

Research Article

Solving Optimization Problems on Hermitian Matrix Functions with Applications

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We consider the extremal inertias and ranks of the matrix expressions $f(X, Y) = A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^*$, where $A_3 = A_3^*$, B_3 , C_3 , and D_3 are known matrices and Y and X are the solutions to the matrix equations $A_1Y = C_1$, $YB_1 = D_1$, and $A_2X = C_2$, respectively. As applications, we present necessary and sufficient condition for the previous matrix function $f(X, Y)$ to be positive (negative), non-negative (positive) definite or nonsingular. We also characterize the relations between the Hermitian part of the solutions of the above-mentioned matrix equations. Furthermore, we establish necessary and sufficient conditions for the solvability of the system of matrix equations $A_1Y = C_1$, $YB_1 = D_1$, $A_2X = C_2$, and $B_3X + (B_3X)^* + C_3YD_3 + (C_3YD_3)^* = A_3$, and give an expression of the general solution to the above-mentioned system when it is solvable.

1. Introduction

Throughout, we denote the field of complex numbers by \mathbb{C} , the set of all $m \times n$ matrices over \mathbb{C} by $\mathbb{C}^{m \times n}$, and the set of all $m \times m$ Hermitian matrices by $\mathbb{C}_h^{m \times m}$. The symbols A^* and $\mathcal{R}(A)$ stand for the conjugate transpose, the column space of a complex matrix A respectively. I_n denotes the $n \times n$ identity matrix. The Moore-Penrose inverse [1] A^\dagger of A , is the unique solution X to the four matrix equations:

- (i) $AXA = A$,
- (ii) $XAX = X$,
- (iii) $(AX)^* = AX$,
- (iv) $(XA)^* = XA$.

Moreover, L_A and R_A stand for the projectors $L_A = I - A^\dagger A$, $R_A = I - AA^\dagger$ induced by A . It is well known that the eigenvalues of a Hermitian matrix $A \in \mathbb{C}^{n \times n}$ are real, and the inertia of A is defined to be the triplet

$$I_n(A) = \{i_+(A), i_-(A), i_0(A)\}, \quad (2)$$

where $i_+(A)$, $i_-(A)$, and $i_0(A)$ stand for the numbers of positive, negative, and zero eigenvalues of A , respectively. The

symbols $i_+(A)$ and $i_-(A)$ are called the positive index and the negative index of inertia, respectively. For two Hermitian matrices A and B of the same sizes, we say $A \geq B$ ($A \leq B$) in the Löwner partial ordering if $A - B$ is positive (negative) semidefinite. The Hermitian part of X is defined as $H(X) = X + X^*$. We will say that X is Re-nnd (Re-nonnegative semidefinite) if $H(X) \geq 0$, X is Re-pd (Re-positive definite) if $H(X) > 0$, and X is Re-ns if $H(X)$ is nonsingular.

It is well known that investigation on the solvability conditions and the general solution to linear matrix equations is very active (e.g., [2–9]). In 1999, Braden [10] gave the general solution to

$$BX + (BX)^* = A. \quad (3)$$

In 2007, Djordjević [11] considered the explicit solution to (3) for linear bounded operators on Hilbert spaces. Moreover, Cao [12] investigated the general explicit solution to

$$BXC + (BXC)^* = A. \quad (4)$$

Xu et al. [13] obtained the general expression of the solution of operator equation (4). In 2012, Wang and He [14] studied

some necessary and sufficient conditions for the consistence of the matrix equation

$$A_1X + (A_1X)^* + B_1YC_1 + (B_1YC_1)^* = E_1 \quad (5)$$

and presented an expression of the general solution to (5).

Note that (5) is a special case of the following system:

$$\begin{aligned} A_1Y = C_1, \quad YB_1 = D_1, \quad A_2X = C_2, \\ B_3X + (B_3X)^* + C_3YD_3 + (C_3YD_3)^* = A_3. \end{aligned} \quad (6)$$

To our knowledge, there has been little information about (6). One goal of this paper is to give some necessary and sufficient conditions for the solvability of the system of matrix (6) and present an expression of the general solution to system (6) when it is solvable.

In order to find necessary and sufficient conditions for the solvability of the system of matrix equations (6), we need to consider the extremal ranks and inertias of (10) subject to (13) and (11).

There have been many papers to discuss the extremal ranks and inertias of the following Hermitian expressions:

$$p(X) = A_3 - B_3X - (B_3X)^*, \quad (7)$$

$$g(Y) = A - BYC - (BYC)^*, \quad (8)$$

$$h(X, Y) = A_1 - B_1XB_1^* - C_1YC_1^*, \quad (9)$$

$$f(X, Y) = A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^*. \quad (10)$$

Tian has contributed much in this field. One of his works [15] considered the extremal ranks and inertias of (7). He and Wang [16] derived the extremal ranks and inertias of (7) subject to $A_1X = C_1, A_2XB_2 = C_2$. Liu and Tian [17] studied the extremal ranks and inertias of (8). Chu et al. [18] and Liu and Tian [19] derived the extremal ranks and inertias of (9). Zhang et al. [20] presented the extremal ranks and inertias of (9), where X and Y are Hermitian solutions of

$$A_2X = C_2, \quad (11)$$

$$YB_2 = D_2, \quad (12)$$

respectively. He and Wang [16] derived the extremal ranks and inertias of (10). We consider the extremal ranks and inertias of (10) subject to (11) and

$$A_1Y = C_1, \quad YB_1 = D_1, \quad (13)$$

which is not only the generalization of the above matrix functions, but also can be used to investigate the solvability conditions for the existence of the general solution to the system (6). Moreover, it can be applied to characterize the relations between Hermitian part of the solutions of (11) and (13).

The remainder of this paper is organized as follows. In Section 2, we consider the extremal ranks and inertias of (10) subject to (11) and (13). In Section 3, we characterize the relations between the Hermitian part of the solution to (11) and (13). In Section 4, we establish the solvability conditions for the existence of a solution to (6) and obtain an expression of the general solution to (6).

2. Extremal Ranks and Inertias of Hermitian Matrix Function (10) with Some Restrictions

In this section, we consider formulas for the extremal ranks and inertias of (10) subject to (11) and (13). We begin with the following Lemmas.

