+)$ are minimum and maximum eigenvalues of the positive definite matrix G_3^+ , respectively. Then, we have

$$\lambda_m(G_3^+) (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2) \leq V_1 \leq \lambda_M(G_3^+) (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2). \quad (33)$$

Let $f(\tilde{x}) = x \ln \sqrt{1+x^2} - x^* \ln \sqrt{1+x^{*2}}$. Obviously, $x \ln \sqrt{1+x^2}$ is a monotonically increasing odd function. Then,

(i) if $\tilde{x} \geq 0$, then $f(\tilde{x}) \geq 0$ and

$$0 \leq \tilde{x}f(\tilde{x}) \leq +\infty, \quad (34)$$

(ii) if $\tilde{x} \leq 0$, then $f(\tilde{x}) \leq 0$ and

$$0 \leq \tilde{x}f(\tilde{x}) \leq +\infty. \quad (35)$$

Thus

$$0 \leq \tilde{x}f(\tilde{x}) \leq +\infty, \quad \forall \tilde{x} \in \mathbb{R}. \quad (36)$$

Differentiating V_1 with respect to time yields

$$\begin{aligned} \left. \frac{dV_1}{dt} \right|_{(26)} &= \frac{2}{p} \tilde{x} \dot{\tilde{x}} + 2\tilde{y} \dot{\tilde{y}} + \frac{2}{q} \tilde{z} \dot{\tilde{z}} - \epsilon \dot{\tilde{y}} \tilde{z} - \epsilon \tilde{y} \dot{\tilde{z}} \\ &= \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix}^T G_3^- \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} - 2\tilde{x}f(\tilde{x}) \\ &\leq \mu_M(G_3^-) (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2) \\ &= \mu_M(G_3^-) \frac{\lambda_M(G_3^+)}{\lambda_m(G_3^+)} (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2) \leq \frac{\mu_M(G_3^-)}{\lambda_m(G_3^+)} V_1, \end{aligned} \quad (37)$$

where $\mu_M(G_3^-)$ is the maximum eigenvalues of the negative definite matrix G_3^- . Then,

$$\begin{aligned} V_1(\tilde{X}(t, t_0, \tilde{X}_0)) &\leq V_1(\tilde{X}_0) e^{(\mu_M(G_3^-)/\lambda_m(G_3^+))(t-t_0)}, \\ \tilde{x}^2(t) + \tilde{y}^2(t) + \tilde{z}^2(t) &\leq \frac{V_1(\tilde{X}(t, t_0, \tilde{X}_0))}{\lambda_m(G_3^+)} \\ &\leq \frac{V_1(\tilde{X}_0)}{\lambda_m(G_3^+)} e^{(\mu_M(G_3^-)/\lambda_m(G_3^+))(t-t_0)}. \end{aligned} \quad (38)$$

Hence, $\tilde{x}^2(t)$, $\tilde{y}^2(t)$, and $\tilde{z}^2(t)$ converge to zero exponentially. According to Definition 5, the equilibrium point X^* is globally exponentially stable. The proof is complete. \square

Remark 7. A constructive method to stabilize the unstable equilibrium points is proposed in this section, matrices G_3^+ , G_3^- , and Lyapunov function candidate V_1 are given. Then, a linear controller is obtained to solve the problem. Comparing with nonlinear controller, linear controller is easy to implement in reality.

4. Globally Exponential Synchronization of Two Chua's Systems

In this section, the globally exponential synchronization of two Chua's systems will be discussed. The drive system is given by

$$\begin{aligned} \dot{x}_d &= p \left(x_d + y_d - x_d \ln \sqrt{1+x_d^2} \right), \\ \dot{y}_d &= x_d - y_d + z_d, \\ \dot{z}_d &= -qy_d, \end{aligned} \quad (39)$$

and the response system is described as follows:

$$\begin{aligned} \dot{x}_r &= p \left(x_r + y_r - x_r \ln \sqrt{1+x_r^2} \right) + u_1, \\ \dot{y}_r &= x_r - y_r + z_r + u_2, \\ \dot{z}_r &= -qy_r + u_3, \end{aligned} \quad (40)$$

where the subscripts d and r denote the drive and response systems and u_i ($i = 1, 2, 3$) is feedback control input which satisfies $u_i(0, 0, 0) = 0$.

