Research Article

Self-Consistent Sources and Conservation Laws for Nonlinear Integrable Couplings of the Li Soliton Hierarchy

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New explicit Lie algebras are introduced for which the nonlinear integrable couplings of the Li soliton hierarchy are obtained. Then, the nonlinear integrable couplings of Li soliton hierarchy with self-consistent sources are established. Finally, we present the infinitely many conservation laws for the nonlinear integrable coupling of Li soliton hierarchy.

1. Introduction

Soliton theory has achieved great success during the last decades, it is being applied to many fields. The diversity and complexity of soliton theory enables investigators to do research from different views, such as binary nonlinearization of soliton hierarchy [1] and Bäcklund transformations of soliton systems from symmetry constraints [2].

Recently, with the development of integrable systems, integrable couplings have attracted much attention. Integrable couplings [3, 4] are coupled systems of integrable equations, which have been introduced when we study of Virasoro symmetric algebras. It is an important topic to look for integrable couplings because integrable couplings have much richer mathematical structures and better physical meanings. In recent years, many methods of searching for integrable couplings have been developed [5–13], but all the integrable couplings obtained are linear for the $v = (v_1, \ldots, v_m)^T$. As for how to generate nonlinear integrable couplings, Ma proposed a general scheme [14]. Suppose that an integrable system

$$u_t = K(u) \tag{1}$$

has a Lax pair U and V, which belong to semisimple matrix Lie algebras. Introduce an enlarged spectral matrix

$$\overline{U} = \overline{U}(\overline{u}) = \begin{bmatrix} U(u) & 0 \\ U_a(v) & U(u) + U_a(v) \end{bmatrix} \tag{2}$$

from a zero curvature representation

$$\overline{U}_t - \overline{V}_x + [\overline{U}, \overline{V}] = 0, \tag{3}$$

where

$$\overline{V} = \overline{V}(\overline{u}) = \begin{bmatrix} V(u) & 0 \\ V_a(\overline{u}) & V(u) + V_a(\overline{u}) \end{bmatrix}, \tag{4}$$

then we can give rise to

$$\begin{aligned} U_t - V_x + [U, V] &= 0, \\ U_{a,t} - V_{a,x} + [U, V_a] + [U_a, V] + [U_a, V_a] &= 0. \end{aligned} \tag{5}$$

This is an integrable coupling of (1), and it is a nonlinear integrable coupling because the commutator $[U_a, V_a]$ can generate nonlinear terms.

Soliton equation with self-consistent sources (SESCS) [15–22] is an important part in soliton theory. Physically,

the sources may result in solitary waves with a nonconstant velocity and therefore lead to a variety of dynamics of physical models. For applications, these kinds of systems are usually used to describe interactions between different solitary waves and are relevant to some problems of hydrodynamics, solid state physics, plasma physics, and so forth. How to obtain an integrable coupling of the SESCS is an interesting topic; in this paper, we will use new formula [23] presented by us to generalize soliton hierarchy with self-consistent sources.

The conservation laws play an important role in discussing the integrability for soliton hierarchy. An infinite number of conservation laws for KdV equation was first discovered by Miura et al. [24]. The direct construction method of multipliers for the conservation laws was presented [25], the Lagrangian approach for evolution equations was considered in [26], Wang and Xia established the infinitely many conservation laws for the integrable super G-J hierarchy [27], and the infinite conservation laws of the generalized quasilinear hyperbolic equations were derived in [28]. Comparatively, the less nonlinear integrable couplings of the soliton equations have been considered for their conservation laws.

This paper is organized as follows. In Section 2, a kind of explicit Lie algebras with the forms of blocks is introduced to generate nonlinear integrable couplings of Li soliton hierarchy. In Section 3, a new nonlinear integrable coupling of Li soliton hierarchy with self-consistent sources is derived. In Section 4, we obtain the conservation laws for the nonlinear integrable couplings of Li hierarchy. Finally, some conclusions are given.

2. Lie Algebras for Constructing Nonlinear Integrable Couplings of Li Soliton Hierarchy

Tu [29] presented a base of the Li algebra sl(2) as follows:

$$G_1 = \text{span} \{e_1, e_2, e_3\}, \tag{6}$$

where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{7}$$

which have the commutative relations

$$[e_1, e_2] = 2e_2, \qquad [e_1, e_3] = -2e_3, \qquad [e_2, e_3] = e_1. \tag{8}$$

Let us introduce a Lie algebra with matrix blocks by using G_1 in order to get nonlinear couplings of soliton hierarchy as follows:

$$G = \text{span} \{g_1, \ldots, g_6\}, \tag{9}$$

where

$$g_1 = \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} e_2 & 0 \\ 0 & e_2 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} e_3 & 0 \\ 0 & e_3 \end{pmatrix}, \qquad g_4 = \begin{pmatrix} 0 & 0 \\ e_1 & e_1 \end{pmatrix}, \tag{10}$$

$$g_5 = \begin{pmatrix} 0 & 0 \\ e_2 & e_2 \end{pmatrix}, \qquad g_6 = \begin{pmatrix} 0 & 0 \\ e_3 & e_3 \end{pmatrix}.$$