Lemma 1 (see [21]). (a) Let A_1, C_1, B_1 , and D_1 be given. Then the following statements are equivalent:

(1) system (13) is consistent,

(2)

$$R_{A_1}C_1 = 0, \quad D_1L_{B_1} = 0, \quad A_1D_1 = C_1B_1. \quad (14)$$

(3)

$$\begin{aligned} r[A_1 \ C_1] &= r(A_1), \\ \begin{bmatrix} D_1 \\ B_1 \end{bmatrix} &= r(B_1), \\ A_1D_1 &= C_1B_1. \end{aligned} \quad (15)$$

In this case, the general solution can be written as

$$Y = A_1^\dagger C_1 + L_{A_1} D_1 B_1^\dagger + L_{A_1} V R_{B_1}, \quad (16)$$

where V is arbitrary.

(b) Let A_2 and C_2 be given. Then the following statements are equivalent:

(1) equation (11) is consistent,

(2)

$$R_{A_2}C_2 = 0, \quad (17)$$

(3)

$$r[A_2 \ C_2] = r(A_2). \quad (18)$$

In this case, the general solution can be written as

$$X = A_2^\dagger C_2 + L_{A_2} W, \quad (19)$$

where W is arbitrary.

Lemma 2 ([22, Lemma 1.5, Theorem 2.3]). Let $A \in \mathbb{C}_h^{m \times m}$, $B \in \mathbb{C}^{m \times n}$, and $D \in \mathbb{C}_h^{n \times n}$, and denote that

$$\begin{aligned} M &= \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \\ N &= \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix}, \\ L &= \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}, \end{aligned} \quad (20)$$

$$G = \begin{bmatrix} P & ML_N \\ L_N M^* & 0 \end{bmatrix}.$$

Then one has the following

(a) the following equalities hold

$$i_{\pm}(M) = r(B) + i_{\pm}(R_B A R_B), \quad (21)$$

$$i_{\pm}(N) = r(Q), \quad (22)$$

(b) if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, then $i_{\pm}(L) = i_{\pm}(A) + i_{\pm}(D - B^* A^{\dagger} B)$. Thus $i_{\pm}(L) = i_{\pm}(A)$ if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $i_{\pm}(D - B^* A^{\dagger} B) = 0$,

(c)

$$i_{\pm}(G) = \begin{bmatrix} P & M & 0 \\ M^* & 0 & N^* \\ 0 & N & 0 \end{bmatrix} - r(N). \quad (23)$$

Lemma 3 (see [23]). Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, and $C \in \mathbb{C}^{l \times n}$. Then they satisfy the following rank equalities:

- (a) $r[A \ B] = r(A) + r(E_A B) = r(B) + r(E_B A)$,
- (b) $r \begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C F_A) = r(C) + r(A F_C)$,
- (c) $r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C)$,
- (d) $r[B \ AF_C] = r \begin{bmatrix} B & A \\ 0 & C \end{bmatrix} - r(C)$,
- (e) $r \begin{bmatrix} C \\ E_B A \end{bmatrix} = r \begin{bmatrix} C & 0 \\ A & B \end{bmatrix} - r(B)$,
- (f) $r \begin{bmatrix} A & B F_D \\ E_C & 0 \end{bmatrix} = r \begin{bmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{bmatrix} - r(D) - r(E)$,

Lemma 4 (see [15]). Let $A \in \mathbb{C}_h^{m \times m}$, $B \in \mathbb{C}^{m \times n}$, $C \in \mathbb{C}_h^{n \times n}$, $Q \in \mathbb{C}^{m \times n}$, and $P \in \mathbb{C}^{p \times n}$ be given, and $T \in \mathbb{C}^{m \times m}$ be nonsingular. Then one has the following

- (1) $i_{\pm}(T A T^*) = i_{\pm}(A)$,
- (2) $i_{\pm} \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} = i_{\pm}(A) + i_{\pm}(C)$,
- (3) $i_{\pm} \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q)$,
- (4) $i_{\pm} \begin{bmatrix} A & B L_P \\ L_P B^* & 0 \end{bmatrix} + r(P) = i_{\pm} \begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix}$.

Lemma 5 (see [22, Lemma 1.4]). Let S be a set consisting of (square) matrices over $\mathbb{C}^{m \times m}$, and let H be a set consisting of (square) matrices over $\mathbb{C}_h^{m \times m}$. Then one has the following

- (a) S has a nonsingular matrix if and only if $\max_{X \in S} r(X) = m$;
- (b) any $X \in S$ is nonsingular if and only if $\min_{X \in S} r(X) = m$;
- (c) $\{0\} \in S$ if and only if $\min_{X \in S} r(X) = 0$;
- (d) $S = \{0\}$ if and only if $\max_{X \in S} r(X) = 0$;
- (e) H has a matrix $X > 0$ ($X < 0$) if and only if $\max_{X \in H} i_{+}(X) = m$ ($\max_{X \in H} i_{-}(X) = m$);
- (f) any $X \in H$ satisfies $X > 0$ ($X < 0$) if and only if $\min_{X \in H} i_{+}(X) = m$ ($\min_{X \in H} i_{-}(X) = m$);
- (g) H has a matrix $X \geq 0$ ($X \leq 0$) if and only if $\min_{X \in H} i_{-}(X) = 0$ ($\min_{X \in H} i_{+}(X) = 0$);

(h) any $X \in H$ satisfies $X \geq 0$ ($X \leq 0$) if and only if $\max_{X \in H} i_{-}(X) = 0$ ($\max_{X \in H} i_{+}(X) = 0$).