Let $e_x = x_r - x_d$, $e_y = y_r - y_d$, and $e_z = z_r - z_d$; one obtains that

$$\begin{aligned} \dot{e}_x &= p \left(e_x + e_y - x_r \ln \sqrt{1+x_r^2} + x_d \ln \sqrt{1+x_d^2} \right) + u_1, \\ \dot{e}_y &= e_x - e_y + e_z + u_2, \\ \dot{e}_z &= -qe_y + u_3. \end{aligned} \quad (41)$$

Theorem 8. *If the following controller is added to the error system (41),*

$$u_1 = -p\delta_x \tilde{x}, \quad u_2 = u_3 = 0, \quad (42)$$

where δ_x is any parameter given beforehand with $\delta_x > 2$; then the zero solution of (41) is globally exponentially stable and the systems (39) and (40) are globally exponentially synchronized.

Proof. Since the proof of this theorem is parallel to that of Theorem 6, we omit it here. \square

If the parameters p and q in the drive system are uncertain, one can construct the following controlled response system:

$$\begin{aligned} \dot{x}_r &= \hat{p} \left(x_r + y_r - x_r \ln \sqrt{1+x_r^2} \right) + u_1, \\ \dot{y}_r &= x_r - y_r + z_r + u_2, \\ \dot{z}_r &= -\hat{q}y_r + u_3, \end{aligned} \quad (43)$$

where \hat{p} and \hat{q} are the estimates of the uncertain parameters p and q , respectively. Let $e_x = x_r - x_d$, $e_y = y_r - y_d$, $e_z = z_r - z_d$,

$\tilde{p} = \hat{p} - p$, and $\tilde{q} = \hat{q} - q$, then, the error system of (39) and (43) is given by

$$\begin{aligned} \dot{e}_x &= \tilde{p} \left(e_x + e_y - x_r \ln \sqrt{1 + x_r^2} + x_d \ln \sqrt{1 + x_d^2} \right) \\ &\quad + \tilde{p} \left(x_d + y_d - x_d \ln \sqrt{1 + x_d^2} \right) + u_1, \\ \dot{e}_y &= e_x - e_y + e_z + u_2, \\ \dot{e}_z &= -\tilde{q}e_y - \tilde{q}y_d + u_3. \end{aligned} \tag{44}$$

Theorem 9. *If the following adaptive controller is added to the error system (44),*

$$\begin{aligned} u_1 &= -\tilde{p} \left(e_x + e_y - x_r \ln \sqrt{1 + x_r^2} + x_d \ln \sqrt{1 + x_d^2} \right) - e_x, \\ u_2 &= -e_x - e_z, \\ u_3 &= \tilde{q}e_y - e_z, \\ \dot{\tilde{p}} &= \dot{\hat{p}} = -e_x \left(x_d + y_d - x_d \ln \sqrt{1 + x_d^2} \right), \\ \dot{\tilde{q}} &= \dot{\hat{q}} = e_z y_d, \end{aligned} \tag{45}$$

then, we have the following.

- (1) *The equilibrium points ($\tilde{p} = 0, \tilde{q} = 0, e_x = 0, e_y = 0, e_z = 0$) of the system (44) with adaptive control law (45) are globally stable; in addition, $\lim_{t \rightarrow \infty} e_x = 0, \lim_{t \rightarrow \infty} e_y = 0$, and $\lim_{t \rightarrow \infty} e_z = 0$. And thus, the two systems (39) and (44) are globally synchronized.*
- (2) *The parameter estimates \hat{p} and \hat{q} will, respectively, converge to p and q as t tends to infinity.*

Proof. The proof contains two steps.

Firstly, a Lyapunov function candidate is constructed as follows:

$$V_2 = \frac{1}{2} \left(e_x^2 + e_y^2 + e_z^2 + \tilde{p}^2 + \tilde{q}^2 \right). \tag{46}$$

Then, one has

$$\begin{aligned} \left. \frac{dV_2}{dt} \right|_{(44)} &= e_x \dot{e}_x + e_y \dot{e}_y + e_z \dot{e}_z + \tilde{p} \dot{\tilde{p}} + \tilde{q} \dot{\tilde{q}} \\ &= e_x \left(\tilde{p} \left(e_x + e_y - x_r \ln \sqrt{1 + x_r^2} \right. \right. \\ &\quad \left. \left. + x_d \ln \sqrt{1 + x_d^2} \right) + u_1 \right) \\ &\quad + e_y \left(e_x - e_y + e_z + u_2 \right) \\ &\quad + e_z \left(-\tilde{q}e_y + u_3 \right) - \tilde{q} \left(e_z y_d - \dot{\tilde{q}} \right) \\ &\quad + \tilde{p} \left(e_x \left(x_d + y_d - x_d \ln \sqrt{1 + x_d^2} \right) + \dot{\tilde{p}} \right) \\ &= -e_x^2 - e_y^2 - e_z^2 \leq 0. \end{aligned} \tag{47}$$

By LaSalle-Yoshizawa theorem [20], all the equilibrium points of the closed-loop systems are globally stable. Additionally, $\lim_{t \rightarrow \infty} e_x = 0, \lim_{t \rightarrow \infty} e_y = 0$, and $\lim_{t \rightarrow \infty} e_z = 0$. And thus, the two systems (39) and (43) are globally synchronized.