Define a commutator as follows:

$$[a, b] = ab - ba, \qquad a, b \in G. \tag{11}$$

A direct verification exhibits that

$$[g_1, g_2] = 2g_3, \qquad [g_1, g_3] = 2g_2,$$

$$[g_2, g_3] = -2g_1, \qquad [g_1, g_5] = 2g_6,$$

$$[g_1, g_6] = 2g_5, \qquad [g_2, g_4] = -2g_6,$$

$$[g_2, g_6] = -2g_4, \tag{12}$$

$$[g_3, g_4] = -2g_5, \qquad [g_3, g_5] = 2g_4,$$

$$[g_4, g_5] = 2g_6, \qquad [g_4, g_6] = 2g_5,$$

$$[g_5, g_6] = -2g_4, \qquad [g_1, g_4] = [g_3, g_6] = 0.$$

Set

$$\widetilde{G_1} = \text{span} \{g_1, g_2, g_3\}, \qquad \widetilde{G_2} = \text{span} \{g_4, g_5, g_6\}, \tag{13}$$

then we find that

$$G = \widetilde{G_1} \oplus \widetilde{G_2}, \qquad \widetilde{G_1} \cong G_1, \qquad \left[\widetilde{G_1}, \widetilde{G_2}\right] \subseteq \widetilde{G_2}, \tag{14}$$

and $\widetilde{G_1}$ and $\widetilde{G_2}$ are all simple Lie subalgebras.

While we use Lie algebras to generate integrable hierarchies of evolution equations, we actually employ their loop algebras $\widetilde{G} = G \otimes C(\lambda, \lambda^{-1})$ to establish Lax pairs, where $C(\lambda, \lambda^{-1})$ represents a set of Laurent ploynomials in λ and G is a Lie algebra. Based on this, we give the loop algebras of (9) as follows:

$$\widetilde{G} = \text{span} \{g_1(n), \ldots, g_6(n)\}, \tag{15}$$

where $g_i(n) = g_i \lambda^n, [g_i(m), g_j(n)] = [g_i, g_j]\lambda^{m+n}, 1 \le i, j \le 6, m, n \in Z$.

We consider an auxiliary linear problem as follows:

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}_x = U(u, \lambda) \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix},$$

$$U(u, \lambda) = R_1 + \sum_{i=1}^{6} u_i g_i(\lambda), \tag{16}$$

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}_{t_n} = V_n(u, \lambda) \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix},$$

where $u = (u_1, \ldots, u_s)^T, U_n = R_1 + u_1 g_1 + \cdots + u_6 g_6, R_1$ is a pseudoregular element, $u_i(n, t) = u_i$ $(i = 1, 2, \ldots, 6)$, and $\varphi_i = \varphi(x, t)$ are field variables defined on $x \in R, t \in R, g_i(\lambda) \in \widetilde{G}$.

The compatibility of (16) gives rise to the well-known zero curvature equation

$$U_t - V_x + [U, V] = 0, \qquad \lambda_t = 0. \tag{17}$$

The general scheme of searching for the consistent V_n, and generating a hierarchy of zero curvature equations was proposed in [30]. Solving the following equation:

$$V_x = [U, V],$$

$$V = \sum_{n=0}^{\infty} V_n \lambda^{-n}$$

$$= \begin{pmatrix} a & b+c & 0 & 0 \\ b-c & -a & 0 & 0 \\ e & f+g & a+e & b+c+f+g \\ f-g & -e & b-c+f-g & -(a+e) \end{pmatrix},$$

(18)

then we sesrch for $\triangle_n \in \widetilde{G}$, the new V_n can be constructed by

$$V_n = \sum_{m=0}^{n} V_m(u) \lambda^{n-m} + \triangle_n(u, \lambda).$$

(19)

Solving zero curvature (17), we could get evolution equation as follows:

$$u_t = K\left(u, u_x, \ldots, \frac{\partial^p u}{\partial x^p}\right).$$

(20)

Now, we consider Li soliton hierarchy [31]. In order to set up nonlinear integrable couplings of the Li soliton hierarchy with self-consistent sources, we first consider the following matrix spectral problem:

$$\varphi_x = U(u, \lambda)\varphi,$$

$$U(u, \lambda) = -g_1(1) + vg_1(0) + ug_2(0) + vg_3(0)$$

(21)

$$- g_4(1) + p_2 g_4(0) + p_1 g_5(0) + p_2 g_6(0),$$

that is,

$$U(u, \lambda)$$

$$= \begin{pmatrix} -\lambda+v & u+v & 0 & 0 \\ u-v & \lambda-v & 0 & 0 \\ -\lambda+p_2 & p_1+p_2 & -2\lambda+v+p_2 & u+v+p_1+p_2 \\ p_1-p_2 & \lambda-p_2 & u-v+p_1-p_2 & 2\lambda-v-p_2 \end{pmatrix}$$

$$= \begin{pmatrix} U_1 & 0 \\ U_0 & U_1+U_0 \end{pmatrix},$$

(22)

where λ is a spectral parameter and U_1 satisfies $\varphi_x = U_1\varphi$ which is matrix spectral problem of the Li soliton hierarchy [31].