Lemma 6 (see [16]). Let $p(X, Y) = A - BX - (BX)^* - CYD - (CYD)^*$, where A, B, C , and D are given with appropriate sizes, and denote that

$$\begin{aligned} M_1 &= \begin{bmatrix} A & B & C \\ B^* & 0 & 0 \\ C^* & 0 & 0 \end{bmatrix}, \\ M_2 &= \begin{bmatrix} A & B & D^* \\ B^* & 0 & 0 \\ D & 0 & 0 \end{bmatrix}, \\ M_3 &= \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \end{bmatrix}, \\ M_4 &= \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \\ C^* & 0 & 0 & 0 \end{bmatrix}, \\ M_5 &= \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \\ D & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (24)$$

Then one has the following:

(1) the maximal rank of $p(X, Y)$ is

$$\max_{X \in \mathbb{C}^{n \times m}, Y} r[p(X, Y)] = \min \{m, r(M_1), r(M_2), r(M_3)\}, \quad (25)$$

(2) the minimal rank of $p(X, Y)$ is

$$\begin{aligned} \min_{X \in \mathbb{C}^{n \times m}, Y} r[p(X, Y)] \\ = 2r(M_3) - 2r(B) \\ + \max \{u_{+} + u_{-}, v_{+} + v_{-}, u_{+} + v_{-}, u_{-} + v_{+}\}, \end{aligned} \quad (26)$$

(3) the maximal inertia of $p(X, Y)$ is

$$\max_{X \in \mathbb{C}^{n \times m}, Y} i_{\pm}[p(X, Y)] = \min \{i_{\pm}(M_1), i_{\pm}(M_2)\}, \quad (27)$$

(4) the minimal inertias of $p(X, Y)$ is

$$\begin{aligned} \min_{X \in \mathbb{C}^{n \times m}, Y} i_{\pm}[p(X, Y)] = r(M_3) - r(B) \\ + \max \{i_{\pm}(M_1) - r(M_4), \\ i_{\pm}(M_2) - r(M_5)\}, \end{aligned} \quad (28)$$

where

$$u_{\pm} = i_{\pm}(M_1) - r(M_4), \quad v_{\pm} = i_{\pm}(M_2) - r(M_5). \quad (29)$$

Now we present the main theorem of this section.

Theorem 7. Let $A_1 \in \mathbb{C}^{m \times n}$, $C_1 \in \mathbb{C}^{m \times k}$, $B_1 \in \mathbb{C}^{k \times l}$, $D_1 \in \mathbb{C}^{n \times l}$, $A_2 \in \mathbb{C}^{l \times q}$, $C_2 \in \mathbb{C}^{l \times p}$, $A_3 \in \mathbb{C}^{p \times p}$, $B_3 \in \mathbb{C}^{p \times q}$, $C_3 \in \mathbb{C}^{p \times n}$, and $D_3 \in \mathbb{C}^{p \times n}$ be given, and suppose that the system of matrix equations (13) and (11) is consistent, respectively. Denote the set of all solutions to (13) by S and (11) by G . Put

$$\begin{aligned}
 E_1 &= \begin{bmatrix} A_3 & C_3 & D_3^* C_1^* & B_3 & C_2^* \\ C_3^* & 0 & A_1^* & 0 & 0 \\ C_1 D_3 & A_1 & 0 & 0 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix}, \\
 E_2 &= r \begin{bmatrix} A_3 & D_3^* & C_3 D_1 & B_3 & C_2^* \\ D_3 & 0 & B_1 & 0 & 0 \\ D_1^* C_3^* & B_1^* & 0 & 0 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix}, \\
 E_3 &= \begin{bmatrix} A_3 & B_3 & C_3^* & D_3^* & C_2^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_1^* C_3^* & 0 & 0 & B_1^* & 0 \\ C_1 D_3 & 0 & A_1 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 \end{bmatrix}, \\
 E_4 &= \begin{bmatrix} A_3 & B_3 & C_3 & D_3^* & C_2^* & D_3^* C_1^* \\ B_3^* & 0 & 0 & 0 & A_2^* & 0 \\ C_3^* & 0 & 0 & 0 & 0 & A_1^* \\ 0 & 0 & 0 & B_1^* & 0 & 0 \\ C_1 D_3 & 0 & A_1 & 0 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
 E_5 &= \begin{bmatrix} A_3 & B_3 & C_3 & D_3^* & C_2^* & C_3 D_1 \\ B_3^* & 0 & 0 & 0 & A_2^* & 0 \\ D_3 & 0 & 0 & 0 & 0 & B_1 \\ D_1^* C_3^* & 0 & 0 & A_1^* & 0 & 0 \\ 0 & 0 & A_1 & 0 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix}.
 \end{aligned} \tag{30}$$

Then one has the following:

(a) the maximal rank of (10) subject to (13) and (11) is

$$\begin{aligned}
 & \max_{X \in G, Y \in S} r[f(X, Y)] \\
 &= \min \{p, r(E_1) - 2r(A_1) - 2r(A_2), \\
 & \quad r(E_2) - 2r(B_1) - 2r(A_2), \\
 & \quad r(E_3) - 2r(A_2) - r(A_1) - r(B_1)\},
 \end{aligned} \tag{31}$$

(b) the minimal rank of (10) subject to (13) and (11) is

$$\begin{aligned}
 & \min_{X \in G, Y \in S} r[f(X, Y)] \\
 &= 2r(E_3) - 2r \begin{bmatrix} B_3 \\ A_2 \end{bmatrix} \\
 &+ \max \{r(E_1) - 2r(E_4), r(E_2) - 2r(E_5), \\
 & \quad i_+(E_1) + i_-(E_2) - r(E_4) - r(E_5), \\
 & \quad i_-(E_1) + i_+(E_2) - r(E_4) - r(E_5)\},
 \end{aligned} \tag{32}$$

(c) the maximal inertia of (10) subject to (13) and (11) is

$$\begin{aligned}
 & \max_{X \in G, Y \in S} i_{\pm}[f(X, Y)] = \min \{i_{\pm}(E_1) - r(A_1) - r(A_2), \\
 & \quad i_{\pm}(E_2) - r(B_1) - r(A_2)\},
 \end{aligned} \tag{33}$$

(d) the minimal inertia of (10) subject to (13) and (11) is

$$\begin{aligned}
 & \min_{X \in G, Y \in S} i_{\pm}[f(X, Y)] = r(E_3) - r \begin{bmatrix} B_3 \\ A_2 \end{bmatrix} \\
 & \quad + \max \{i_{\pm}(E_1) - r(E_4), \\
 & \quad i_{\pm}(E_2) - r(E_5)\}.
 \end{aligned} \tag{34}$$

