

Research Article

A Fixed Point Approach to the Stability of an Integral Equation Related to the Wave Equation

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We will apply the fixed point method for proving the generalized Hyers-Ulam stability of the integral equation $(1/2c) \int_{x-ct}^{x+ct} u(\tau, t_0) d\tau = u(x, t)$ which is strongly related to the wave equation.

1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive functions was solved by Hyers [2] under the assumption that G_1 and G_2 are the Banach spaces. Indeed, he proved that each solution of the inequality $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$, for all x and y , can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, is said to have the Hyers-Ulam stability.

Rassias [3] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p) \quad (1)$$

and proved the Hyers theorem. That is, Rassias proved the generalized Hyers-Ulam stability (or Hyers-Ulam-Rassias stability) of the Cauchy additive functional equation. (Aoki

[4] has provided a proof of a special case of Rassias' theorem just for the stability of the additive function. Aoki did not prove the stability of the linear function, which was implied by Rassias' theorem.) Since then, the stability of several functional equations has been extensively investigated [5–12].

The terminologies generalized Hyers-Ulam stability, Hyers-Ulam-Rassias stability, and Hyers-Ulam stability can also be applied to the case of other functional equations, differential equations, and various integral equations.

Let c and t_0 be fixed real numbers with $c > 0$. For any differentiable function $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$, the function defined as

$$u(x, t) := \frac{1}{2c} \int_{x-ct}^{x+ct} h(\tau, t_0) d\tau \quad (2)$$

is a solution of the wave equation

$$u_{tt}(x, t) = c^2 u_{xx}(x, t), \quad (3)$$

as we see

$$\begin{aligned} u_t(x, t) &= \frac{1}{2c} \frac{\partial}{\partial t} \int_{x-ct}^{x+ct} h(\tau, t_0) d\tau \\ &= \frac{1}{2} h(x+ct, t_0) + \frac{1}{2} h(x-ct, t_0), \\ u_{tt}(x, t) &= \frac{c}{2} h_x(x+ct, t_0) - \frac{c}{2} h_x(x-ct, t_0), \end{aligned}$$

$$\begin{aligned}
 u_x(x, t) &= \frac{1}{2c}h(x + ct, t_0) - \frac{1}{2c}h(x - ct, t_0), \\
 u_{xx}(x, t) &= \frac{1}{2c}h_x(x + ct, t_0) - \frac{1}{2c}h_x(x - ct, t_0),
 \end{aligned}
 \tag{4}$$

from which we know that $u(x, t)$ satisfies the wave equation (3).

Conversely, we know that every solution $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ of the wave equation (3) can be expressed by

$$u(x, t) = f(x + ct) + g(x - ct), \tag{5}$$

where $f, g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ are arbitrary twice differentiable functions. If these $f(x, t)$ and $g(x, t)$ satisfy

$$\begin{aligned}
 \frac{1}{2c} \int_{x-ct}^{x+ct} f(\tau) d\tau &= f(x + ct), \\
 \frac{1}{2c} \int_{x-ct}^{x+ct} g(\tau) d\tau &= g(x - ct)
 \end{aligned}
 \tag{6}$$

for all $x, t \in \mathbb{R}$, then $u(x, t)$ expressed by (5) satisfies the integral equation (7). These facts imply that the integral equation (7) is strongly connected with the wave equation (3).

Cădariu and Radu [13] applied the fixed point method to the investigation of the Cauchy additive functional equation. Using such a clever idea, they could present another proof for the Hyers-Ulam stability of that equation [14–19].

In this paper, we introduce the integral equation:

$$\frac{1}{2c} \int_{x-ct}^{x+ct} u(\tau, t_0) d\tau = u(x, t), \tag{7}$$

which may be considered as a special form of (2), and prove the generalized Hyers-Ulam stability of the integral equation (7) by using ideas from [13, 15, 19, 20]. More precisely, assume that $\varphi(x, t)$ is a given function and $u(x, t)$ is an arbitrary and continuous function which satisfies the integral inequality:

$$\left| \frac{1}{2c} \int_{x-ct}^{x+ct} u(\tau, t_0) d\tau - u(x, t) \right| \leq \varphi(x, t). \tag{8}$$

If there exist a function $u_0(x, t)$ and a constant $C > 0$ such that

$$\begin{aligned}
 \frac{1}{2c} \int_{x-ct}^{x+ct} u_0(\tau, t_0) d\tau &= u_0(x, t), \\
 |u(x, t) - u_0(x, t)| &\leq C\varphi(x, t),
 \end{aligned}
 \tag{9}$$

then we say that the integral equation (7) has the generalized Hyers-Ulam stability.

