

## Research Article

# New Iterative Method: An Application for Solving Fractional Physical Differential Equations

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The new iterative method with a powerful algorithm is developed for the solution of linear and nonlinear ordinary and partial differential equations of fractional order as well. The analysis is accompanied by numerical examples where this method, in solving them, is used without linearization or small perturbation which confirm the power, accuracy, and simplicity of the given method compared with some of the other methods.

## 1. Introduction

Considerable attention has been devoted to the study of the fractional calculus during the past three decades and its numerous applications in the area of physics and engineering. The applications of fractional calculus used in many fields such as electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry, chemical physics, optics, and signal processing can be successfully modelled by linear or nonlinear fractional differential equations.

So far there have been several fundamental works on the fractional derivative and fractional differential equations [1–3]. These works are to be considered as an introduction to the theory of fractional derivative and fractional differential equations and provide a systematic understanding of the fractional calculus such as the existence and uniqueness [4, 5]. Recently, many other researchers have paid attention to existence result of solution of the initial value problem and boundary problem for fractional differential equations [4–6].

Finding approximate or exact solutions of fractional differential equations is an important task. Except for a limited number of these equations, we have difficulty in finding their analytical solutions. Therefore, there have been attempts to develop new methods for obtaining analytical solutions

which reasonably approximate the exact solutions. Several such techniques have drawn special attention, such as Adomian's decomposition method [7], homotopy perturbation method [8–10], homotopy analysis method [11, 12], variational iteration method [13–17], Chebyshev spectral method [18, 19], and new iterative method [20–22]. Among them, the new iterative method provides an effective procedure for explicit and numerical solutions of a wide and general class of differential systems representing real physical problems. The new iterative method is more superior than the other nonlinear methods, such as the perturbation methods where this method does not depend on small parameters, such that it can find wide application in nonlinear problems without linearization or small perturbation.

The motivation of this paper is to extend the application of the new iterative method proposed by Daftardar-Gejji and Jafari [20–22] to solve linear and nonlinear ordinary and partial differential equations of fractional order. This motivation is based on the importance of these equations and their applications in various subjects in physical branches [10, 11, 14, 23–25].

There are several definitions of a fractional derivative of order  $\alpha > 0$  [3, 26]. The two most commonly used definitions are Riemann-Liouville and Caputo. Each definition uses Riemann-Liouville fractional integration and derivative of whole order. The difference between the two definitions

is in the order of evaluation. Riemann-Liouville fractional integration of order  $\alpha$  is defined as

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \xi)^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0, \quad x > 0. \tag{1}$$

The next two equations define Riemann-Liouville and Caputo fractional derivatives of order  $\alpha$ , respectively, as

$$D^\alpha f(x) = \frac{d^m}{dx^m} (I^{m-\alpha} f(x)), \tag{2a}$$

$$D_*^\alpha f(x) = I^{m-\alpha} \left( \frac{d^m}{dx^m} f(x) \right), \tag{2b}$$

where  $m - 1 < \alpha < m, m \in N$ .

Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. Riemann-Liouville fractional derivative is computed in the reverse order. Therefore, Caputo fractional derivative allows traditional initial and boundary conditions to be included in the formulation of the problem.

From properties of  $D_*^\alpha$  and  $I^\alpha$ , it is important to note that

$$D_*^\alpha x^\beta = \frac{\Gamma(\beta + 1) x^{\beta-\alpha}}{\Gamma(\beta + 1 - \alpha)}, \quad \beta \geq \alpha, \tag{3a}$$

where  $D_*^\alpha$  is Caputo derivative operator of order  $\alpha$ ,

$$I^\alpha x^\beta = \frac{\Gamma(\beta + 1) x^{\beta+\alpha}}{\Gamma(\beta + 1 + \alpha)}. \tag{3b}$$

## 2. Basic Idea of New Iterative Method

For the basic idea of the new iterative method, we consider the following general functional equation [20–22]:

$$u(x) = f(x) + N(u(x)), \tag{4}$$

where  $N$  is a nonlinear operator from a Banach space  $B \rightarrow B$  and  $f$  is a known function. We have been looking for a solution of (4) having the series form

$$u(x) = \sum_{i=0}^\infty u_i(x). \tag{5}$$

The nonlinear operator  $N$  can be decomposed as

$$N\left(\sum_{i=0}^\infty u_i\right) = N(u_0) + \sum_{i=1}^\infty \left\{ N\left(\sum_{j=0}^i u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\}. \tag{6}$$

From (5) and (6), (4) is equivalent to

$$\sum_{i=0}^\infty u_i = f + N(u_0) + \sum_{i=1}^\infty \left\{ N\left(\sum_{j=0}^i u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\}. \tag{7}$$

We define the following recurrence relation:

$$\begin{aligned} u_0 &= f, \\ u_1 &= N(u_0), \\ u_{n+1} &= N(u_0 + u_1 + \dots + u_n) \\ &\quad - N(u_0 + u_1 + \dots + u_{n-1}), \quad n = 1, 2, \dots \end{aligned} \tag{8}$$