Proof. By Lemma 1, the general solutions to (13) and (11) can be written as

$$\begin{aligned}
 X &= A_2^\dagger C_2 + L_{A_2} W, \\
 Y &= A_1^\dagger C_1 + L_{A_1} D_1 B_1^\dagger + L_{A_1} Z R_{B_1},
 \end{aligned} \tag{35}$$

where W and Z are arbitrary matrices with appropriate sizes. Put

$$\begin{aligned}
 Q &= B_3 L_{A_2}, \quad T = C_3 L_{A_1}, \quad J = R_{B_1} D_3, \\
 P &= A_3 - B_3 A_2^\dagger C_2 - (B_3 A_2^\dagger C_2)^* \\
 & \quad - C_3 (A_1^\dagger C_1 + L_{A_1} D_1 B_1^\dagger) D_3 \\
 & \quad - (C_3 (A_1^\dagger C_1 + L_{A_1} D_1 B_1^\dagger) D_3)^*.
 \end{aligned} \tag{36}$$

Substituting (36) into (10) yields

$$f(X, Y) = P - QW - (QW)^* - TZJ - (TZJ)^*. \tag{37}$$

Clearly P is Hermitian. It follows from Lemma 6 that

$$\begin{aligned}
 & \max_{X \in G, Y \in S} r[f(X, Y)] \\
 &= \max_{W, Z} r(P - QW - (QW)^* - TZJ - (TZJ)^*) \\
 &= \min \{m, r(N_1), r(N_2), r(N_3)\},
 \end{aligned} \tag{38}$$

$$\begin{aligned}
 & \min_{X \in G, Y \in S} r[f(X, Y)] \\
 &= \max_{W, Z} r(P - QW - (QW)^* - TZJ - (TZJ)^*) \\
 &= 2r(N_3) - 2r(Q) \\
 & \quad + \max \{s_+ + s_-, t_+ + t_-, s_+ + t_-, s_- + t_+\},
 \end{aligned} \tag{39}$$

$$\begin{aligned} & \max_{X \in G, Y \in S} i_{\pm} [f(X, Y)] \\ & = \max_{W, Z} r(P - QW - (QW)^* - TZJ - (TZJ)^*) \end{aligned} \quad (40)$$

$$= \min \{i_{\pm}(N_1), i_{\pm}(N_2)\},$$

$$\begin{aligned} & \min_{X \in G, Y \in S} i_{\pm} [f(X, Y)] \\ & = \max_{W, Z} r(P - QW - (QW)^* - TZJ - (TZJ)^*) \end{aligned} \quad (41)$$

$$= r(N_3) - r(Q) + \max \{s_{\pm}, t_{\pm}\},$$

where

$$\begin{aligned} N_1 &= \begin{bmatrix} P & Q & T \\ Q^* & 0 & 0 \\ T^* & 0 & 0 \end{bmatrix}, \\ N_2 &= \begin{bmatrix} P & Q & J^* \\ Q^* & 0 & 0 \\ J & 0 & 0 \end{bmatrix}, \\ N_3 &= \begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 & 0 \end{bmatrix}, \\ N_4 &= \begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 & 0 \\ T^* & 0 & 0 & 0 \end{bmatrix}, \\ N_5 &= \begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 & 0 \\ J & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \quad (42)$$

$$s_{\pm} = i_{\pm}(N_1) - r(N_4), \quad t_{\pm} = i_{\pm}(N_2) - r(N_5).$$

Now, we simplify the ranks and inertias of block matrices in (38)–(41).

By Lemma 4, block Gaussian elimination, and noting that

$$L_S^* = (I - S^\dagger S)^* = I - S^*(S^*)^\dagger = R_{S^*}, \quad (43)$$

we have the following:

$$\begin{aligned} r(N_1) &= r \begin{bmatrix} P & Q & T \\ Q^* & 0 & 0 \\ T^* & 0 & 0 \end{bmatrix} \\ &= r \begin{bmatrix} A_3 & C_3 & D_3^* C_1^* & B_3 & C_2^* \\ C_3^* & 0 & A_1^* & 0 & 0 \\ C_1 D_3 & A_1 & 0 & 0 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} \\ &\quad - 2r(A_1) - 2r(A_2). \end{aligned} \quad (44)$$

By $C_1 B_1 = A_1 D_1$, we obtain

$$\begin{aligned} r(N_2) &= r \begin{bmatrix} P & Q & J^* \\ Q^* & 0 & 0 \\ J & 0 & 0 \end{bmatrix} \\ &= r \begin{bmatrix} A_3 & D_3^* & C_3 D_1 & B_3 & C_2^* \\ D_3 & 0 & B_1 & 0 & 0 \\ D_1^* C_3^* & B_1^* & 0 & 0 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} \\ &\quad - 2r(B_1) - 2r(A_2), \end{aligned}$$

$$\begin{aligned} r(N_3) &= r \begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 & 0 \end{bmatrix} \\ &= r \begin{bmatrix} A_3 & B_3 & C_3^* & D_3^* & C_2^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_1^* C_3^* & 0 & 0 & B_1^* & 0 \\ C_1 D_3 & 0 & A_1 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 \end{bmatrix} \\ &\quad - r(B_1) - 2r(A_2) - r(A_1), \end{aligned}$$

$$\begin{aligned} r(N_4) &= r \begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 & 0 \\ T^* & 0 & 0 & 0 \end{bmatrix} \\ &= r \begin{bmatrix} A_3 & B_3 & C_3 & D_3^* & C_2^* & D_3^* C_1^* \\ B_3^* & 0 & 0 & 0 & A_2^* & 0 \\ C_3^* & 0 & 0 & 0 & 0 & A_1^* \\ 0 & 0 & 0 & B_1^* & 0 & 0 \\ C_1 D_3 & 0 & A_1 & 0 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\quad - r(B_1) - 2r(A_2) - 2r(A_1), \end{aligned}$$

$$\begin{aligned} r(N_5) &= r \begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 & 0 \\ T^* & 0 & 0 & 0 \end{bmatrix} \\ &= r \begin{bmatrix} A_3 & B_3 & C_3 & D_3^* & C_2^* & C_3 D_1 \\ B_3^* & 0 & 0 & 0 & A_2^* & 0 \\ D_3 & 0 & 0 & 0 & 0 & B_1 \\ D_1^* C_3^* & 0 & 0 & A_1^* & 0 & 0 \\ 0 & 0 & A_1 & 0 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\quad - 2r(B_1) - 2r(A_2) - r(A_1). \end{aligned} \quad (45)$$