To establish the nonlinear integrable coupling system of the Li soliton hierarchy, the adjoint equation $V_x = [U, V]$ of

the spectral problem (21) is firstly solved, we assume that a solution V is given by the following:

$$V = \begin{pmatrix} a & b+c & 0 & 0 \\ b-c & -a & 0 & 0 \\ e & f+g & a+e & b+c+f+g \\ f-g & -e & b-c+f-g & -(a+e) \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} V_n \lambda^{-n}$$

$$= \sum_{n=0}^{\infty}$$

$$\times \begin{pmatrix} a_n & b_n+c_n & 0 & 0 \\ b_n-c_n & -a_n & 0 & 0 \\ e_n & f_n+g_n & a_n+e_n & b_n+c_n+f_n+g_n \\ f_n-g_n & -e_n & b_n-c_n+f_n-g_n & -(a_n+e_n) \end{pmatrix} \lambda^{-n}.$$

(23)

Therefore, the condition (18) becomes the following recursion relation:

$$a_{n,x} = 2vb_n - 2uc_n,$$

$$b_{n,x} = -2c_{n+1} + 2vc_n - 2va_n,$$

$$c_{n,x} = -2b_{n+1} + 2vb_n - 2ua_n,$$

$$e_{n,x} = -2ug_n + 2vf_n - 2p_1c_n$$
$$+ 2p_2b_n - 2p_1g_n + 2p_2f_n,$$

(24)

$$f_{n,x} = -2g_{n+1} + 2vg_n - 2ve_n$$
$$+ 2p_2c_n - 2p_2a_n + 2p_2g_n - 2p_2e_n,$$

$$g_{n,x} = -2f_{n+1} + 2vf_n - 2ue_n + 2p_2b_n$$
$$- 2p_1a_n + 2p_2f_n - 2p_1e_n.$$

Choose the initial data

$$a_0 = e_0 = \beta, \qquad b_0 = c_0 = f_0 = g_0 = 0,$$

(25)

we see that all sets of functions a_n, b_n, c_n, e_n, f_n, and g_n are uniquely determined. In particular, the first few sets are as follows:

$$a_1 = 0, \qquad b_1 = -u\beta, \qquad c_1 = -v\beta, \qquad e_1 = 0,$$

$$f_1 = -p_1\beta, \qquad g_1 = -p_2\beta, \qquad a_2 = \frac{1}{2}(v^2 - u^2),$$

$$b_2 = \left(\frac{1}{2}v_x - uv\right)\beta, \qquad c_2 = \left(\frac{1}{2}u_x - v^2\right)\beta,$$

$$e_2 = \left(\frac{1}{2}vp_2 - \frac{1}{2}up_1 + \frac{1}{4}u^2 - \frac{1}{4}v^2 - \frac{1}{4}p_1^2 + \frac{1}{4}p_2^2\right)\beta,$$

$$f_2 = \left(\frac{1}{4}p_{2,x} - \frac{1}{4}v_x + \frac{1}{2}uv - \frac{1}{2}vp_1 - \frac{1}{2}up_2 - \frac{1}{2}p_1p_2\right)\beta,$$

$$g_2 = \left(\frac{1}{4}p_{1,x} - \frac{1}{4}u_x + \frac{1}{2}v^2 - \frac{1}{2}p_2^2 - vp_2\right)\beta, \dots .$$

$$(26)$$

Considering

$$V_n = V + \triangle_n,$$

$$\triangle_n$$

$$= \begin{pmatrix} -(a_n - c_n) & 0 & 0 & 0 \\ 0 & a_n - c_n & 0 & 0 \\ -(e_n - g_n) & 0 & -(a_n - c_n) - (e_n - g_n) & 0 \\ 0 & e_n - g_n & 0 & a_n - c_n + e_n - g_n \end{pmatrix}.$$

$$(27)$$

From the zero curvature equation $U_t - V_x + [U, V] = 0$, we obtain the nonlinear integrable coupling system

$$u_{t_n} = K_n = \begin{pmatrix} u \\ v \\ p_1 \\ p_2 \end{pmatrix}_{t_n} = \begin{pmatrix} b_{n,x} \\ -(a_n - c_n)_x \\ f_{n,x} \\ -(e_n - g_n)_x \end{pmatrix}$$

$$= J \begin{pmatrix} b_n \\ a_n - c_n \\ f_n \\ e_n - g_n \end{pmatrix} = JL^n \begin{pmatrix} 0 \\ \beta \\ 0 \\ \beta \end{pmatrix}, \quad n \geq 0,$$

$$(28)$$

with the Hamiltonian operator J and the hereditary recursion operator L, respectively, as follows:

$$J = \begin{pmatrix} \partial & 0 & 0 & 0 \\ 0 & -\partial & 0 & 0 \\ 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & -\partial \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & \frac{1}{2}\partial - u & 0 & 0 \\ \partial^{-1}u\partial + \frac{1}{2}\partial & \partial^{-1}v\partial + v & 0 & 0 \\ 0 & M_1 & 0 & M_2 \\ M_3 & M_4 & M_5 & M_6 \end{pmatrix},$$

$$(29)$$

where

$$M_1 = -\frac{1}{4}\partial - \frac{1}{2}p_1 + \frac{1}{2}u,$$

$$M_2 = \frac{1}{4}\partial - \frac{1}{2}p_1 - \frac{1}{2}u,$$

$$M_3 = -\frac{1}{2}\partial^{-1}u - \frac{1}{2}\partial^{-1}p_1 - \partial^{-1}p_1\partial - \frac{1}{4}\partial,$$

$$M_4 = -\frac{1}{2}\partial^{-1}v\partial + \frac{1}{2}\partial^{-1}p_2\partial + \partial^{-1}p_2u - \frac{1}{2}v + \frac{1}{2}p_2, \quad (30)$$

$$M_5 = \frac{1}{2}\partial^{-1}u\partial + \frac{1}{2}\partial^{-1}p_1\partial + \frac{1}{4}\partial,$$

$$M_6 = -2\partial^{-1}uv - \frac{3}{2}\partial^{-1}p_1v + \frac{1}{2}\partial^{-1}v\partial$$

$$+ \frac{1}{2}\partial^{-1}p_2\partial - \frac{1}{2}v + \frac{1}{2}p_2.$$

Obviously, when $p_1 = p_2 = 0$ in (28), the above results become Li soliton hierarchy. So, we can say that (28) is integrable coupling of the Li soliton hierarchy.