Then,

$$\begin{aligned} &(u_1 + u_2 + \dots + u_{n+1}) \\ &= N(u_0 + u_1 + \dots + u_n), \quad n = 1, 2, \dots, \\ u &= \sum_{i=0}^\infty u_i = f + N\left(\sum_{i=0}^\infty u_i\right). \end{aligned} \tag{9}$$

If  $\|N(x) - N(y)\| < k \|x - y\|, 0 < k < 1$ , then

$$\begin{aligned} &\|u_{n+1}\| \\ &= \|N(u_0 + \dots + u_n) - N(u_0 + \dots + u_{n-1})\| \\ &\leq k \|u_n\| \leq \dots \leq k^{n+1} \|u_0\|, \quad n = 0, 1, 2, \dots, \end{aligned} \tag{10}$$

and the series  $\sum_{i=0}^\infty u_i$  absolutely and uniformly converges to a solution of (4) [27], which is unique, in view of the Banach fixed point theorem [28]. The  $n$ -term approximate solution of (4) and (5) is given by  $u(x) = \sum_{i=0}^{n-1} u_i$ .

*2.1. Convergence of the Method.* Now we analyze the convergence of the new iterative method for solving any general functional equation (4). Let  $e = u^* - u$ , where  $u^*$  is the exact solution,  $u$  is the approximate solution, and  $e$  is the error in the solution of (4); obviously  $e$  satisfies (4), that is,

$$e(x) = f(x) + N(e(x)) \tag{11}$$

and the recurrence relation (8) becomes

$$\begin{aligned} e_0 &= f, \\ e_1 &= N(e_0), \\ e_{n+1} &= N(e_0 + e_1 + \dots + e_n) \\ &\quad - N(e_0 + e_1 + \dots + e_{n-1}), \quad n = 1, 2, \dots \end{aligned} \tag{12}$$

If  $\|N(x) - N(y)\| \leq k \|x - y\|$ ,  $0 < k < 1$ , then

$$\begin{aligned}
 e_0 &= f, \\
 \|e_1\| &= \|N(e_0)\| \leq k \|e_0\|, \\
 \|e_2\| &= \|N(e_0 + e_1) - N(e_0)\| \\
 &\leq k \|e_1\| \leq k^2 \|e_0\|, \\
 \|e_3\| &= \|N(e_0 + e_1 + e_2) - N(e_0 + e_1)\| \\
 &\leq k \|e_2\| \leq k^3 \|e_0\|, \\
 &\vdots \\
 \|e_{n+1}\| &= \|N(e_0 + \dots + e_n) - N(e_0 + \dots + e_{n-1})\| \\
 &\leq k \|e_n\| \leq k^{n+1} \|e_0\|, \quad n = 0, 1, 2, \dots
 \end{aligned}
 \tag{13}$$

Thus  $e_{n+1} \rightarrow 0$  as  $n \rightarrow \infty$ , which proves the convergence of the new iterative method for solving the general functional equation (4). For more details, you can see [29].

### 3. Suitable Algorithm

In this section, we introduce a suitable algorithm for solving nonlinear partial differential equations using the new iterative method. Consider the following nonlinear partial differential equation of arbitrary order:

$$\begin{aligned}
 D_{*t}^\alpha u(x, t) &= A(u, \partial u) + B(x, t), \\
 m - 1 < \alpha < m, \quad m \in N,
 \end{aligned}
 \tag{14a}$$

$$\frac{\partial^k}{\partial t^k} u(x, 0) = h_k(x), \quad k = 0, 1, \dots, m - 1,
 \tag{14b}$$

where  $A$  is a nonlinear function of  $u$  and  $\partial u$  (partial derivatives of  $u$  with respect to  $x$  and  $t$ ) and  $B$  is the source function. In view of the new iterative method, the initial value problem (14a) and (14b) is equivalent to the integral equation

$$u(x, t) = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B + I_t^\alpha A = f + N(u),
 \tag{15}$$

where

$$f = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I_t^\alpha B,
 \tag{16a}$$

$$N(u) = I_t^\alpha A.
 \tag{16b}$$

*Remark 1.* When the general functional equation (4) is linear, the recurrence relation (8) can be simplified in the form

$$\begin{aligned}
 u_0 &= f, \\
 u_{n+1} &= N(u_n), \quad n = 0, 1, 2, \dots
 \end{aligned}
 \tag{17}$$

*Proof.* From the properties of integration and by using (8) and (16b), we have

$$\begin{aligned}
 u_{n+1} &= N(u_0 + \dots + u_{n-1} + u_n) \\
 &\quad - N(u_0 + \dots + u_{n-1}) \\
 &= I_t^\alpha [u_0 + \dots + u_{n-1} + u_n] \\
 &\quad - I_t^\alpha [u_0 + \dots + u_{n-1}] \\
 &= I_t^\alpha [u_0] + \dots + I_t^\alpha [u_{n-1}] \\
 &\quad + I_t^\alpha [u_n] - I_t^\alpha [u_0] - I_t^\alpha [u_{n-1}] \\
 &= I_t^\alpha [u_n] = N(u_n), \quad n = 0, 1, 2, \dots
 \end{aligned}
 \tag{18}$$

Therefore, we get the solution of (15) by employing the recurrence relation (8) or (17).  $\square$

### 4. Applications

To illustrate the effectiveness of the proposed method, several test examples are carried out in this section.