By Lemma 2, we can get the following:

$$i_{\pm}(N_1) = i_{\pm} \begin{bmatrix} P & Q & T \\ Q^* & 0 & 0 \\ T^* & 0 & 0 \end{bmatrix}$$

$$= i_{\pm} \begin{bmatrix} A_3 & C_3 & D_3^* C_1^* & B_3 & C_2^* \\ C_3^* & 0 & A_1^* & 0 & 0 \\ C_1 D_3 & A_1 & 0 & 0 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} \\ -r(A_1) - r(A_2), \quad (46)$$

$$i_{\pm}(N_2) = i_{\pm} \begin{bmatrix} P & Q & J^* \\ Q^* & 0 & 0 \\ J & 0 & 0 \end{bmatrix} \\ = i_{\pm} \begin{bmatrix} A_3 & D_3^* & C_3 D_1 & B_3 & C_2^* \\ D_3 & 0 & B_1 & 0 & 0 \\ D_1^* C_3^* & B_1^* & 0 & 0 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} \quad (47) \\ -r(B_1) - r(A_2).$$

Substituting (44)-(47) into (38) and (41) yields (31)-(34), respectively. \square

Corollary 8. Let $A_1, C_1, B_1, D_1, A_2, C_2, A_3, B_3, C_3, D_3$, and E_i , ($i = 1, 2, \dots, 5$) be as in Theorem 7, and suppose that the system of matrix equations (13) and (11) is consistent, respectively. Denote the set of all solutions to (13) by S and (11) by G . Then, one has the following:

(a) there exist $X \in G$ and $Y \in S$ such that $A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* > 0$ if and only if

$$i_+(E_1) - r(A_1) - r(A_2) \geq p, \\ i_+(E_2) - r(B_1) - r(A_2) \geq p. \quad (48)$$

(b) there exist $X \in G$ and $Y \in S$ such that $A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* < 0$ if and only if

$$i_-(E_1) - r(A_1) - r(A_2) \geq p, \\ i_-(E_2) - r(B_1) - r(A_2) \geq p, \quad (49)$$

(c) there exist $X \in G$ and $Y \in S$ such that $A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* \geq 0$ if and only if

$$r(E_3) - r \begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_-(E_1) - r(E_4) \leq 0, \\ r(E_3) - r \begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_-(E_2) - r(E_5) \leq 0. \quad (50)$$

(d) there exist $X \in G$ and $Y \in S$ such that $A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* \leq 0$ if and only if

$$r(E_3) - r \begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_+(E_1) - r(E_4) \leq 0, \\ r(E_3) - r \begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_+(E_2) - r(E_5) \leq 0, \quad (51)$$

(e) $A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* > 0$ for all $X \in G$ and $Y \in S$ if and only if

$$r(E_3) - r \begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_+(E_1) - r(E_4) = p \\ \text{or } r(E_3) - r \begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_+(E_2) - r(E_5) = p, \quad (52)$$

(f) $A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* < 0$ for all $X \in G$ and $Y \in S$ if and only if

$$r(E_3) - r \begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_-(E_1) - r(E_4) = p \\ \text{or } r(E_3) - r \begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_-(E_2) - r(E_5) = p, \quad (53)$$

(g) $A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* \geq 0$ for all $X \in G$ and $Y \in S$ if and only if

$$i_-(E_1) - r(A_1) - r(A_2) \leq 0 \\ \text{or } i_-(E_2) - r(B_1) - r(A_2) \leq 0, \quad (54)$$

(h) $A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^* \leq 0$ for all $X \in G$ and $Y \in S$ if and only if

$$i_+(E_1) - r(A_1) - r(A_2) \leq 0 \\ \text{or } i_+(E_2) - r(B_1) - r(A_2) \leq 0, \quad (55)$$

(i) there exist $X \in G$ and $Y \in S$ such that $A_3 - B_3 X - (B_3 X)^* - C_3 Y D_3 - (C_3 Y D_3)^*$ is nonsingular if and only if

$$r(E_1) - 2r(A_1) - 2r(A_2) \geq p, \\ r(E_2) - 2r(B_1) - 2r(A_2) \geq p, \\ r(E_3) - 2r(A_2) - r(A_1) - r(B_1) \geq p. \quad (56)$$

3. Relations between the Hermitian Part of the Solutions to (13) and (11)

Now we consider the extremal ranks and inertias of the difference between the Hermitian part of the solutions to (13) and (11).

Theorem 9. Let $A_1 \in \mathbb{C}^{m \times p}$, $C_1 \in \mathbb{C}^{m \times p}$, $B_1 \in \mathbb{C}^{p \times l}$, $D_1 \in \mathbb{C}^{p \times l}$, $A_2 \in \mathbb{C}^{l \times p}$, and $C_2 \in \mathbb{C}^{l \times p}$, be given. Suppose that the system of matrix equations (13) and (11) is consistent,

respectively. Denote the set of all solutions to (13) by S and (11) by G . Put

$$\begin{aligned}
 H_1 &= \begin{bmatrix} 0 & I & C_1^* & -I & C_2^* \\ I & 0 & A_1^* & 0 & 0 \\ C_1 & A_1 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix}, \\
 H_2 &= r \begin{bmatrix} 0 & I & D_1 & -I & C_2^* \\ I & 0 & B_1 & 0 & 0 \\ D_1^* & B_1^* & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix}, \\
 H_3 &= \begin{bmatrix} 0 & -I & I & I & C_2^* \\ -I & 0 & 0 & 0 & A_2^* \\ D_1^* & 0 & 0 & B_1^* & 0 \\ C_1 & 0 & A_1 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 \end{bmatrix}, \\
 H_4 &= \begin{bmatrix} 0 & -I & I & I & C_2^* & C_1^* \\ -I & 0 & 0 & 0 & A_2^* & 0 \\ I & 0 & 0 & 0 & 0 & A_1^* \\ 0 & 0 & 0 & B_1^* & 0 & 0 \\ C_1 & 0 & A_1 & 0 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
 H_5 &= \begin{bmatrix} 0 & -I & I & I & C_2^* & D_1 \\ -I & 0 & 0 & 0 & A_2^* & 0 \\ I & 0 & 0 & 0 & 0 & B_1 \\ D_1^* & 0 & 0 & A_1^* & 0 & 0 \\ 0 & 0 & A_1 & 0 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix}.
 \end{aligned} \tag{57}$$