Taking $n = 2$, we get that the nonlinear integrable couplings of Li soliton hierarchy are as follows:

$$u_{t_2} = \left(-\frac{1}{2}v_{xx} - u_xv - uv_x\right)\beta,$$

$$v_{t_2} = \left(\frac{1}{2}u_{xx} - 3vv_x + uu_x\right)\beta,$$

$$p_{1,t_2} = \left(\frac{1}{4}p_{2,x} - \frac{1}{4}v_x + \frac{1}{2}uv - \frac{1}{2}vp_1\right.$$

$$\left. -\frac{1}{2}up_2 - \frac{1}{2}p_1p_2\right)_x\beta, \quad (31)$$

$$p_{2,t_2} = \left(\frac{1}{4}p_{1,x} - \frac{1}{4}u_x + \frac{1}{2}v^2 - \frac{3}{4}p_2^2 - \frac{3}{2}vp_2\right.$$

$$\left. +\frac{1}{2}up_1 - \frac{1}{4}u^2 + \frac{1}{4}v^2 + \frac{1}{4}p_1^2\right)_x\beta.$$

So, we can say that the system in (28) with $n \geq 2$ provides a hierarchy of nonlinear integrable couplings for the Li hierarchy of the soliton equation.

3. Self-Consistent Sources for the Nonlinear Integrable Couplings of Li Soliton Hierarchy

According to (16), now we consider a new auxiliary linear problem. For N distinct λ_j, $j = 1, 2, \dots, N$ and the systems of (16) become in the following form:

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{pmatrix}_x = U(u, \lambda_j) \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{pmatrix}$$

$$= \sum_{i=1}^{6} u_i g_i(\lambda) \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{pmatrix}, \quad j = 1, \dots, N,$$

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{pmatrix}_{t_n} = V_n\left(u, \lambda_j\right) \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{pmatrix}$$

$$= \left[\sum_{m=0}^{n} V_m(u)\, \lambda_j^{n-m} + \Delta_n\left(u, \lambda_j\right) \right]$$

$$\times \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{pmatrix}, \quad j = 1, \ldots, N. \tag{32}$$

Based on the result in [32], we show that the following equation

$$\frac{\delta H_k}{\delta u} + \sum_{j=1}^{N} \alpha_j \frac{\delta \lambda_j}{\delta u} = 0 \tag{33}$$

holds true, where α_j is a constant. From (32), we may know that

$$\frac{\delta \lambda_j}{\delta u_i} = \alpha_j \operatorname{Tr}\left(\Psi_j \frac{\partial U\left(u, \lambda_j\right)}{\partial u_i} \right)$$

$$= \alpha_j \operatorname{Tr}\left(\Psi_j g_i\left(\lambda_j\right) \right), \quad i = 1, 2, \tag{34}$$

where Tr denotes the trace of a matrix and

$$\Psi_j = \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 & \phi_{3j}\phi_{4j} & -\phi_{3j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} & \phi_{4j}^2 & -\phi_{3j}\phi_{4j} \\ 0 & 0 & \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ 0 & 0 & \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix},$$

$$j = 1, \ldots, N. \tag{35}$$

For $i = 3, 4$ we define that

$$\frac{\delta \lambda_j}{\delta u_i} = \beta_j \operatorname{Tr}\left(\Psi_{jA} \frac{\partial U_0\left(u, \lambda_j\right)}{\partial u_i} \right), \tag{36}$$

where

$$U = \begin{pmatrix} U_1 & 0 \\ U_0 & U_1 + U_0 \end{pmatrix},$$

$$\Psi_{jA} = \begin{pmatrix} \phi_{3j}\phi_{4j} & -\phi_{3j}^2 \\ \phi_{4j}^2 & -\phi_{3j}\phi_{4j} \end{pmatrix}, \tag{37}$$

and β_j is a constant.

According to (34) and (36), we obtain a kind of nonlinear integrable couplings with self-consistent sources as follows:

$$u_{t_n} = J \frac{\delta H_{n+1}}{\delta u_i} + J \sum_{j=1}^{N} \alpha_j \frac{\delta \lambda_j}{\delta u}$$

$$= J L^n \frac{\delta H_1}{\delta u_i} + J \sum_{j=1}^{N} \alpha_j \frac{\delta \lambda_j}{\delta u}, \quad n = 1, 2, \ldots. \tag{38}$$