*Example 2.* In this example, we consider the following initial value problem in the case of the inhomogeneous Bagely-Torvik equation [23, 24]:

$$\begin{aligned}
 D_{*x}^2 u(x) + D_{*x}^{1.5} u(x) + u(x) &= g(x), \\
 u(0) = 1, \quad u'(0) = 1, \quad x \in [0, L],
 \end{aligned}
 \tag{19}$$

where  $g(x) = 1 + x$ . The exact solution of this problem is  $u(x) = 1 + x$ .

By applying the technique described in Sections 2 and 3, the initial value problem (19) is equivalent to the integral equation

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - I_x^2 [D_{*x}^{1.5} u(x) + u(x)].
 \tag{20}$$

Let  $N(u) = -I_x^2 [D_{*x}^{1.5} u(x) + u(x)]$ . In view of recurrence relation (17), we have the following first approximations:

$$\begin{aligned}
 u_0(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}, \\
 u_1(x) &= N(u_0) = -\frac{8x^{2.5}}{15\sqrt{\pi}} - \frac{16x^{3.5}}{105\sqrt{\pi}} \\
 &\quad - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} - \frac{x^5}{120}, \\
 u_2(x) &= N(u_1) = \frac{8x^{2.5}}{15\sqrt{\pi}} + \frac{16x^{3.5}}{105\sqrt{\pi}} \\
 &\quad + \frac{64x^{4.5}}{945\sqrt{\pi}} + \frac{128x^{5.5}}{10395\sqrt{\pi}} + \frac{x^3}{6} + \frac{x^4}{12} \\
 &\quad + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040},
 \end{aligned}
 \tag{21}$$

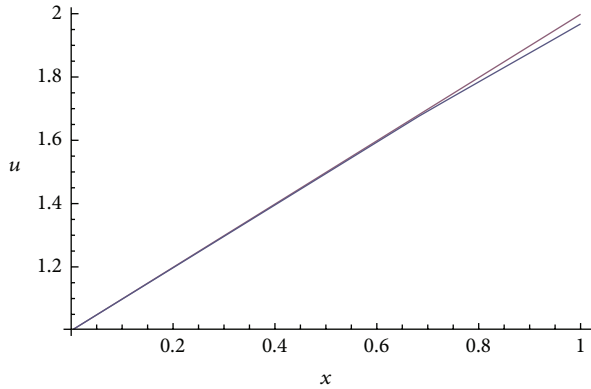


FIGURE 1: Plots of the approximate solution and the exact solution for (19).

and so on. In the same manner the rest of components can be obtained. The 6-term approximate solution for (19) is

$$\begin{aligned}
 u(x) &= \sum_{i=0}^5 u_i \\
 &= 1 + x - \frac{x^6}{144} - \frac{5x^7}{5040} - \frac{x^9}{36288} \\
 &\quad - \frac{x^{10}}{362880} - \frac{x^{12}}{479001600} - \frac{x^{13}}{6227020800} \\
 &\quad - \frac{32x^{4.5}}{945\sqrt{\pi}} - \frac{64x^{5.5}}{10395\sqrt{\pi}} - \frac{512x^{7.5}}{405405\sqrt{\pi}} \\
 &\quad - \frac{1024x^{8.5}}{6891885\sqrt{\pi}} - \frac{512x^{10.5}}{687465529\sqrt{\pi}} - \frac{128x^{11.5}}{1976463395\sqrt{\pi}}.
 \end{aligned} \tag{22}$$

*Remark 3.* In Example 2, we have used the recurrence relation (17). If we used the recurrence relation (8) in place of (17), we obtain the same result.

In Figure 1, we have plotted the 6-term approximate solution with the corresponding exact solution for (19). It is remarkable to note that the two solutions are almost equal.

Comparing these obtained results with those obtained by new Jacobi operational matrix in [23, 24], we can confirm the simplicity and accuracy of the given method.

*Example 4.* Consider the following fractional Riccati equation [10]:

$$D_{*x}^\alpha u(x) + u^2(x) = 1, \quad u(0) = 0, \quad x > 0, \quad 0 < \alpha \leq 1. \tag{23}$$

The exact solution when  $\alpha = 1$  is  $u(x) = (e^{2x} - 1)/(e^{2x} + 1)$ .

By applying the technique described in Sections 2 and 3, the initial value problem (23) is equivalent to the integral equation

$$u(x) = \frac{x^\alpha}{\Gamma(1 + \alpha)} - I_x^\alpha [u^2(x)]. \tag{24}$$

Let  $N(u) = -I_x^\alpha [u^2(x)]$ . In view of recurrence relation (8), we have the following first approximations:

$$\begin{aligned}
 u_0(x) &= \frac{x^\alpha}{\Gamma(1 + \alpha)}, \\
 u_1(x) &= N(u_0) = -\frac{\Gamma(1 + 2\alpha) x^{3\alpha}}{\Gamma(1 + 3\alpha) \Gamma(1 + \alpha)^2}, \\
 u_2(x) &= N(u_0 + u_1) - N(u_0) \\
 &= \frac{2\Gamma(1 + 2\alpha) \Gamma(1 + 4\alpha) x^{5\alpha}}{\Gamma(1 + 3\alpha) \Gamma(1 + 5\alpha) \Gamma(1 + \alpha)^3} \\
 &\quad - \frac{\Gamma(1 + 2\alpha)^2 \Gamma(1 + 6\alpha) x^{7\alpha}}{\Gamma(1 + 3\alpha)^2 \Gamma(1 + 7\alpha) \Gamma(1 + \alpha)^4},
 \end{aligned} \tag{25}$$

and so on. The 4-term approximate solution for (23) is

$$\begin{aligned}
 u(x) &= \sum_{i=0}^3 u_i \\
 &= \frac{x^\alpha}{\Gamma(1 + \alpha)} \\
 &\quad - \frac{\Gamma(1 + 2\alpha) x^{3\alpha}}{\Gamma(1 + 3\alpha) \Gamma(1 + \alpha)^2} \\
 &\quad + \frac{2\Gamma(1 + 2\alpha) \Gamma(1 + 4\alpha) x^{5\alpha}}{\Gamma(1 + 3\alpha) \Gamma(1 + 5\alpha) \Gamma(1 + \alpha)^3} \\
 &\quad - \frac{\Gamma(1 + 2\alpha)^2 \Gamma(1 + 6\alpha) x^{7\alpha}}{\Gamma(1 + 3\alpha)^2 \Gamma(1 + 7\alpha) \Gamma(1 + \alpha)^4} \\
 &\quad - \frac{4\Gamma(1 + 2\alpha) \Gamma(1 + 4\alpha) \Gamma(1 + 6\alpha) x^{9\alpha}}{\Gamma(1 + 3\alpha) \Gamma(1 + 5\alpha) \Gamma(1 + 7\alpha) \Gamma(1 + \alpha)^4} \\
 &\quad + \frac{4\Gamma(1 + 2\alpha)^2 \Gamma(1 + 4\alpha)}{\Gamma(1 + 3\alpha)^2 \Gamma(1 + 5\alpha)} \\
 &\quad \cdot \frac{\Gamma(1 + 8\alpha) x^{9\alpha}}{\Gamma(1 + 9\alpha) \Gamma(1 + \alpha)^5} \\
 &\quad + \frac{2\Gamma(1 + 2\alpha)^2 \Gamma(1 + 6\alpha) \Gamma(1 + 8\alpha) x^{9\alpha}}{\Gamma(1 + 3\alpha)^2 \Gamma(1 + 7\alpha) \Gamma(1 + 9\alpha) \Gamma(1 + \alpha)^5} \\
 &\quad - \frac{4\Gamma(1 + 2\alpha)^2 \Gamma(1 + 4\alpha)^2 \Gamma(1 + 10\alpha) x^{11\alpha}}{\Gamma(1 + 3\alpha)^2 \Gamma(1 + 5\alpha)^2 \Gamma(1 + 11\alpha) \Gamma(1 + \alpha)^6} \\
 &\quad - \frac{2\Gamma(1 + 2\alpha)^3}{\Gamma(1 + 3\alpha)^3 \Gamma(1 + 7\alpha)} \\
 &\quad \cdot \frac{\Gamma(1 + 6\alpha) \Gamma(1 + 10\alpha) x^{11\alpha}}{\Gamma(1 + 11\alpha) \Gamma(1 + \alpha)^6}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{4\Gamma(1+2\alpha)^3\Gamma(1+4\alpha)\Gamma(1+6\alpha)}{\Gamma(1+3\alpha)^3\Gamma(1+5\alpha)\Gamma(1+7\alpha)} \\
 & \cdot \frac{\Gamma(1+12\alpha)x^{13\alpha}}{\Gamma(1+13\alpha)\Gamma(1+\alpha)^7} \\
 & - \frac{\Gamma(1+2\alpha)^4\Gamma(1+6\alpha)^2\Gamma(1+14\alpha)x^{15\alpha}}{\Gamma(1+3\alpha)^4\Gamma(1+7\alpha)^2\Gamma(1+15\alpha)\Gamma(1+\alpha)^8}. \tag{26}
 \end{aligned}$$

In Figure 2, we have plotted the 4-term approximate solution for (23) for different values of  $\alpha$  with the corresponding exact solution. It is remarkable to note that the approximate solution, in case  $\alpha = 1$ , and the exact solution are almost equal (continuous curve) whenever the approximate solution, in cases  $\alpha = 0.9, 0.8$ , is of high agreement with the exact solution (dashed and dotted curves, resp.).

Comparing the obtained results with those obtained by homotopy analysis method, in case  $h = -1$ , in [10], we can confirm the simplicity and accuracy of the given method.

*Example 5.* Consider the following initial value problem with fractional order [23, 24]:

$$\begin{aligned}
 D_{*x}^3 u(x) + D_{*x}^{2.5} u(x) + u^2(x) &= x^4, \\
 u(0) = u'(0) = 0, \quad u''(0) &= 2. \tag{27}
 \end{aligned}$$

The exact solution for this problem is  $u(x) = x^2$ .

