Research Article

A New Series of Three-Dimensional Chaotic Systems with Cross-Product Nonlinearities and Their Switching

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This paper introduces a new series of three-dimensional chaotic systems with cross-product nonlinearities. Based on some conditions, we analyze the globally exponentially or globally conditional exponentially attractive set and positive invariant set of these chaotic systems. Moreover, we give some known examples to show our results, and the exponential estimation is explicitly derived. Finally, we construct some three-dimensional chaotic systems with cross-product nonlinearities and study the switching system between them.

1. Introduction

Since Lorenz discovered the well-known Lorenz chaotic system, many other chaotic systems have been found, including the well-known Rössler system and Chua's circuit, which serve as models of the study of chaos [1–12].

The Lorenz system plays an important role in the study of nonlinear science and chaotic dynamics [13–18]. We know that it is extremely difficult to obtain the information of chaotic attractor directly from system. Most of the results in the literature are based on computer simulations. When calculating the Lyapunov exponents of the system, one needs to assume that the system is bounded in order to conclude chaos. Therefore, the study of the globally attractive set of the Lorenz system is not only theoretically significant but also practically important. Moreover, Liao et al. [19, 20] gave globally exponentially attractive set and positive invariant set for the classical Lorenz system and the generalized system by constructive proofs. In addition, Yu et al. [21] studied the problem of invariant set of systems, which was considered as a more generalized Lorenz system.

In this paper, we consider the following three-dimensional autonomous systems with cross-product nonlinearities:

$$\dot{x} = Ax + f(x) + C, \tag{1}$$

where $x^T = (x_1, x_2, x_3)$ and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}; \quad f(x) = \begin{bmatrix} x^T B_1 x \\ x^T B_2 x \\ x^T B_3 x \end{bmatrix};$$

$$B_i = \begin{bmatrix} b_{i11} & b_{i12} & b_{i13} \\ b_{i21} & b_{i22} & b_{i23} \\ b_{i31} & b_{i32} & b_{i33} \end{bmatrix}, \quad i = 1, 2, 3 \tag{2}$$

with $a_{ij}, b_{ijk}, c_i \in R$, $i, j, k = 1, 2, 3$. This second-order dynamic system may be regarded as the most general Lorenz system. For such system, we can choose Lyapunov function:

$$V(X(t)) = \frac{1}{2} \left[(\lambda_1 x_1 - d_1)^2 + (\lambda_2 x_2 - d_2)^2 + (\lambda_3 x_3 - d_3)^2 \right], \tag{3}$$

which is obviously positive definite and radially unbounded, where d_i, λ_i, $i = 1, 2, 3$ are undetermined parameters. In this paper, we will study this more general Lorenz system (1) than the classical system and the generalized Lorenz system. The result obtained contains earlier results as its special cases.

This paper is organized as follows. In Section 2, we define the globally exponentially attractive set and positive invariant set

set and the globally conditional exponentially attractive set and positive invariant set of the three-dimensional chaotic systems with cross-product nonlinearities. In Section 3, the qualitative analysis of the exponentially attractive set and positive invariant set of the chaotic systems has been done. In Section 4, we also suggest an idea to construct chaotic systems, and some new chaotic systems and switched chaotic systems are illustrated.

2. Preliminaries

In this section, we present some basic definitions which are needed for proving all theorems in the next section. For convenience, denote $X := (x_1, x_2, x_3)$ and $X(t) := X(t, t_0, X_0)$.

Definition 1. For the three-dimensional autonomous systems with cross-product nonlinearities (1), if there exists compact (bounded and closed) set $\Omega \in R^3$ such that for all $X_0 \in R^3$, the following condition: $\rho(X(t), \Omega) := \inf_{Y \in \Omega} \|X(t) - Y\| \to 0$ as $t \to +\infty$, holds, then the set Ω is said to be globally attractive. That is, system (1) is ultimately bounded; namely, system (1) is globally stable in the sense of Lagrange or dissipative with ultimate bound.

Furthermore, if for all $X_0 \in \Omega_0 \subseteq \Omega \subset R^3$, $X(t, t_0, X_0) \subseteq \Omega_0$, then Ω_0 for $t \geq 0$ is called the positive invariant set of the system (1).

Definition 2. For the three-dimensional autonomous systems with cross-product nonlinearities (1), if there exist compact set $\Omega \subset R^3$ such that for all $X_0 \in R^3$ and constants $M > 0$, $\alpha > 0$ such that $\rho(X(t), \Omega) \leq Me^{-\alpha(t-t_0)}$, then the three-dimensional autonomous systems with cross-product non-linearities system (1) are said to have globally exponentially attractive set, or the system (1) is globally exponentially stable in the sense of Lagrange, and Ω is called the globally exponentially attractive set.