2. Preliminaries

For a nonempty set X , we introduce the definition of the generalized metric on X . A function $d : X \times X \rightarrow [0, \infty]$ is called a generalized metric on X if and only if d satisfies

$$(M_1) \quad d(x, y) = 0 \text{ if and only if } x = y;$$

$$(M_2) \quad d(x, y) = d(y, x) \text{ for all } x, y \in X;$$

$$(M_3) \quad d(x, z) \leq d(x, y) + d(y, z) \text{ for all } x, y, z \in X.$$

We remark that the only one difference of the generalized metric from the usual metric is that the range of the former is permitted to include the infinity.

We now introduce one of fundamental results of fixed point theory. For the proof, we refer to [21]. This theorem will play an important role in proving our main theorems.

Theorem 1. *Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \rightarrow X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^kx) < \infty$ for some $x \in X$, then the following are true:*

- (a) the sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ ;
- (b) x^* is the unique fixed point of Λ in

$$X^* = \{y \in X \mid d(\Lambda^k x, y) < \infty\}; \tag{10}$$

- (c) if $y \in X^*$, then

$$d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y). \tag{11}$$

3. The Generalized Hyers-Ulam Stability

In the following theorem, for given real numbers a, b, c , and t_0 satisfying $c > 0, t_0 > 0$, and $a + ct_0 < b - ct_0$, let $I := [a, b], T := (0, t_0]$, and $I_0 := [a + ct_0, b - ct_0]$ be finite intervals. Assume that L and M are positive constants with $0 < L < 1$. Moreover, let $\varphi : I \times T \rightarrow (0, 1]$ be a continuous function satisfying

$$\frac{1}{2c} \int_{x-ct}^{x+ct} \varphi(\tau, t_0) d\tau \leq L\varphi(x, t) \tag{12}$$

for all $x \in I_0$ and $t \in T$.

We denote by X the set of all functions $f : I \times T \rightarrow \mathbb{C}$ with the following properties:

- (a) $f(x, t)$ is continuous for all $x \in I_0$ and $t \in T$;
- (b) $f(x, t) = 0$ for all $x \in I \setminus I_0$ and $t \in T$;
- (c) $|f(x, t)| \leq M\varphi(x, t)$ for all $x \in I_0$ and $t \in T$.

Moreover, we introduce a generalized metric on X as follows:

$$\begin{aligned}
 d(f, g) &:= \inf \{C \in [0, \infty] \mid |f(x, t) - g(x, t)| \\
 &\leq C\varphi(x, t) \quad \forall x \in I_0, t \in T\}.
 \end{aligned}
 \tag{13}$$

Theorem 2. *If a function $u \in X$ satisfies the integral inequality:*

$$\left| \frac{1}{2c} \int_{x-ct}^{x+ct} u(\tau, t_0) d\tau - u(x, t) \right| \leq \varphi(x, t) \tag{14}$$

for all $x \in I_0$ and $t \in T$, then there exists a unique function $u_0 \in X$ which satisfies

$$\frac{1}{2c} \int_{x-ct}^{x+ct} u_0(\tau, t_0) d\tau = u_0(x, t), \tag{15}$$

$$|u(x, t) - u_0(x, t)| \leq \frac{1}{1-L} \varphi(x, t) \tag{16}$$

for all $x \in I_0$ and $t \in T$.

Proof. First, we show that (X, d) is complete. Let $\{h_n\}$ be a Cauchy sequence in (X, d) . Then, for any $\varepsilon > 0$ there exists an integer $N_\varepsilon > 0$ such that $d(h_m, h_n) \leq \varepsilon$ for all $m, n \geq N_\varepsilon$. In view of (13), we have

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall m, n \geq N_\varepsilon, \forall x \in I_0, \forall t \in T: \tag{17}$$

$$|h_m(x, t) - h_n(x, t)| \leq \varepsilon \varphi(x, t).$$

If x and t are fixed, (17) implies that $\{h_n(x, t)\}$ is a Cauchy sequence in \mathbb{C} . Since \mathbb{C} is complete, $\{h_n(x, t)\}$ converges for any $x \in I_0$ and $t \in T$. Thus, considering (b), we can define a function $h : I \times T \rightarrow \mathbb{C}$ by

$$h(x, t) := \lim_{n \rightarrow \infty} h_n(x, t), \quad (x \in I, t \in T). \tag{18}$$

Since φ is bounded on $I_0 \times T$, (17) implies that $\{h_n|_{I_0 \times T}\}$ converges uniformly to $h|_{I_0 \times T}$ in the usual topology of \mathbb{C} . Hence, h is continuous and $|h|$ is bounded on $I_0 \times T$ with an upper bound $M\varphi(x, t)$; that is, $h \in X$. (It has not been proved yet that $\{h_n\}$ converges to h in (X, d) .)