As in Example 4, the initial value problem (27) is equivalent to the integral equation

$$u(x) = x^2 + \frac{x^7}{210} - I_x^3 [D_{*x}^{2.5} u(x) + u^2(x)]. \tag{28}$$

Let  $N(u) = -I_x^3 [D_{*x}^{2.5} u(x) + u^2(x)]$ . In view of recurrence relation (8), we have the following first approximations:

$$\begin{aligned}
 u_0(x) &= x^2 + \frac{x^7}{210}, \\
 u_1(x) = N(u_0) &= \frac{-129024x^{7.5}}{42567525\sqrt{\pi}} - \frac{x^7}{210} \\
 &\quad - \frac{x^{12}}{138600} - \frac{x^{17}}{179928000}, \\
 u_2(x) = N(u_0 + u_1) - N(u_0) & \\
 &= 1.71008^{-3} x^{7.5} + 5.95252^{-4} x^8 \\
 &\quad + 7.21501^{-6} x^{12} + 4.28636^{-6} x^{12.5} \\
 &\quad + 9.09455^{-9} x^{17} + 1.31911^{-9} x^{17.5} \\
 &\quad - 5.97301^{-10} x^{18} + 1.20298^{-12} x^{22} \\
 &\quad - 2.48834^{-12} x^{22.5} - 2.96617^{-15} x^{27} \\
 &\quad - 1.02289^{-15} x^{27.5} - 2.69485^{-18} x^{32} \\
 &\quad - 6.62568^{-28} x^{37}, \tag{29}
 \end{aligned}$$

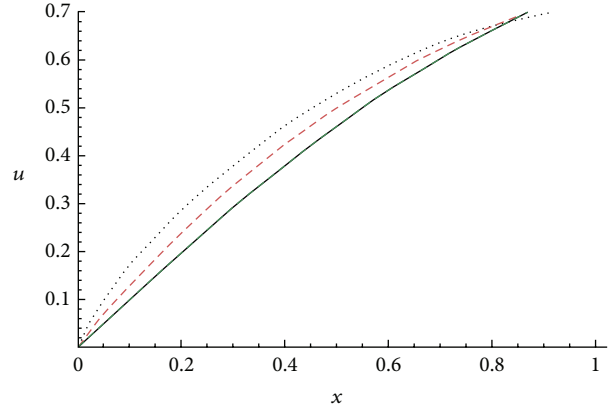


FIGURE 2: Plots of the approximate solution for different values of  $\alpha$  and the exact solution for (23).

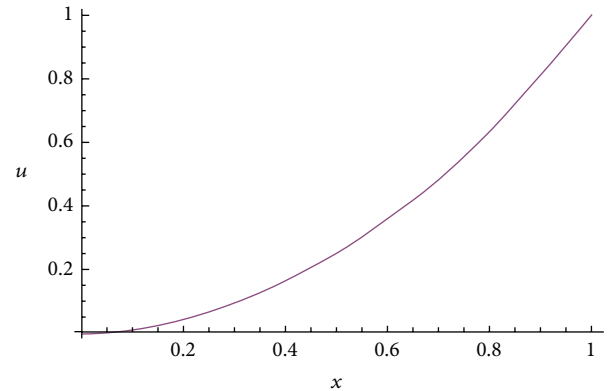


FIGURE 3: Plots of the approximate solution and the exact solution for (27).

and so on. The 4-term approximate solution and the corresponding exact solution for (27) are plotted in Figure 3. It is remarkable to note that the two solutions are almost equal.

Comparing these obtained results with those obtained by new Jacobi operational matrix in [23, 24], we can confirm the simplicity and accuracy of the given method.

*Example 6.* Consider the following fractional order wave equation in 2-dimensional space [14]:

$$\begin{aligned}
 D_{*t}^\alpha u(x, y, t) + c(u_x + u_y) &= 0, \\
 u(x, y, 0) = \sin \left[ \pi \left( \frac{x+y}{l} \right) \right], \quad & 0 < \alpha \leq 1. \tag{30}
 \end{aligned}$$

The exact solution for this problem when  $\alpha = 1$  is

$$u(x, y, t) = \sin \left[ \pi \left( \frac{x+y-2ct}{l} \right) \right]. \tag{31}$$

The initial value problem (30) is equivalent to the integral equation

$$u(x, y, t) = \sin \left[ \pi \left( \frac{x+y}{l} \right) \right] - I_t^\alpha [c(u_x + u_y)]. \tag{32}$$