Definition 3. For the three-dimensional autonomous systems with cross-product nonlinearities (1), if there exist compact set $\Omega \subset R^3$, a constant $\alpha > 0$, and a bounded function $M(x_0, t) > 0$ on t, as $t \geq t_0$, such that $\rho(X(t), \Omega) \leq M(x_0)e^{-\alpha(t-t_0)}$, where $M(x_0) = \sup M(x_0, t)$, $t \geq t_0$, then the system (1) is said to have globally conditional exponentially attractive set, or the system (1) is globally conditional exponentially stable in the sense of Lagrange, and Ω is called the globally conditional exponentially attractive set.

In general, from the definition we see that a globally exponential attractive set is not necessarily a positive invariant set. But our results obtained in the next section indeed show that a globally exponentially attractive set is a positive invariant set.

Note that it is difficult to verify the existence of Ω in Definition 2. Since the Lyapunov direct method is still a powerful tool in the study of asymptotic behaviour of nonlinear dynamical systems, the following definition is more useful in applications.

Definition 4. For three-dimensional autonomous systems with cross-product nonlinearities (1), if there exist a positive definite and radially unbounded Lyapunov function $V(X(t))$ and positive numbers $L > 0$, $\alpha > 0$ such that the following inequality

$$V(X(t)) - L \leq (V(X_0) - L)e^{-\alpha(t-t_0)} \tag{4}$$

is valid for $V(X(t)) > L(t \geq t_0)$, then the system (1) is said to be globally exponentially attractive or globally exponentially stable in the sense of Lagrange, and $\Omega := \{X \mid V(X(t)) \leq L, t \geq t_0\}$ is called the globally exponentially attractive set.

Definition 5. For the three-dimensional autonomous systems with cross-product nonlinearities (1), if there exist a positive definite and radially unbounded Lyapunov function $V(X(t))$ and a bounded function $L(x_0, t) > 0$, on t, as $t \geq t_0$, and $\alpha > 0$ such that the following inequality

$$V(X(t)) - L(x_0) \leq (V(X_0) - L(x_0))e^{-\alpha(t-t_0)} \tag{5}$$

is valid for $V(X(t)) > L(x_0)$, $t \geq t_0$, where $L(x_0) = \sup L(x_0, t)$, $t \geq t_0$, then the system (1) is said to be globally conditional exponentially attractive or globally conditional exponentially stable in the sense of Lagrange, and $\Omega := \{X \mid V(X(t)) \leq L(x_0), t \geq t_0\}$ is called the globally conditional exponentially attractive set.

3. Qualitative Analysis

We call the dynamic system (1) the first class three-dimensional chaotic system with cross-product nonlinearities (1), if there are some nonzero numbers $\{\lambda_1, \lambda_2, \lambda_3\}$ so as to satisfy conditions

$$\lambda_1^2 b_{111} = 0, \qquad \lambda_1^2 (b_{112} + b_{121}) + \lambda_2^2 b_{211} = 0,$$

$$\lambda_2^2 b_{222} = 0, \qquad \lambda_2^2 (b_{221} + b_{212}) + \lambda_1^2 b_{122} = 0,$$

$$\lambda_3^2 b_{333} = 0, \qquad \lambda_3^2 (b_{313} + b_{331}) + \lambda_1^2 b_{133} = 0,$$

$$\lambda_1^2 (b_{113} + b_{131}) + \lambda_3^2 b_{311} = 0, \tag{6}$$

$$\lambda_2^2 (b_{223} + b_{232}) + \lambda_3^2 b_{322} = 0,$$

$$\lambda_3^2 (b_{323} + b_{332}) + \lambda_2^2 b_{233} = 0,$$

$$\lambda_1^2 (b_{123} + b_{132}) + \lambda_2^2 (b_{213} + b_{231}) + \lambda_3^2 (b_{321} + b_{312}) = 0.$$

Condition (6) is satisfied by some known three-dimensional quadratic autonomous chaotic systems, the well-known Lorenz system [1–3], the Rössler system [5], the Rucklidge system [6], and the Chen system [7, 8]. Lorenz systems are widely studied and the references therein [9–12, 19–21]. For example, consider the classical Lorenz system

$$\dot{x} = \sigma(y - x),$$

$$\dot{y} = \rho x - \gamma y - xz, \tag{7}$$

$$\dot{z} = xy - \beta z,$$

and the general Lorenz systems

$$
\begin{aligned}
\dot{x} &= -ax + by + yz, \\
\dot{y} &= cx - y - xz, \\
\dot{z} &= dy - z + xy,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\dot{x} &= -z, \\
\dot{y} &= -y - x^2, \\
\dot{z} &= 1.7x + y + 1.7.
\end{aligned}
\tag{9}
$$

Thus it can be seen that condition (6) is very important in qualitative analysis of the exponentially attractive set and positive invariant set of Lorenz systems.