By referring to Lemma 4.1 in [21], $\lim_{t \rightarrow \infty} \tilde{p} = 0, \lim_{t \rightarrow \infty} \tilde{q} = 0$ if there exist two functions $f_1(t), f_2(t)$ which satisfy persistency of excitation condition, such that

$$\begin{aligned} \lim_{t \rightarrow \infty} \dot{\tilde{p}} &= 0, & \lim_{t \rightarrow \infty} \dot{\tilde{q}} &= 0, \\ \lim_{t \rightarrow \infty} \tilde{p} f_1(t) &= 0, & \lim_{t \rightarrow \infty} \tilde{q} f_2(t) &= 0. \end{aligned} \tag{48}$$

Since $\lim_{t \rightarrow \infty} e_x = 0, \lim_{t \rightarrow \infty} e_z = 0$, it is easy to obtain from (45) that

$$\begin{aligned} \lim_{t \rightarrow \infty} \dot{\tilde{p}} &= \lim_{t \rightarrow \infty} -e_x \left(x_d + y_d - x_d \ln \sqrt{1 + x_d^2} \right) = 0, \\ \lim_{t \rightarrow \infty} \dot{\tilde{q}} &= \lim_{t \rightarrow \infty} e_z y_d = 0. \end{aligned} \tag{49}$$

Add u_1 and u_3 from (45) to (44), we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \dot{e}_x &= \lim_{t \rightarrow \infty} -e_x + \tilde{p} \left(x_d + y_d - x_d \ln \sqrt{1 + x_d^2} \right) \\ &= \lim_{t \rightarrow \infty} \tilde{p} \left(x_d + y_d - x_d \ln \sqrt{1 + x_d^2} \right) \\ &= 0, \\ \lim_{t \rightarrow \infty} \dot{e}_z &= \lim_{t \rightarrow \infty} -e_z - \tilde{q}y_d \\ &= \lim_{t \rightarrow \infty} -\tilde{q}y_d \\ &= 0. \end{aligned} \tag{50}$$

Let $f_1(t) \triangleq x_d + y_d - x_d \ln \sqrt{1 + x_d^2}$ and $f_2(t) \triangleq y_d$; it is easy to see that $f_1(t)$ and $f_2(t)$ satisfy persistency of excitation condition [21]; thus, we have

$$\lim_{t \rightarrow \infty} \tilde{p} = 0, \quad \lim_{t \rightarrow \infty} \tilde{q} = 0. \tag{51}$$

The proof is complete. □

Remark 10. Synchronization control methods for two Chua's systems are proposed in this section. A linear controller is given when parameters p and q are known. In the case of uncertain parameters p and q , an adaptive controller is proposed to solve the problem. Comparing with the results in reference [22, 23], the output tracking error only converges to a small neighborhood of the origin, yet the synchronization errors of Chua's systems can exponentially approach zero. On the other hand, compared with [23], a novel exponentially convergent method is used to solve the convergence of the estimation error of the uncertain parameters.

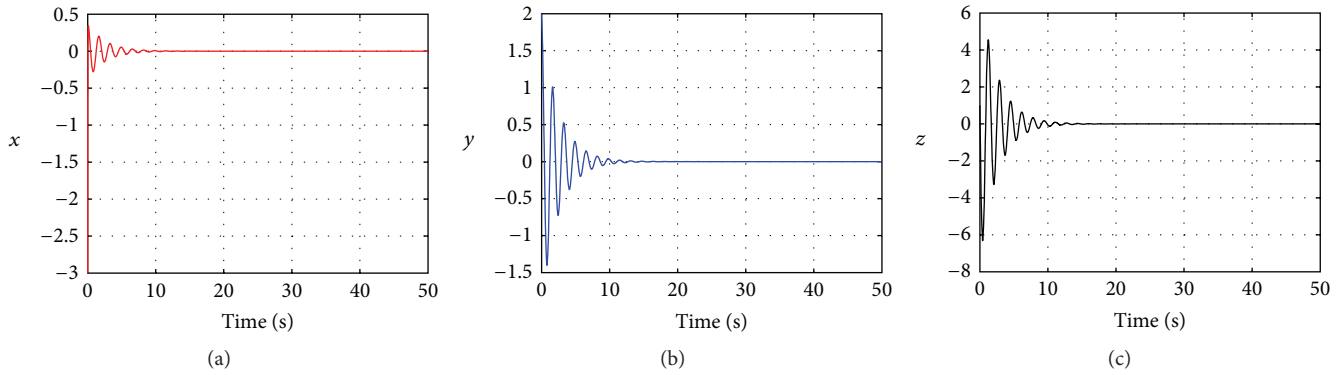


FIGURE 2: The asymptotic stability of S_0 with controller (28).