Let  $N(u) = -I_t^\alpha [c(u_x + u_y)]$ . In view of recurrence relation (17), we have the following first approximations:

$$\begin{aligned} u_0(x, y, t) &= \sin \left[ \pi \left( \frac{x+y}{l} \right) \right], \\ u_1(x, y, t) &= \frac{-1}{\Gamma(1+\alpha)} \left( \frac{2c\pi t^\alpha}{l} \right) \cdot \cos \left[ \pi \left( \frac{x+y}{l} \right) \right], \\ u_2(x, y, t) &= \frac{-1}{\Gamma(1+2\alpha)} \left( \frac{2c\pi t^\alpha}{l} \right)^2 \cdot \sin \left[ \pi \left( \frac{x+y}{l} \right) \right], \\ u_3(x, y, t) &= \frac{1}{\Gamma(1+3\alpha)} \left( \frac{2c\pi t^\alpha}{l} \right)^3 \cdot \cos \left[ \pi \left( \frac{x+y}{l} \right) \right], \end{aligned} \quad (33)$$

and so on. The  $n$ -term approximate solution for (30) is

$$\begin{aligned} u(x, y, t) &= \sum_{i=0}^{n-1} u_i \\ &= \sin \left[ \pi \left( \frac{x+y}{l} \right) \right] \\ &\quad \times \left[ 1 - \frac{1}{\Gamma(1+2\alpha)} \left( \frac{2c\pi t^\alpha}{l} \right)^2 \right. \\ &\quad \left. + \frac{1}{\Gamma(1+4\alpha)} \left( \frac{2c\pi t^\alpha}{l} \right)^4 \right. \\ &\quad \left. - \frac{1}{\Gamma(1+6\alpha)} \left( \frac{2c\pi t^\alpha}{l} \right)^6 + \dots \right] \\ &\quad - \cos \left[ \pi \left( \frac{x+y}{l} \right) \right] \\ &\quad \times \left[ \frac{1}{\Gamma(1+\alpha)} \left( \frac{2c\pi t^\alpha}{l} \right) \right. \\ &\quad \left. - \frac{1}{\Gamma(1+3\alpha)} \left( \frac{2c\pi t^\alpha}{l} \right)^3 \right. \\ &\quad \left. + \frac{1}{\Gamma(1+5\alpha)} \left( \frac{2c\pi t^\alpha}{l} \right)^5 \right. \\ &\quad \left. - \frac{1}{\Gamma(1+7\alpha)} \left( \frac{2c\pi t^\alpha}{l} \right)^7 + \dots \right]. \end{aligned} \quad (34)$$

In closed form this gives:

$$\begin{aligned} u(x, y, t) &= \sin \left[ \pi \left( \frac{x+y}{l} \right) \right] \\ &\quad \cdot \cos \left( \frac{2c\pi t^\alpha}{l} \right) \\ &\quad - \cos \left[ \pi \left( \frac{x+y}{l} \right) \right], \\ \sin \left( \frac{2c\pi t^\alpha}{l} \right) &= \sin \left[ \pi \left( \frac{x+y-2ct^\alpha}{l} \right) \right] \end{aligned} \quad (35)$$

which is the exact solution for the given problem. When  $\alpha = 1$ , the above  $n$ -term approximate solution for (30) becomes

$$\begin{aligned} u(x, y, t) &= \sum_{i=0}^{n-1} u_i \\ &= \sin \left[ \pi \left( \frac{x+y}{l} \right) \right] \\ &\quad \times \left[ 1 - \frac{1}{2!} \left( \frac{2c\pi t}{l} \right)^2 + \frac{1}{4!} \left( \frac{2c\pi t}{l} \right)^4 \right. \\ &\quad \left. - \frac{1}{6!} \left( \frac{2c\pi t}{l} \right)^6 + \dots \right] \\ &\quad - \cos \left[ \pi \left( \frac{x+y}{l} \right) \right] \\ &\quad \times \left[ \left( \frac{2c\pi t}{l} \right) - \frac{1}{3!} \left( \frac{2c\pi t}{l} \right)^3 \right. \\ &\quad \left. + \frac{1}{5!} \left( \frac{2c\pi t}{l} \right)^5 - \frac{1}{7!} \left( \frac{2c\pi t}{l} \right)^7 + \dots \right]. \end{aligned} \quad (36)$$

In closed form, this gives

$$\begin{aligned} u(x, y, t) &= \sin \left[ \pi \left( \frac{x+y}{l} \right) \right] \\ &\quad \cdot \cos \left( \frac{2c\pi t}{l} \right) - \cos \left[ \pi \left( \frac{x+y}{l} \right) \right], \\ \sin \left( \frac{2c\pi t}{l} \right) &= \sin \left[ \pi \left( \frac{x+y-2ct}{l} \right) \right], \end{aligned} \quad (37)$$

which is the same result obtained by variational iteration method in [14].

*Example 7.* Consider the following fractional order heat equation in 2-dimensional space [11]:

$$\begin{aligned} D_{*t}^\beta u(x, y, t) &= \alpha(u_{xx} + u_{yy}), \\ u(x, y, 0) &= c \left[ \sin \left( \frac{\pi x}{l} \right) + \sin \left( \frac{\pi y}{l} \right) \right], \quad 0 < \beta \leq 1. \end{aligned} \quad (38)$$

The exact solution for this problem when  $\beta = 1$  is

$$u(x, y, t) = ce^{-\alpha\pi^2 t/l^2} \left[ \sin \left( \frac{\pi x}{l} \right) + \sin \left( \frac{\pi y}{l} \right) \right]. \quad (39)$$

The initial value problem (38) is equivalent to the integral equation

$$u(x, y, t) = c \left[ \sin \left( \frac{\pi x}{l} \right) + \sin \left( \frac{\pi y}{l} \right) \right] + I_t^\beta [\alpha(u_{xx} + u_{yy})]. \quad (40)$$