Then one has the following:

$$\begin{aligned}
 &\max_{X \in G, Y \in S} r [(X + X^*) - (Y + Y^*)] \\
 &= \min \{p, r(H_1) - 2r(A_1) - 2r(A_2), r(H_2) - 2r(B_1) \\
 &\quad - 2r(A_2), r(H_3) - 2r(A_2) - r(A_1) - r(B_1)\}, \\
 &\min_{X \in G, Y \in S} r [(X + X^*) - (Y + Y^*)] \\
 &= 2r(H_3) - 2p \\
 &\quad + \max \{r(H_1) - 2r(H_4), r(H_2) - 2r(H_5), \\
 &\quad i_+(H_1) + i_-(H_2) - r(H_4) - r(H_5), \\
 &\quad i_-(H_1) + i_+(H_2) - r(H_4) - r(H_5)\}, \\
 &\max_{X \in G, Y \in S} i_{\pm} [(X + X^*) - (Y + Y^*)] \\
 &= \min \{i_{\pm}(H_1) - r(A_1) - r(A_2), \\
 &\quad i_{\pm}(H_2) - r(B_1) - r(A_2)\},
 \end{aligned}$$

$$\begin{aligned}
 &\min_{X \in G, Y \in S} i_{\pm} [(X + X^*) - (Y + Y^*)] \\
 &= r(E_3) - p + \max \{i_{\pm}(H_1) - r(H_4), i_{\pm}(H_2) - r(H_5)\}.
 \end{aligned} \tag{58}$$

Proof. By letting $A_3 = 0, B_3 = -I, C_3 = I,$ and $D_3 = I$ in Theorem 7, we can get the results. \square

Corollary 10. Let $A_1, C_1, B_1, D_1, A_2, C_2,$ and $H_i, (i = 1, 2, \dots, 5)$ be as in Theorem 9, and suppose that the system of matrix equations (13) and (11) is consistent, respectively. Denote the set of all solutions to (13) by S and (11) by G . Then, one has the following:

(a) there exist $X \in G$ and $Y \in S$ such that $(X + X^*) > (Y + Y^*)$ if and only if

$$\begin{aligned}
 &i_+(H_1) - r(A_1) - r(A_2) \geq p, \\
 &i_+(H_2) - r(B_1) - r(A_2) \geq p.
 \end{aligned} \tag{59}$$

(b) there exist $X \in G$ and $Y \in S$ such that $(X + X^*) < (Y + Y^*)$ if and only if

$$\begin{aligned}
 &i_-(H_1) - r(A_1) - r(A_2) \geq p, \\
 &i_-(H_2) - r(B_1) - r(A_2) \geq p,
 \end{aligned} \tag{60}$$

(c) there exist $X \in G$ and $Y \in S$ such that $(X + X^*) \geq (Y + Y^*)$ if and only if

$$\begin{aligned}
 &r(H_3) - p + i_-(H_1) - r(H_4) \leq 0, \\
 &r(H_3) - p + i_-(H_1) - r(H_4) \leq 0,
 \end{aligned} \tag{61}$$

(d) there exist $X \in G$ and $Y \in S$ such that $(X + X^*) \leq (Y + Y^*)$ if and only if

$$\begin{aligned}
 &r(H_3) - p + i_+(H_1) - r(H_4) \leq 0, \\
 &r(H_3) - p + i_+(H_2) - r(H_5) \leq 0,
 \end{aligned} \tag{62}$$

(e) $(X + X^*) > (Y + Y^*)$ for all $X \in G$ and $Y \in S$ if and only if

$$\begin{aligned}
 &r(H_3) - p + i_+(H_1) - r(H_4) = p \\
 &\text{or } r(H_3) - p + i_+(H_2) - r(H_5) = p,
 \end{aligned} \tag{63}$$

(f) $(X + X^*) < (Y + Y^*)$ for all $X \in G$ and $Y \in S$ if and only if

$$\begin{aligned}
 &r(H_3) - p + i_-(H_1) - r(H_4) = p \\
 &\text{or } r(H_3) - p + i_-(H_2) - r(H_5) = p,
 \end{aligned} \tag{64}$$

(g) $(X + X^*) \geq (Y + Y^*)$ for all $X \in G$ and $Y \in S$ if and only if

$$\begin{aligned}
 &i_-(H_1) - r(A_1) - r(A_2) \leq 0 \\
 &\text{or } i_-(H_2) - r(B_1) - r(A_2) \leq 0,
 \end{aligned} \tag{65}$$

(h) $(X + X^*) \leq (Y + Y^*)$ for all $X \in G$ and $Y \in S$ if and only if

$$i_+(H_1) - r(A_1) - r(A_2) \leq 0 \tag{66}$$

or $i_+(H_2) - r(B_1) - r(A_2) \leq 0,$

(i) there exist $X \in G$ and $Y \in S$ such that $(X + X^*) - (Y + Y^*)$ is nonsingular if and only if

$$r(H_1) - 2r(A_1) - 2r(A_2) \geq p, \tag{67}$$

$$r(H_2) - 2r(B_1) - 2r(A_2) \geq p,$$

$$r(H_3) - 2r(A_2) - r(A_1) - r(B_1) \geq p.$$

4. The Solvability Conditions and the General Solution to System (6)

We now turn our attention to (6). We in this section use Theorem 9 to give some necessary and sufficient conditions for the existence of a solution to (6) and present an expression of the general solution to (6). We begin with a lemma which is used in the latter part of this section.

Lemma 11 (see [14]). Let $A_1 \in \mathbb{C}^{m \times n_1}, B_1 \in \mathbb{C}^{m \times n_2}, C_1 \in \mathbb{C}^{q \times m},$ and $E_1 \in \mathbb{C}_h^{m \times m}$ be given. Let $A = R_{A_1}B_1, B = C_1R_{A_1}, E = R_{A_1}E_1R_{A_1}, M = R_A B^*, N = A^*L_B,$ and $S = B^*L_M.$ Then the following statements are equivalent:

(1) equation (5) is consistent,

(2)

$$R_M R_A E = 0, \quad R_A E R_A = 0, \quad L_B E L_B = 0, \tag{68}$$

(3)

$$r \begin{bmatrix} E_1 & B_1 & C_1^* & A_1 \\ A_1^* & 0 & 0 & 0 \end{bmatrix} = r [B_1 \ C_1^* \ A_1] + r(A_1),$$

$$r \begin{bmatrix} E_1 & B_1 & A_1 \\ A_1^* & 0 & 0 \\ B_1^* & 0 & 0 \end{bmatrix} = 2r [B_1 \ A_1], \tag{69}$$

$$r \begin{bmatrix} E_1 & C_1^* & A_1 \\ A_1^* & 0 & 0 \\ C_1 & 0 & 0 \end{bmatrix} = 2r [C_1^* \ A_1].$$