Therefore, according to formulas (34) and (36), we have the following results by direct computations:

$$\sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = \sum_{j=1}^{N} \begin{pmatrix} \dfrac{\delta \lambda_j}{\delta u} \\ \dfrac{\delta \lambda_j}{\delta v} \\ \dfrac{\delta \lambda_j}{\delta p_1} \\ \dfrac{\delta \lambda_j}{\delta p_1} \end{pmatrix} \tag{39}$$

$$= \begin{pmatrix} 2\left(\langle \Phi_2, \Phi_2 \rangle - \langle \Phi_1, \Phi_1 \rangle\right) \\ 2\left(\langle \Phi_1, \Phi_1 \rangle + \langle \Phi_2, \Phi_2 \rangle + 2\langle \Phi_1, \Phi_2 \rangle\right) \\ \langle \Phi_4, \Phi_4 \rangle - \langle \Phi_3, \Phi_3 \rangle \\ \langle \Phi_3, \Phi_3 \rangle + \langle \Phi_4, \Phi_4 \rangle + 2\langle \Phi_3, \Phi_4 \rangle \end{pmatrix},$$

by taking $\alpha_j = 1$ and $\beta_j = 1$ in formulas (34) and (36). Therefore, we have nonlinear integrable coupling system of the Li equations hierarchy with self-consistent sources as follows:

$$u_{t_n} = K_n$$

$$= \begin{pmatrix} u \\ v \\ p_1 \\ p_2 \end{pmatrix}_{t_n}$$

$$= J L^n \begin{pmatrix} 0 \\ \beta \\ 0 \\ \beta \end{pmatrix}$$

$$+ J \begin{pmatrix} 2\left(\langle \Phi_2, \Phi_2 \rangle - \langle \Phi_1, \Phi_1 \rangle\right) \\ 2\left(\langle \Phi_1, \Phi_1 \rangle + \langle \Phi_2, \Phi_2 \rangle + 2\langle \Phi_1, \Phi_2 \rangle\right) \\ \langle \Phi_4, \Phi_4 \rangle - \langle \Phi_3, \Phi_3 \rangle \\ \langle \Phi_3, \Phi_3 \rangle + \langle \Phi_4, \Phi_4 \rangle + 2\langle \Phi_3, \Phi_4 \rangle \end{pmatrix}$$

$$= J L^n \begin{pmatrix} 0 \\ \beta \\ 0 \\ \beta \end{pmatrix} + J \begin{pmatrix} 2\sum_{j=1}^{N}\left(\varphi_{2j}^2 - \varphi_{1j}^2\right) \\ 2\sum_{j=1}^{N}\left(\varphi_{1j}^2 + \varphi_{2j}^2 + 2\varphi_{1j}\varphi_{2j}\right) \\ \sum_{j=1}^{N}\left(\varphi_{4j}^2 - \varphi_{3j}^2\right) \\ \sum_{j=1}^{N}\left(\varphi_{3j}^2 + \varphi_{4j}^2 + 2\varphi_{3j}\varphi_{4j}\right) \end{pmatrix}, \tag{40}$$

with

$$\varphi_{1j,x} = (-\lambda + v)\,\varphi_{1j} + (u + v)\,\varphi_{2j},$$

$$\varphi_{2j,x} = (u - v)\,\varphi_{1j} + (\lambda - v)\,\varphi_{2j},$$

$$\varphi_{3j,x} = (-\lambda + p_2)\,\varphi_{1j} + (p_1 + p_2)\,\varphi_{2j}$$

$$+ (-2\lambda + v + p_2)\,\varphi_{3j} + (u + v + p_1 + p_2)\,\varphi_{4j}, \quad (41)$$

$$\varphi_{4j,x} = (p_1 - p_2)\,\varphi_{1j} + (\lambda - p_2)\,\varphi_{2j}$$

$$+ (u - v + p_1 - p_2)\,\varphi_{3j}$$

$$+ (2\lambda - v - p_2)\,\varphi_{4j}, \quad j = 1, \ldots, N,$$

where $\Phi_i = (\varphi_{i1}, \ldots, \varphi_{iN})$, $i = 1, 2, 3, 4$, and $\langle \cdot, \cdot \rangle$ is the standard inner product in R^N.

When $n = 2$ and $\beta = 2$, we obtain nonlinear integrable couplings of Li hierarchy with self-consistent sources

$$u_{t_2} = -\frac{1}{2}v_{xx} - u_x v - u v_x + 2\partial\sum_{j=1}^{N}\left(\varphi_{2j}^2 - \varphi_{1j}^2\right),$$

$$v_{t_2} = \frac{1}{2}u_{xx} - 3v v_x + u u_x$$

$$- 2\partial\sum_{j=1}^{N}\left(\varphi_{1j}^2 + \varphi_{2j}^2 + 2\varphi_{1j}\varphi_{2j}\right),$$

$$p_{1t_2} = \left(\frac{1}{4}p_{2,x} - \frac{1}{4}v_x + \frac{1}{2}uv - \frac{1}{2}v p_1 - \frac{1}{2}u p_2\right.$$

$$\left. - \frac{1}{2}p_1 p_2\right)_x + \partial\sum_{j=1}^{N}\left(\varphi_{4j}^2 - \varphi_{3j}^2\right),$$

$$p_{2t_2} = \left(\frac{1}{4}p_{1,x} - \frac{1}{4}u_x + \frac{1}{2}v^2 - \frac{3}{4}p_2^2 - \frac{3}{2}v p_2 + \frac{1}{2}u p_1\right.$$

$$\left. - \frac{1}{4}u^2 + \frac{1}{4}v^2 + \frac{1}{4}p_1^2\right)_x$$

$$- \partial\sum_{j=1}^{N}\left(\varphi_{3j}^2 + \varphi_{4j}^2 + 2\varphi_{3j}\varphi_{4j}\right),$$