Let  $N(u) = I_t^\beta [\alpha(u_{xx} + u_{yy})]$ . In view of recurrence relation (17), we have the following first approximations:

$$\begin{aligned}
 u_0(x, y, t) &= c \left[ \sin\left(\frac{\pi x}{l}\right) + \sin\left(\frac{\pi y}{l}\right) \right], \\
 u_1(x, y, t) &= \frac{-c}{\Gamma(1 + \beta)} \left( \frac{\alpha \pi^2 t^\beta}{l^2} \right) \\
 &\quad \times \left[ \sin\left(\frac{\pi x}{l}\right) + \sin\left(\frac{\pi y}{l}\right) \right], \\
 u_2(x, y, t) &= \frac{c}{\Gamma(1 + 2\beta)} \left( \frac{\alpha \pi^2 t^\beta}{l^2} \right)^2 \\
 &\quad \times \left[ \sin\left(\frac{\pi x}{l}\right) + \sin\left(\frac{\pi y}{l}\right) \right], \\
 u_3(x, y, t) &= \frac{-c}{\Gamma(1 + 3\beta)} \left( \frac{\alpha \pi^2 t^\beta}{l^2} \right)^3 \\
 &\quad \times \left[ \sin\left(\frac{\pi x}{l}\right) + \sin\left(\frac{\pi y}{l}\right) \right],
 \end{aligned} \tag{41}$$

and so on. The  $n$ -term approximate solution for (38) is

$$\begin{aligned}
 u(x, y, t) &= \sum_{i=0}^{n-1} u_i \\
 &= c \left[ \sin\left(\frac{\pi x}{l}\right) + \sin\left(\frac{\pi y}{l}\right) \right] \\
 &\quad \times \left[ 1 - \frac{1}{\Gamma(1 + \beta)} \left( \frac{\alpha \pi^2 t^\beta}{l^2} \right) \right. \\
 &\quad \left. + \frac{1}{\Gamma(1 + 2\beta)} \left( \frac{\alpha \pi^2 t^\beta}{l^2} \right)^2 \right. \\
 &\quad \left. - \frac{1}{\Gamma(1 + 3\beta)} \left( \frac{\alpha \pi^2 t^\beta}{l^2} \right)^3 + \dots \right].
 \end{aligned} \tag{42}$$

When  $\beta = 1$ , The  $n$ -term approximate solution for (38) becomes

$$\begin{aligned}
 u(x, y, t) &= \sum_{i=0}^{n-1} u_i \\
 &= c \left[ \sin\left(\frac{\pi x}{l}\right) + \sin\left(\frac{\pi y}{l}\right) \right] \\
 &\quad \times \left[ 1 - \frac{1}{1!} \left( \frac{\alpha \pi^2 t}{l^2} \right) + \frac{1}{2!} \left( \frac{\alpha \pi^2 t}{l^2} \right)^2 \right. \\
 &\quad \left. - \frac{1}{3!} \left( \frac{\alpha \pi^2 t}{l^2} \right)^3 + \dots \right].
 \end{aligned} \tag{43}$$

In closed form, this gives

$$u(x, y, t) = ce^{-\alpha \pi^2 t / l^2} \left[ \sin\left(\frac{\pi x}{l}\right) + \sin\left(\frac{\pi y}{l}\right) \right] \tag{44}$$

which is the exact solution for the given problem.

The obtained results in this example are the same as these obtained in [11] by the homotopy perturbation method, in case  $\beta = 1$ , but with the simplicity of the given method.

*Example 8.* In this last example, we consider the following fractional order nonlinear wave equation [25]:

$$\begin{aligned}
 D_{*x}^\alpha u(x, t) - uu_{tt} &= \frac{x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{(x^2 + t^2)}{2}, \\
 u(0, t) &= \frac{t^2}{2}, \quad u_x(0, t) = 0, \quad 1 < \alpha \leq 2.
 \end{aligned} \tag{45}$$

The exact solution for this problem when  $\alpha = 2$  is  $u(x, t) = (1/2)(x^2 + t^2)$  where  $0 \leq x, t \leq 1$ .

The initial value problem (45) is equivalent to the integral equation

$$u(x, t) = \frac{1}{2} (x^2 + t^2) - \frac{x^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{t^2 x^\alpha}{2\Gamma(1+\alpha)} + I_x^\alpha [uu_{tt}]. \tag{46}$$

Let  $N(u) = I_x^\alpha [uu_{tt}]$ . In view of recurrence relation (8), we have

$$\begin{aligned}
 u_0(x, t) &= \frac{1}{2} (x^2 + t^2) \\
 &\quad - \frac{x^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{t^2 x^\alpha}{2\Gamma(1+\alpha)}, \\
 u_1(x, t) &= \frac{x^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{x^{2+2\alpha}}{\Gamma(3+2\alpha)} \\
 &\quad - \frac{\Gamma(3+\alpha) x^{2+2\alpha}}{2\Gamma(1+\alpha)\Gamma(3+2\alpha)} \\
 &\quad + \frac{\Gamma(3+2\alpha) x^{2+3\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)} \\
 &\quad + t^2 \left( \frac{x^\alpha}{2\Gamma(1+\alpha)} - \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \right. \\
 &\quad \left. + \frac{\Gamma(1+2\alpha) x^{3\alpha}}{2\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} \right),
 \end{aligned} \tag{47}$$

and so on. The 3-term approximate solution and the corresponding exact solution for (45) are plotted in Figure 4(a), in case  $t = 1/2$ , for  $\alpha = 1.8, 1.9, 2.$ , in Figure 4(b), in case  $t = 1$ , for  $\alpha = 1.8, 1.9, 2.$ , and in Figure 4(c), in case  $\alpha = 2$ . It is remarkable to note that in the first two figures all the solutions are almost equal.