In this case, the general solution of (5) can be expressed as

$$Y = \frac{1}{2} \left[A^\dagger E B^\dagger - A^\dagger B^* M^\dagger E B^\dagger - A^\dagger S (B^\dagger)^* E N^\dagger A^* B^\dagger \right. \\ \left. + A^\dagger E (M^\dagger)^* + (N^\dagger)^* E B^\dagger S^\dagger S \right] + L_A V_1 + V_2 R_B \\ + U_1 L_S L_M + R_N U_2^* L_M - A^\dagger S U_2 R_N A^* B^\dagger,$$

$$X = A_1^\dagger [E_1 - B_1 Y C_1 - (B_1 Y C_1)^*] \\ - \frac{1}{2} A_1^\dagger [E_1 - B_1 Y C_1 - (B_1 Y C_1)^*] A_1 A_1^\dagger \\ - A_1^\dagger W_1 A_1^* + W_1^* A_1 A_1^\dagger + L_{A_1} W_2, \tag{70}$$

where $U_1, U_2, V_1, V_2, W_1,$ and W_2 are arbitrary matrices over \mathbb{C} with appropriate sizes.

Now we give the main theorem of this section.

Theorem 12. Let $A_i, C_i, (i = 1, 2, 3), B_j,$ and $D_j, (j = 1, 3)$ be given. Set

$$A = B_3 L_{A_2}, \quad B = C_3 L_{A_1}, \\ C = R_{B_1} D_3, \quad F = R_A B, \tag{71}$$

$$G = C R_A, \quad M = R_F G^*,$$

$$N = F^* L_G, \quad S = G^* L_M,$$

$$D = A_3 - B_3 A_2^\dagger C_2 - (B_3 A_2^\dagger C_2)^* \\ - C_3 (A_1^\dagger C_1 + L_{A_1} D_1 B_1^\dagger) D_3 \tag{72}$$

$$- D_3^* (A_1^\dagger C_1 + L_{A_1} D_1 B_1^\dagger)^* C_3^*,$$

$$E = R_A D R_A. \tag{73}$$

Then the following statements are equivalent:

(1) system (6) is consistent,

(2) the equalities in (14) and (17) hold, and

$$R_M R_F E = 0, \quad R_F E R_F = 0, \quad L_G E L_G = 0, \tag{74}$$

(3) the equalities in (15) and (18) hold, and

$$r \begin{bmatrix} A_3 & C_3 & D_3^* & B_3 & C_2^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_1 D_3 & A_1 & 0 & 0 & 0 \\ D_1^* C_3^* & 0 & B_1^* & 0 & 0 \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} = r \begin{bmatrix} C_3 & D_3^* & B_3 \\ A_1 & 0 & 0 \\ 0 & B_1^* & 0 \\ 0 & 0 & A_2 \end{bmatrix} + r \begin{bmatrix} A_2 \\ B_3 \end{bmatrix},$$

$$r \begin{bmatrix} A_3 & C_3 & B_3 & D_3^* C_1^* & C_2^* \\ C_3^* & 0 & 0 & A_1^* & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_1 D_3 & A_1 & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} = 2r \begin{bmatrix} C_3 & B_3 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

$$r \begin{bmatrix} A_3 & D_3^* & B_3 & C_3 D_1 & C_2^* \\ D_3 & 0 & 0 & B_1 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_1^* C_3^* & B_1^* & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} = 2r \begin{bmatrix} D_3^* & B_3 \\ B_1^* & 0 \\ 0 & A_2 \end{bmatrix}. \tag{75}$$

In this case, the general solution of system (6) can be expressed as

$$X = A_2^\dagger C_2 + L_{A_2} U, \tag{76}$$

$$Y = A_1^\dagger C_1 + L_{A_1} D_1 B_1^\dagger + L_{A_1} V R_{B_1},$$

where

$$\begin{aligned}
 V &= \frac{1}{2} \left[F^\dagger EG^\dagger - F^\dagger G^* M^\dagger EG^\dagger - F^\dagger S(G^\dagger)^* EN^\dagger F^* G^\dagger \right. \\
 &\quad \left. + F^\dagger E(M^\dagger)^* + (N^\dagger)^* EG^\dagger S^\dagger S \right] + L_F V_1 \\
 &\quad + V_2 R_G + U_1 L_S L_M + R_N U_2^* L_M - F^\dagger S U_2 R_N F^* G^\dagger, \\
 U &= A^\dagger [D - BVC - (BVC)^*] \\
 &\quad - \frac{1}{2} A^\dagger [D - BVC - (BVC)^*] AA^\dagger \\
 &\quad - A^\dagger W_1 A^* + W_1^* AA^\dagger + L_A W_2,
 \end{aligned} \tag{77}$$

where U_1, U_2, V_1, V_2, W_1 , and W_2 are arbitrary matrices over \mathbb{C} with appropriate sizes.

Proof. (2) \Leftrightarrow (3): Applying Lemma 3 and Lemma 11 gives

$$\begin{aligned}
 R_M R_F E = 0 &\Leftrightarrow r(R_M R_F E) \\
 &= 0 \Leftrightarrow r \begin{bmatrix} D & B & C^* & A \\ A^* & 0 & 0 & 0 \end{bmatrix} \\
 &= r[B \ C^* \ A] + r(A) \\
 &\Leftrightarrow r \begin{bmatrix} D & C_3 & D_3^* & B_3 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & B_1^* & 0 & 0 \\ 0 & 0 & 0 & A_2 & 0 \end{bmatrix} \\
 &= r \begin{bmatrix} C_3 & D_3^* & B_3 \\ A_1 & 0 & 0 \\ 0 & B_1^* & 0 \\ 0 & 0 & A_2 \end{bmatrix} + r \begin{bmatrix} A_2 \\ B_3 \end{bmatrix} \\
 &\Leftrightarrow r \begin{bmatrix} A_3 & C_3 & D_3^* & B_3 & C_2^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_1 D_3 & A_1 & 0 & 0 & 0 \\ D_1^* C_3^* & 0 & B_1^* & 0 & 0 \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} \\
 &= r \begin{bmatrix} C_3 & D_3^* & B_3 \\ A_1 & 0 & 0 \\ 0 & B_1^* & 0 \\ 0 & 0 & A_2 \end{bmatrix} + r \begin{bmatrix} A_2 \\ B_3 \end{bmatrix}.
 \end{aligned} \tag{78}$$