We will research this dynamic system in two cases.

First, supposing $b_{111} = b_{122} = b_{133} = b_{211} = b_{222} = b_{233} = b_{311} = b_{322} = b_{333} = 0$, $b_{iij} = -b_{iji}$, $i, j = 1, 2, 3$, $\exists b_{ijk} \neq -b_{ikj}$, $i \neq j \neq k$, the dynamic system (1) can be rewritten as

$$
\begin{aligned}
\dot{x}_1 &= \sum_{j=1}^{3} a_{1j} x_j + (b_{123} + b_{132}) x_2 x_3 + c_1, \\
\dot{x}_2 &= \sum_{j=1}^{3} a_{2j} x_j + (b_{213} + b_{231}) x_1 x_3 + c_2, \\
\dot{x}_3 &= \sum_{j=1}^{3} a_{3j} x_j + (b_{312} + b_{321}) x_1 x_2 + c_3.
\end{aligned}
\tag{10}
$$

The construction techniques of this kind of Lorenz systems are to pay attention to satisfing formula

$$
\lambda_1^2 (b_{123} + b_{132}) + \lambda_2^2 (b_{213} + b_{231}) + \lambda_3^2 (b_{321} + b_{312}) = 0,
\tag{11}
$$

where λ_i, $i = 1, 2, 3$ are parameters and

$$
\begin{aligned}
f(\mu, X) = &\sum_{i=1}^{3} \lambda_i^2 (a_{ii} + \mu_i) x_i^2 \\
&+ \sum_{i=1}^{3} \left(\lambda_i^2 c_i + \mu_i d_i \lambda_i - \mu_i \lambda_i^2 d_i - \sum_{j=1}^{3} \lambda_j d_j a_{ji} \right) x_i \\
&+ \sum_{i=1}^{3} \left(\lambda_i d_i c_i - \lambda_i d_i^2 \right),
\end{aligned}
\tag{12}
$$

where μ_i, $i = 1, 2, 3$ are undetermined parameters. And we always assume that the supremum $f(\mu, X) < +\infty$ in the paper.

Lemma 6. *Suppose* $\lambda_i > 0$, $i = 1, 2, 3$,

$$
\begin{aligned}
\lambda_1^2 a_{12} + \lambda_2^2 a_{21} + \lambda_3 d_3 (b_{312} + b_{321}) &= 0, \\
\lambda_1^2 a_{13} + \lambda_3^2 a_{31} + \lambda_2 d_2 (b_{213} + b_{231}) &= 0, \\
\lambda_2^2 a_{23} + \lambda_3^2 a_{32} + \lambda_1 d_1 (b_{123} + b_{132}) &= 0, \\
a_{11} + \mu_1 &< 0, \\
a_{22} + \mu_2 &< 0, \\
a_{33} + \mu_3 &< 0.
\end{aligned}
\tag{13}
$$

The function (12) *has maximum.*

Proof. Consider

$$
\begin{aligned}
f'_{x_1}(\mu, X) =\ & 2 \left(\lambda_1^2 a_{11} + \mu_1 \lambda_1^2 \right) x_1 \\
& + \left(\lambda_1^2 c_1 + \mu_1 d_1 \lambda_1 - \mu_1 \lambda_1^2 d_1 - \lambda_1 d_1 a_{11} \right. \\
& \left. - \lambda_2 d_2 a_{21} - \lambda_3 d_3 a_{31} \right), \\
f'_{x_2}(\mu, X) =\ & 2 \left(\lambda_2^2 a_{22} + \mu_2 \lambda_2^2 \right) x_2 \\
& + \left(\lambda_2^2 c_2 + \mu_2 d_2 \lambda_2 - \mu_2 \lambda_2^2 d_2 - \lambda_1 d_1 a_{12} \right. \\
& \left. - \lambda_2 d_2 a_{22} - \lambda_3 d_3 a_{32} \right), \\
f'_{x_3}(\mu, X) =\ & 2 \left(\lambda_3^2 a_{33} + \mu_3 \lambda_3^2 \right) x_3 \\
& + \left(\lambda_3^2 c_3 + \mu_3 d_3 \lambda_3 - \mu_3 \lambda_3^2 d_3 - \lambda_1 d_1 a_{13} \right. \\
& \left. - \lambda_2 d_2 a_{23} - \lambda_3 d_3 a_{33} \right), \\
f''_{x_1^2}(\mu, X) =\ & 2\lambda_1^2 (a_{11} + \mu_1), \\
f''_{x_2^2}(\mu, X) =\ & 2\lambda_2^2 (a_{22} + \mu_2), \\
f''_{x_3^2}(\mu, X) =\ & 2\lambda_3^2 (a_{33} + \mu_3), \\
f''_{x_1 x_2}(\mu, X) =\ & f''_{x_2 x_1}(\mu, X) = f''_{x_1 x_3}(\mu, X) \\
=\ & f''_{x_2 x_3}(\mu, X) = f''_{x_3 x_2}(\mu, X) = 0.
\end{aligned}
\tag{14}
$$