Comparing these results with those obtained by the modification homotopy perturbation method in [25], we can confirm the accuracy and simplicity of the given method.

### 5. Conclusion

In this paper, the new iterative method with suitable algorithm is successfully used to solve linear and nonlinear

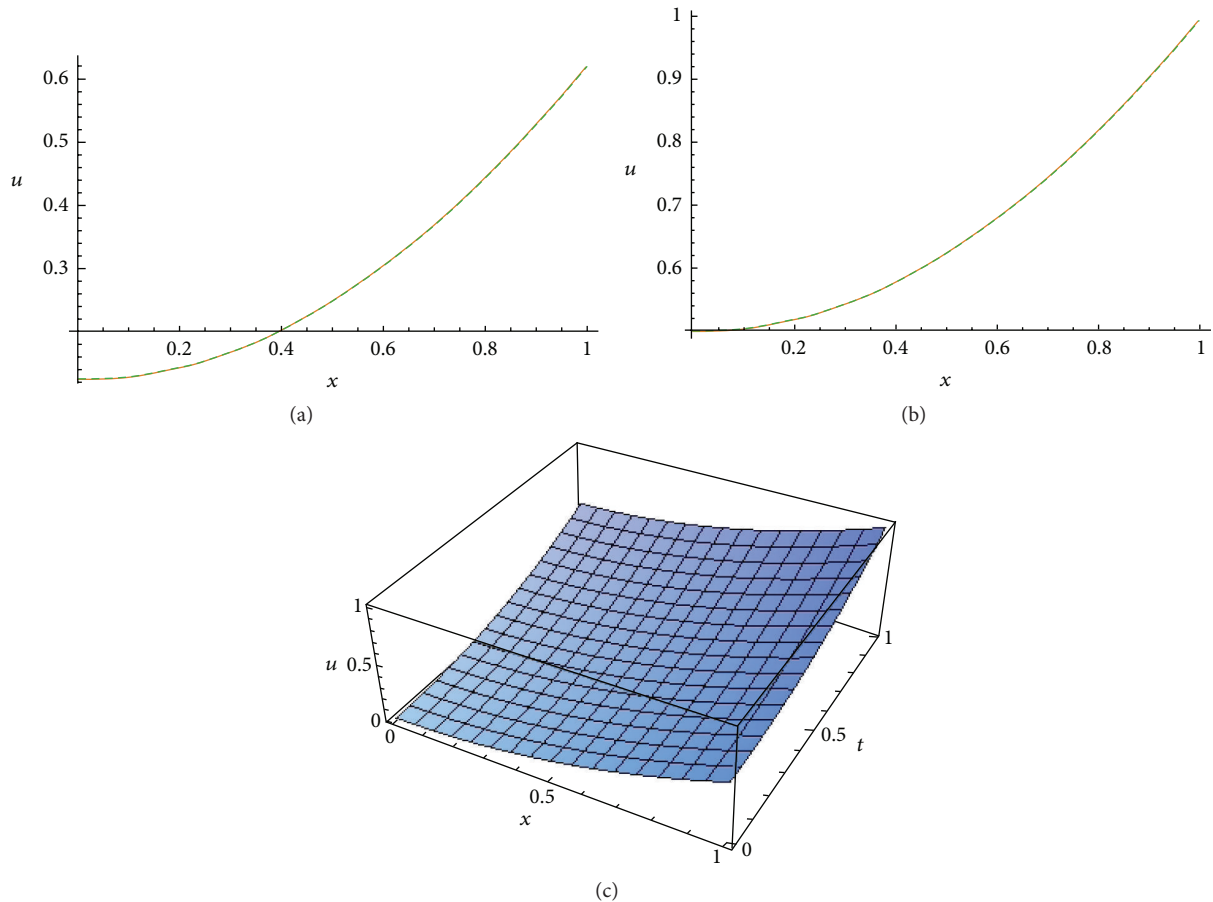


FIGURE 4: (a) Plots of the approximate solution for different values of  $\alpha$  and the exact solution, in case  $t = 1/2$ ; for (45). (b) Plots of the approximate solution for different values of  $\alpha$  and the exact solution, in case  $t = 1$ ; for (45). (c) Plots of the approximate solution, in case  $\alpha = 2$  for (45).

ordinary and partial differential equations with fractional order. It is clear that the computations are easy and the solutions agree well with the corresponding exact solutions and more accurate than the solutions obtained by other methods. Moreover, the accuracy is high with little computed terms of the solution which confirm that this method with the given algorithm is a powerful method for handling fractional differential equations.

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