If we let m increase to infinity, it follows from (17) that

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon, \forall x \in I_0, \forall t \in T: \tag{19}$$

$$|h(x, t) - h_n(x, t)| \leq \varepsilon \varphi(x, t).$$

By considering (13), we get

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon : d(h, h_n) \leq \varepsilon. \tag{20}$$

This implies that the Cauchy sequence $\{h_n\}$ converges to h in (X, d) . Hence, (X, d) is complete.

We now define an operator $\Lambda : X \rightarrow X$ by

$$(\Lambda h)(x, t) := \begin{cases} \frac{1}{2c} \int_{x-ct}^{x+ct} h(\tau, t_0) d\tau & (x \in I_0, t \in T), \\ 0 & (\text{otherwise}) \end{cases} \tag{21}$$

for all $h \in X$. Then, according to the fundamental theorem of calculus, Λh is continuous on $I_0 \times T$. Furthermore, it follows from (12), (c), and (21) that

$$\begin{aligned} |(\Lambda h)(x, t)| &\leq \frac{1}{2c} \int_{x-ct}^{x+ct} |h(\tau, t_0)| d\tau \\ &\leq \frac{1}{2c} \int_{x-ct}^{x+ct} M\varphi(\tau, t_0) d\tau \\ &\leq ML\varphi(x, t) < M\varphi(x, t) \end{aligned} \tag{22}$$

for any $x \in I_0$ and $t \in T$. Hence, we conclude that $\Lambda h \in X$.

We assert that Λ is strictly contractive on X . Given any $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f, g) \leq C_{fg}$. That is,

$$|f(x, t) - g(x, t)| \leq C_{fg} \varphi(x, t) \tag{23}$$

for all $x \in I_0$ and $t \in T$. Then, it follows from (12), (21), and (23) that

$$\begin{aligned} &|(\Lambda f)(x, t) - (\Lambda g)(x, t)| \\ &= \frac{1}{2c} \left| \int_{x-ct}^{x+ct} (f(\tau, t_0) - g(\tau, t_0)) d\tau \right| \\ &\leq \frac{1}{2c} \int_{x-ct}^{x+ct} |f(\tau, t_0) - g(\tau, t_0)| d\tau \\ &\leq \frac{C_{fg}}{2c} \int_{x-ct}^{x+ct} \varphi(\tau, t_0) d\tau \\ &\leq LC_{fg} \varphi(x, t) \end{aligned} \tag{24}$$

for all $x \in I_0$ and $t \in T$. That is, $d(\Lambda f, \Lambda g) \leq LC_{fg}$. Hence, we may conclude that $d(\Lambda f, \Lambda g) \leq Ld(f, g)$ for any $f, g \in X$ and we note that $0 < L < 1$.

We prove that the distance between the first two successive approximations of Λ is finite. Let $h_0 \in X$ be given. By (b), (c), and (13) and from the fact that $\Lambda h_0 \in X$, we have

$$\begin{aligned} |(\Lambda h_0)(x, t) - h_0(x, t)| &\leq |(\Lambda h_0)(x, t)| + |h_0(x, t)| \\ &\leq 2M\varphi(x, t) \end{aligned} \tag{25}$$

for any $x \in I_0$ and $t \in T$. Thus, (13) implies that

$$d(\Lambda h_0, h_0) \leq 2M < \infty. \tag{26}$$

Therefore, it follows from Theorem 1(a) that there exists a $u_0 \in X$ such that $\Lambda^n h_0 \rightarrow u_0$ in (X, d) and $\Lambda u_0 = u_0$.

In view of (c) and (13), it is obvious that $\{f \in X \mid d(h_0, f) < \infty\} = X$, where h_0 was chosen with the property (26). Now, Theorem 1(b) implies that u_0 is the unique element of X which satisfies $(\Lambda u_0)(x, t) = u_0(x, t)$ for any $x \in I_0$ and $t \in T$.

Finally, Theorem 1(c), together with (13) and (14), implies that

$$d(u, u_0) \leq \frac{1}{1-L} d(\Lambda u, u) \leq \frac{1}{1-L}, \tag{27}$$

since (14) means that $d(\Lambda u, u) \leq 1$. In view of (13), we can conclude that (16) holds for all $x \in I_0$ and $t \in T$. \square

Remark 3. Even though condition (12) seems to be strict, the condition can be satisfied provided that a and b are chosen so that $|b - a|$ is small enough and c is a large number.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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