By a similar approach, we can obtain that

$$\begin{aligned}
 R_F E R_F = 0 &\Leftrightarrow r \begin{bmatrix} A_3 & C_3 & B_3 & D_3^* C_1^* & C_2^* \\ C_3^* & 0 & 0 & A_1^* & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_1 D_3 & A_1 & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} \\
 &= 2r \begin{bmatrix} C_3 & B_3 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix},
 \end{aligned}$$

$$\begin{aligned}
 L_G E L_G = 0 &\Leftrightarrow r \begin{bmatrix} A_3 & D_3^* & B_3 & C_3 D_1 & C_2^* \\ D_3 & 0 & 0 & B_1 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_1^* C_3^* & B_1^* & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} \\
 &= 2r \begin{bmatrix} D_3^* & B_3 \\ B_1^* & 0 \\ 0 & A_2 \end{bmatrix}.
 \end{aligned} \tag{79}$$

(1) \Leftrightarrow (2): We separate the four equations in system (6) into three groups:

$$A_1 Y = C_1, \quad Y B_1 = D_1, \tag{80}$$

$$A_2 X = C_2, \tag{81}$$

$$B_3 X + (B_3 X)^* + C_3 Y D_3 + (C_3 Y D_3)^* = A_3. \tag{82}$$

By Lemma 1, we obtain that system (80) is solvable if and only if (14), (81) is consistent if and only if (17). The general solutions to system (80) and (81) can be expressed as (16) and (19), respectively. Substituting (16) and (19) into (82) yields

$$AU + (AU)^* + BVC + (BVC)^* = D. \tag{83}$$

Hence, the system (5) is consistent if and only if (80), (81), and (83) are consistent, respectively. It follows from Lemma 11 that (83) is solvable if and only if

$$R_M R_F E = 0, \quad R_F E R_F = 0, \quad L_G E L_G = 0. \tag{84}$$

We know by Lemma 11 that the general solution of (83) can be expressed as (77). \square

In Theorem 12, let A_1 and D_1 vanish. Then we can obtain the general solution to the following system:

$$A_2 X = C_2, \quad Y B_1 = D_1, \tag{85}$$

$$B_3 X + (B_3 X)^* + C_3 Y D_3 + (C_3 Y D_3)^* = A_3.$$

Corollary 13. Let $A_2, C_2, B_1, D_1, B_3, C_3, D_3$, and $A_3 = A_3^*$ be given. Set

$$A = B_3 L_{A_2}, \quad C = R_{B_1} D_3,$$

$$F = R_A C_3, \quad G = C R_{A_1},$$

$$M = R_F G^*, \quad N = F^* L_G,$$

$$S = G^* L_M, \tag{86}$$

$$\begin{aligned}
 D &= A_3 - B_3 A_2^\dagger C_2 - (B_3 A_2^\dagger C_2)^* \\
 &\quad - C_3 D_1 B_1^\dagger D_3 - (C_3 D_1 B_1^\dagger D_3)^*, \\
 E &= R_A D R_{A_1}.
 \end{aligned}$$

Then the following statements are equivalent:

(1) system (85) is consistent

(2)

$$R_{A_2}C_2 = 0, \quad D_1L_{B_1} = 0, \quad R_M R_F E = 0, \quad (87)$$

$$R_F E R_F = 0, \quad L_G E L_G = 0, \quad (88)$$

(3)

$$r [A_2 \ C_2] = r (A_2), \quad \begin{bmatrix} D_1 \\ B_1 \end{bmatrix} = r (B_1),$$

$$r \begin{bmatrix} A_3 & C_3 & D_3^* & B_3 & C_2^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_1^* C_3^* & 0 & B_1^* & 0 & 0 \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} = r \begin{bmatrix} C_3 & D_3^* & B_3 \\ 0 & B_1^* & 0 \\ 0 & 0 & A_2 \end{bmatrix} + r \begin{bmatrix} A_2 \\ B_3 \end{bmatrix},$$

$$r \begin{bmatrix} A_3 & C_3 & B_3 & C_2^* \\ C_3^* & 0 & 0 & 0 \\ B_3^* & 0 & 0 & A_2^* \\ C_2 & 0 & A_2 & 0 \end{bmatrix} = 2r \begin{bmatrix} C_3 & B_3 \\ 0 & A_2 \end{bmatrix},$$

$$r \begin{bmatrix} A_3 & D_3^* & B_3 & C_3 D_1 & C_2^* \\ D_3 & 0 & 0 & B_1 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_1^* C_3^* & B_1^* & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} = 2r \begin{bmatrix} D_3^* & B_3 \\ B_1^* & 0 \\ 0 & A_2 \end{bmatrix}. \quad (89)$$

In this case, the general solution of system (6) can be expressed as

$$X = A_2^\dagger C_2 + L_{A_2} U, \quad (90)$$

$$Y = D_1 B_1^\dagger + V R_{B_1},$$

where

$$\begin{aligned} V &= \frac{1}{2} \left[F^\dagger E G^\dagger - F^\dagger G^* M^\dagger E G^\dagger - F^\dagger S (G^\dagger)^* E N^\dagger F^* G^\dagger \right. \\ &\quad \left. + F^\dagger E (M^\dagger)^* + (N^\dagger)^* E G^\dagger S^\dagger S \right] + L_F V_1 \\ &\quad + V_2 R_G + U_1 L_S L_M + R_N U_2^* L_M - F^\dagger S U_2 R_N F^* G^\dagger, \\ U &= A^\dagger [D - C_3 V C - (C_3 V C)^*] \\ &\quad - \frac{1}{2} A^\dagger [D - C_3 V C - (C_3 V C)^*] A A^\dagger \\ &\quad - A^\dagger W_1 A^* + W_1^* A A^\dagger + L_A W_2, \end{aligned} \quad (91)$$

where $U_1, U_2, V_1, V_2, W_1,$ and W_2 are arbitrary matrices over \mathbb{C} with appropriate sizes.

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