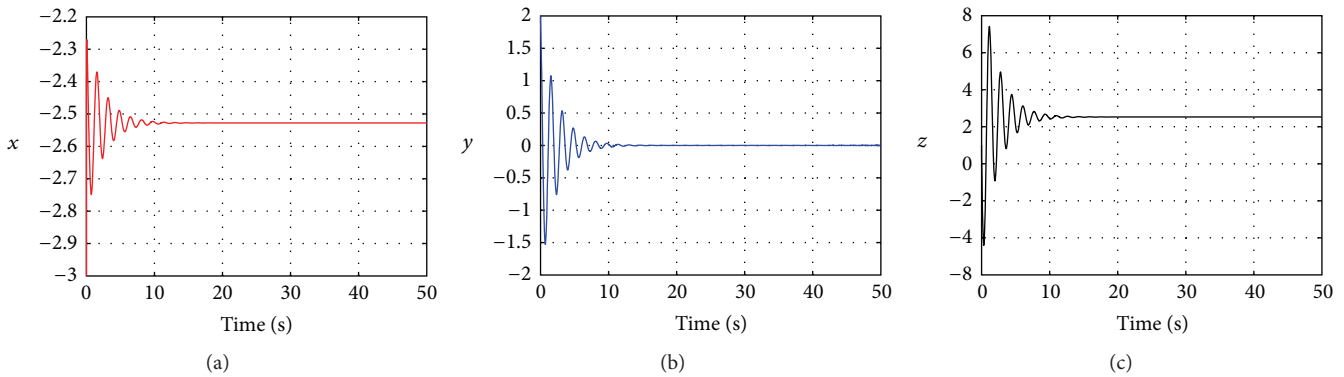


FIGURE 3: The asymptotic stability of S_- with controller (28).

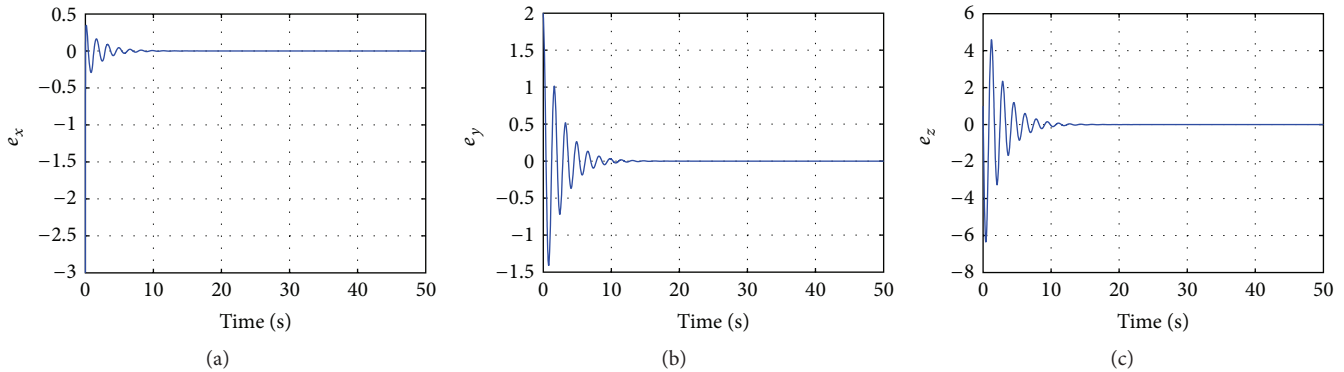


FIGURE 4: The synchronization errors of the system (39) and (40) with linear controller (42).

5. Numerical Simulations

In this section, several examples of numerical simulations are proposed to illustrate the theoretical results obtained in the previous sections. A fourth-order Runge-Kutta method is used to obtain the simulation results with MATLAB.