$$(42)$$

with

$$\varphi_{1j,x} = (-\lambda + v)\,\varphi_{1j} + (u + v)\,\varphi_{2j},$$

$$\varphi_{2j,x} = (u - v)\,\varphi_{1j} + (\lambda - v)\,\varphi_{2j},$$

$$\varphi_{3j,x} = (-\lambda + p_2)\,\varphi_{1j} + (p_1 + p_2)\,\varphi_{2j}$$

$$+ (-2\lambda + v + p_2)\,\varphi_{3j} + (u + v + p_1 + p_2)\,\varphi_{4j},$$

$$\varphi_{4j,x} = (p_1 - p_2)\,\varphi_{1j} + (\lambda - p_2)\,\varphi_{2j}$$

$$+ (u - v + p_1 - p_2)\,\varphi_{3j}$$

$$+ (2\lambda - v - p_2)\,\varphi_{4j}, \quad j = 1, \ldots, N.$$

$$(43)$$

4. Conservation Laws for the Nonlinear Integrable Couplings of Li Soliton Hierarchy

In what follows, we will construct conservation laws for the nonlinear integrable couplings of the Li hierarchy. For the coupled spectral problem of Li hierarchy

$$U(u, \lambda)$$

$$= \begin{pmatrix} -\lambda + v & u + v & 0 & 0 \\ u - v & \lambda - v & 0 & 0 \\ -\lambda + p_2 & p_1 + p_2 & -2\lambda + v + p_2 & u + v + p_1 + p_2 \\ p_1 - p_2 & \lambda + -p_2 & u - v + p_1 - p_2 & 2\lambda - v - p_2 \end{pmatrix},$$

$$(44)$$

we introduce the variables

$$M = \frac{\varphi_2}{\varphi_1}, \qquad N = \frac{\varphi_3}{\varphi_1}, \qquad K = \frac{\varphi_4}{\varphi_1}. \qquad (45)$$

From (44), we have

$$M_x = u - v + 2\lambda M - 2vM - (u + v)M^2,$$

$$N_x = -\lambda + p_2 - \lambda N + (p_1 + p_2)M$$

$$+ p_2 N + (u + v + p_1 + p_2)K - (u + v)NM,$$

$$K_x = p_1 - p_2 + 3\lambda K + \lambda M - p_2 M$$

$$- (2v + p_2)K + (u - v + p_1 - p_2)N - (u + v)KM.$$

$$(46)$$

We expand M, N, and K in powers of λ as follows:

$$M = \sum_{j=1}^{\infty} m_j \lambda^{-j}, \qquad N = \sum_{j=1}^{\infty} n_j \lambda^{-j},$$

$$K = \sum_{j=1}^{\infty} k_j \lambda^{-j}. \qquad (47)$$

Substituting (47) into (46) and comparing the coefficients of the same power of λ, we obtain the following:

$$m_1 = \frac{1}{2}(v - u), \qquad n_1 = p_2,$$

$$k_1 = \frac{1}{3}(p_2 - p_1) + \frac{1}{6}(u - v),$$

$$m_2 = \frac{1}{4}(v - u)_x + \frac{1}{2}\left(v^2 - uv\right),$$

$$n_2 = -p_{2,x} - \frac{1}{3}p_1^2 - \frac{2}{3}up_1 + \frac{2}{3}vp_2 + \frac{4}{3}p_2^2 + \frac{1}{6}u^2 - \frac{1}{6}v^2,$$

$$k_2 = \frac{5}{36}(u - v)_x + \frac{1}{9}(p_2 - p_1)_x - \frac{5}{18}v^2 + \frac{5}{18}uv$$
$$+ \frac{2}{3}vp_2 - \frac{4}{9}up_2 + \frac{4}{9}p_2^2 - \frac{4}{9}p_1p_2 - \frac{2}{9}vp_1,$$

$$m_3 = \frac{1}{8}(v - u)_{xx} + \frac{1}{8}u^3 - \frac{1}{8}u^2v + \frac{3}{4}vv_x$$
$$- \frac{1}{2}vu_x - \frac{1}{4}uv_x + \frac{5}{8}v^3 - \frac{5}{8}uv^2,$$

$$n_3 = p_{2,xx} + \frac{5}{9}(p_1 u)_x - \frac{5}{9}(p_2 v)_x - \frac{32}{9}p_2 p_{2,x} - \frac{7}{36}uu_x$$
$$+ \frac{7}{36}vv_x + \frac{5}{9}p_1 p_{1,x} + \frac{1}{9}p_1 v_x - \frac{1}{9}p_2 u_x - \frac{5}{36}uv_x$$
$$+ \frac{1}{9}up_{2,x} + \frac{5}{36}vu_x - \frac{1}{9}vp_{1,x} + \frac{1}{9}p_1 p_{2,x} - \frac{1}{9}p_2 p_{1,x}$$
$$- \frac{4}{9}p_1 uv + \frac{11}{9}p_2 v^2 - \frac{14}{9}up_1 p_2 + \frac{16}{9}vp_2^2$$
$$+ \frac{16}{9}p_2^3 - \frac{7}{9}p_2 u^2 - \frac{7}{9}p_2 p_1^2 + \frac{5}{18}vu^2 - \frac{5}{18}v^3 - \frac{2}{9}vp_1^2,$$