Chua’s system (2) and the error systems (26), (41), and (44) are considered in this section for the numerical simulations. Let $p = 11$, $q = 14.87$, the initial state $x(0) = -3$, $y(0) = 2$, and $z(0) = 1$. Figure 2 shows the state trajectories

of the closed-loop system (26) with linear control input (28); it is easy to see that the equilibrium point S_0 is asymptotically stable. Similarly, Figure 3 shows that, with the corresponding control input (28), the equilibrium point S_- is asymptotically stable. Figure 4 shows the synchronous errors of the system (39) and (40) with linear controller (42); it is easy to see that the two systems are globally asymptotically synchronized. When parameters p and q are uncertain, by using adaptive controller (45), the synchronous errors are asymptotically convergent to $e_x = 0$, $e_y = 0$, $e_z = 0$ (see Figure 5),

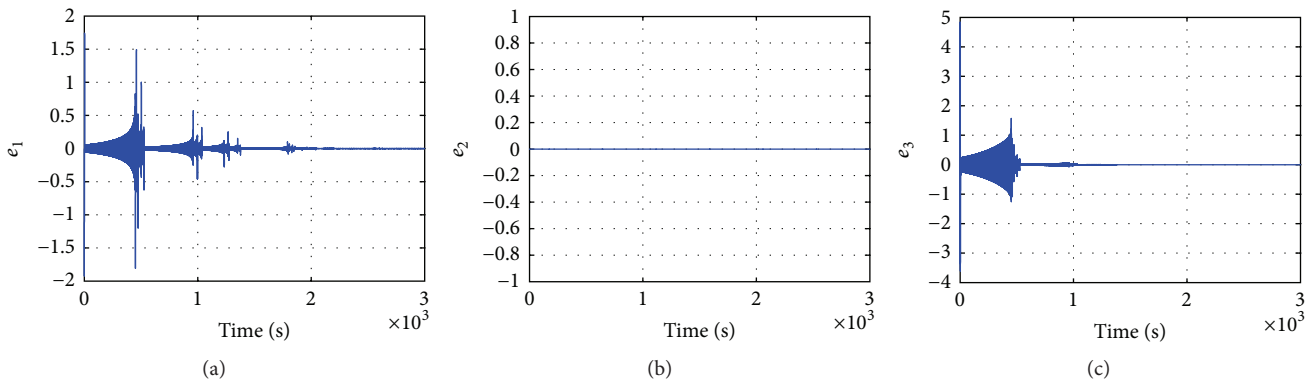


FIGURE 5: The synchronization errors of the system (39) and (43) with adaptive controller (45).

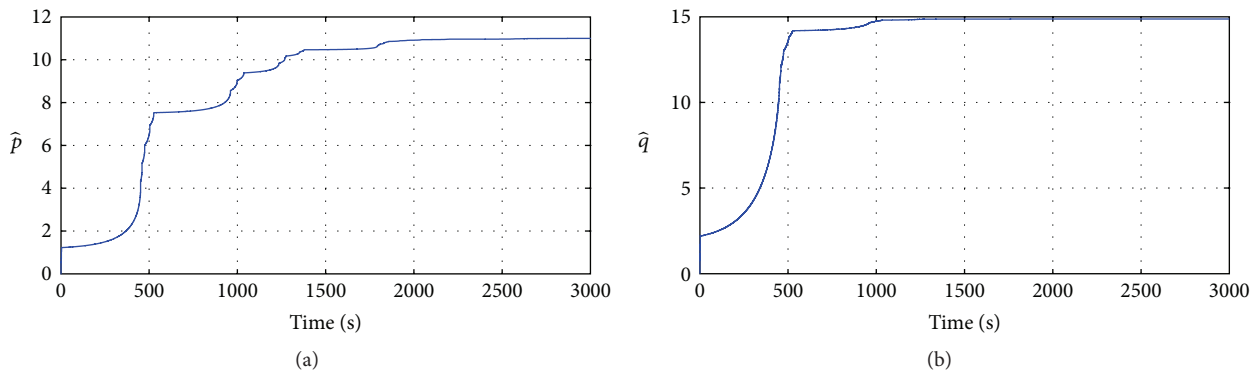


FIGURE 6: The convergence of parameters \hat{p} and \hat{q} .

and the estimate values of the uncertain parameters p and q asymptotically converge to the real values $p = 11, q = 14.87$ (see Figure 6).

6. Conclusions

A new smooth Chua’s system is constructed, and the chaotic characteristics is confirmed by computing the Lyapunov exponents of the system. A Constructive method is used to prove the existence of globally exponential attractive set and positive invariant set. For the three unstable equilibrium points of the system, a linear controller is designed to achieve globally exponential stability of the equilibrium points. Then, a linear controller and an adaptive controller are, respectively, proposed so that two similar types of smooth Chua’s systems are globally synchronized, and the estimate errors of the uncertain parameters converge to zero as t tends to infinity.

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