$$k_3 = \frac{19}{216}(u - v)_{xx} + \frac{1}{27}(p_2 - p_1)_{xx} + \frac{5}{54}(u - v)_x$$
$$- \frac{47}{108}vv_x + \frac{7}{27}vu_x + \frac{19}{108}uv_x - \frac{4}{27}vp_{2,x}$$
$$+ \frac{7}{27}p_2 v_x + \frac{5}{27}up_{2,x} - \frac{5}{27}p_2 u_x - \frac{5}{27}p_2 p_{1,x}$$
$$+ \frac{5}{27}p_1 p_{2,x} - \frac{2}{27}p_1 v_x - \frac{4}{27}vp_{1,x} - \frac{103}{216}v^3$$
$$+ \frac{101}{216}uv^2 + \frac{1}{8}vu^2 + \frac{7}{18}p_2 v^2 - \frac{16}{27}p_2 uv$$
$$- \frac{19}{54}p_2 v^2 + \frac{32}{27}vp_2^2 - \frac{16}{27}p_1 p_2 v - \frac{4}{27}p_1 v^2$$
$$- \frac{16}{27}up_2^2 + \frac{16}{27}p_2^3 - \frac{16}{27}p_1 p_2^2 + \frac{2}{9}p_1 u^2$$
$$+ \frac{1}{18}uv^2 + \frac{1}{3}up_1^2 + \frac{1}{9}p_1^3 - \frac{2}{9}uvp_1 - \frac{2}{9}up_1 p_2$$
$$- \frac{1}{9}vp_1^2 - \frac{1}{9}p_2 p_1^2 - \frac{1}{8}u^3, \dots,$$

$$(48)$$

and a recursion formula for m_j, n_j, and k_j as follows:

$$m_{j+1} = \frac{1}{2}m_{j,x} + vm_j + \frac{1}{2}(u + v)\sum_{l=1}^{j-1}m_l m_{j-l},$$

$$n_{j+1} = -n_{j,x} + (p_1 + p_2)m_j + p_2 n_j$$
$$+ (u + v + p_1 + p_2)k_j - (u + v)\sum_{l=1}^{j-1}m_l n_{j-l},$$

$$k_{j+1} = \frac{1}{3}k_{j,x} - \frac{1}{6}m_{j,x} - \frac{1}{3}vm_j + \frac{1}{3}p_2 m_j$$
$$+ \frac{1}{3}(2v + p_2)k_j - \frac{1}{3}(u - v + p_1 - p_2)n_j$$
$$- \frac{1}{6}(u + v)\sum_{l=1}^{j-1}m_l m_{j-l} + \frac{1}{3}(u + v)\sum_{l=1}^{j-1}m_l k_{j-l}.$$

$$(49)$$

Because of

$$\frac{\partial}{\partial t}\left[-\lambda + v + (u + v)M\right] = \frac{\partial}{\partial x}\left[a + (b + c)M\right],$$

$$\frac{\partial}{\partial t}\left[-\lambda + p_2 + (p_1 + p_2)M + (-2\lambda + v + p_2)N \right.$$
$$\left. + (u + v + p_1 + p_2)K\right]$$
$$= \frac{\partial}{\partial x}\left[e + (f + g)M + (a + e)N \right.$$
$$\left. + (b + c + f + g)K\right],$$

$$(50)$$

where

$$a = \xi_0 \lambda^2 + \xi_1 \lambda + \frac{1}{2}\xi_0\left(v^2 - u^2\right),$$

$$b = -\xi_0 u\lambda + \xi_0\left(-uv + \frac{1}{2}v_x\right) - \xi_1 u,$$

$$c = -\xi_0 v\lambda + \xi_0\left(-v^2 + \frac{1}{2}u_x\right) - \xi_1 v,$$

$$e = \xi_0 \lambda^2 + \xi_1 \lambda$$
$$+ \xi_0\left(\frac{1}{2}vp_2 - \frac{1}{2}up_1 + \frac{1}{4}u^2 - \frac{1}{4}v^2 - \frac{1}{4}p_1^2 + \frac{1}{4}p_2^2\right),$$

$$f = -\xi_0 p_1 \lambda + \xi_0\left(\frac{1}{4}p_{2,x} - \frac{1}{4}v_x + \frac{1}{2}uv - \frac{1}{2}vp_1 \right.$$
$$\left. - \frac{1}{2}up_2 - \frac{1}{2}p_1 p_2\right) - \xi_1 p_1,$$

$$g = -\xi_0 p_2 \lambda + \xi_0\left(\frac{1}{4}p_{1,x} - \frac{1}{4}u_x + \frac{1}{2}v^2 \right.$$
$$\left. - \frac{1}{2}p_2^2 - vp_2\right) - \xi_1 p_2.$$

$$(51)$$

Assume that

$$\sigma = -\lambda + v + (u + v) M,$$

$$\theta = a + (b + c) M,$$

$$\rho = -\lambda + p_2 + (p_1 + p_2) M + (-2\lambda + v + p_2) N \quad (52)$$

$$+ (u + v + p_1 + p_2) K,$$

$$\delta = e + (f + g) M + (a + e) N + (b + c + f + g) K.$$

Then, (50) can be written as $\sigma_t = \theta_x$ and $\rho_t = \delta_x$, which are the right form of conservation laws. We expand $\sigma, \theta, \rho,$ and δ as series in powers of λ with the coefficients, which are called conserved densities and currents, respectively:

$$\sigma = -\lambda + v + (u + v) \sum_{j=1}^{\infty} \sigma_j \lambda^{-j},$$

$$\theta = \xi_0 \lambda^2 + \xi_1 \lambda + \sum_{j=1}^{\infty} \theta_j \lambda^{-j},$$

$$\rho = -\lambda + p_2 + \sum_{j=1}^{\infty} \rho_j \lambda^{-j}, \quad (53)$$

$$\delta = \xi_0 \lambda^2 + \xi_1 \lambda + \sum_{j=1}^{\infty} \delta_j \lambda^{-j},$$

where ξ_0 and ξ_1 are constants of integration. The first conserved densities and currents are read as follows:

$$\sigma_1 = \frac{1}{2} \left(v^2 - u^2 \right),$$

$$\theta_1 = \xi_0 \left(\frac{1}{2} u u_x - \frac{3}{4} u v_x - \frac{3}{4} v v_x \right) - \frac{1}{2} \xi_1 \left(v^2 - u^2 \right),$$

$$\rho_1 = \frac{1}{2} u p_1 + \frac{1}{3} v p_2 - \frac{4}{3} p_2^2 - \frac{1}{6} u^2$$

$$+ \frac{1}{6} v^2 + \frac{1}{3} p_1^2 + \frac{1}{6} u p_1 + 2 p_{2,x},$$

$$\delta_1 = \xi_0 \left(2 p_{2,xx} + \frac{1}{36} p_1 v_x + \frac{41}{36} p_1 u_x - \frac{41}{36} p_2 v_x \right.$$

$$- \frac{1}{36} p_2 u_x - \frac{41}{36} v p_{2,x} - \frac{1}{36} v p_{1,x}$$

$$+ \frac{13}{36} v v_x - \frac{1}{36} v u_x + \frac{1}{36} u p_{2,x}$$

$$+ \frac{41}{36} u p_{1,x} - \frac{13}{36} u u_x - \frac{257}{36} p_2 p_{2,x}$$

$$+ \frac{41}{36} p_1 p_{1,x} + \frac{1}{36} p_1 p_{2,x}$$

$$- \frac{1}{36} p_2 p_{1,x} + \frac{47}{36} p_2 v^2 - \frac{1}{18} v u^2$$

$$+ \frac{11}{36} v p_1 p_2 + \frac{55}{18} v p_2^2 - \frac{43}{36} p_2 u^2$$

$$- \frac{53}{18} u p_1 p_2 + \frac{93}{36} p_2^3$$

$$- \frac{3}{4} p_2 p_1^2 - \frac{1}{18} v p_1^2 - \frac{4}{9} v p_2^2$$

$$- \frac{1}{9} u v p_1 - \frac{4}{9} p_1 p_2^2$$

$$\left. + \frac{1}{18} v^3 + \frac{1}{36} u v_x \right)$$

$$+ \xi_1 \left(-2 p_{2,x} + v p_1 - \frac{5}{3} u p_1 + \frac{5}{3} v p_2 - u p_2 \right.$$

$$\left. + \frac{7}{3} p_2^2 + \frac{1}{6} u^2 - \frac{1}{6} v^2 - \frac{1}{3} p_1^2 \right), \ldots.$$

$$(54)$$

The recursion relations for $\sigma_j, \theta_j, \rho_j,$ and δ_j are as follows:

$$\sigma_j = (u + v) m_j,$$

$$\theta_j = -\xi_0 (u + v) m_{j+1} + \xi_0 \left(\frac{1}{2} u_x + \frac{1}{2} v_x - u v - v^2 \right) m_j$$

$$- \xi_1 (u + v) m_j,$$

$$\rho_j = (p_1 + p_2) m_j - 2 n_{j+1} + (v + p_2) n_j$$

$$+ (u + v + p_1 + p_2) k_j,$$

$$\delta_j = \xi_0 \left[-(p_1 + p_2) m_{j+1} \right.$$

$$+ \left(\frac{1}{4} p_{2,x} + \frac{1}{4} p_{1,x} - \frac{1}{4} v_x \right.$$

$$- \frac{1}{4} u_x + \frac{1}{2} u v - \frac{1}{2} v p_1$$

$$- \frac{1}{2} u p_2 - \frac{1}{2} p_1 p_2 + \frac{1}{2} v^2$$

$$\left. - \frac{1}{2} p_2^2 - p_2 v \right) m_j$$

$$+ 2 n_{j+2} + \left(\frac{1}{2} v^2 - \frac{1}{2} u^2 + \frac{1}{2} v p_2 \right.$$

$$- \frac{1}{2} u p_1 + \frac{1}{4} u^2$$

$$\left. + \frac{1}{4} p_2^2 - \frac{1}{4} v^2 - \frac{1}{4} p_1^2 \right) n_j$$

$$- (u + v + p_1 + p_2) k_{j+1}$$

$$+ \left(\frac{1}{4} u_x + \frac{1}{4} v_x + \frac{1}{4} p_{1,x} + \frac{1}{4} p_{2,x} - \frac{1}{2} uv \right.$$
$$- \frac{1}{2} v^2 - \frac{1}{2} v p_1 - \frac{1}{2} u p_2$$
$$\left. - \frac{1}{2} p_1 p_2 - \frac{1}{2} p_2^2 - v p_2 \right) \Big]$$
$$+ \xi_1 \left[2 n_{j+1} - (p_1 + p_2) m_j \right.$$
$$\left. - (u + v + p_1 + p_2) k_j \right],$$

(55)

where m_j, n_j, and k_j can be calculated from (49). The infinite conservation laws of nonlinear integrable couplings (37) can be easily obtained in (45)–(55), respectively.

5. Conclusions

In this paper, a new explicit Lie algebra was introduced, and a new nonlinear integrable couplings of Li soliton hierarchy with self-consistent sources was worked out. Then, the conservation laws of Li soliton hierarchy were also obtained. The method can be used to other soliton hierarchy with self-consistent sources. In the near future, we will investigate exact solutions of nonlinear integrable couplings of soliton equations with self-consistent sources which are derived by using our method.

Acknowledgments

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