Thus the Hesse matrix H_f of the $f(\mu, X)$ is a negative definite matrix; furthermore $\max_{X \in R^3} f(\mu, X)$ exists.

These parameters d_i, μ_i, λ_i, $i = 1, 2, 3$ will be determined by solving the maximum of $f_1(\mu, X)$ and formula (12), and let

$$
M^+ = \begin{cases} M, & M > 0, \\ 0, & M \leq 0. \end{cases}
\tag{15}
$$

$\square$

Theorem 7. *If condition* (6) *exists,* $\eta = \min\{\mu_1, \mu_2, \mu_3\}$, $M = \max_{X \in R^3} f(\mu, X)$, $\lambda_i > 0$, $i = 1, 2, 3$, *then the estimation*

$$
\left[V(X(t)) - \frac{1}{2\eta} M^+ \right] \leq \left[V(X(t_0)) - \frac{1}{2\eta} M^+ \right] e^{-2\eta(t - t_0)}
\tag{16}
$$

holds, and the set

$$\Omega = \left\{ X \mid V(X(t)) \le \frac{1}{2\eta} M^+ \right\}$$

$$= \left\{ X \mid \left[(\lambda_1 x_1 - d_1)^2 + (\lambda_2 x_2 - d_2)^2 + (\lambda_3 x_3 - d_3)^2 \right] \right.$$

$$\left. \le \frac{1}{\eta} M^+ \right\}$$

(17)

is the globally exponentially attractive set and positive invariant set of system (10); *that is,*

$$\overline{\lim_{t \to \infty}} V(X(t)) \le \frac{1}{2\eta} M^+.$$

(18)

Proof. Differentiating the Lyapunov function $V(X(t))$ in (3) with respect to time t along the trajectory of system (10) yields

$$\dot{V}(X(t))\big|_{(10)} = \lambda_1 (\lambda_1 x_1 - d_1) \dot{x}_1 + \lambda_2 (\lambda_2 x_2 - d_2) \dot{x}_2$$

$$+ \lambda_3 (\lambda_3 x_3 - d_3) \dot{x}_3$$

$$= -\sum_{j=1}^{3} \mu_j (\lambda_j x_j - d_j)^2 + \lambda_1 (\lambda_1 x_1 - d_1)$$

$$\times \left(\sum_{j=1}^{3} a_{1j} x_j + \mu_1 \lambda_1^{-1} (\lambda_1 x_1 - d_1) \right.$$

$$\left. + (b_{123} + b_{132}) x_2 x_3 + c_1 \right)$$

$$+ \lambda_2 (\lambda_2 x_2 - d_2)$$

$$\times \left(\sum_{j=1}^{3} a_{2j} x_j + \mu_2 \lambda_2^{-1} (\lambda_2 x_2 - d_2) \right.$$

$$\left. + (b_{213} + b_{231}) x_1 x_3 + c_2 \right)$$

$$+ \lambda_3 (\lambda_3 x_3 - d_3)$$

$$\times \left(\sum_{j=1}^{3} a_{3j} x_j + \mu_3 \lambda_3^{-1} (\lambda_3 x_3 - d_3) \right.$$

$$\left. + (b_{312} + b_{321}) x_1 x_2 + c_3 \right)$$

$$\le -\eta \sum_{j=1}^{3} (\lambda_j x_j - d_j)^2 + f(\mu, X)$$

$$\le -\eta \sum_{j=1}^{3} (\lambda_j x_j - d_j)^2 + M^+$$

$$\le -2\eta \left[V(X(t)) - \frac{1}{2\eta} M^+ \right] \le 0,$$

$$\text{when } V(X(t)) > \frac{1}{2\eta} M^+,$$

(19)

where $\mu_j > 0$, $j = 1, 2, 3$. Integrating both sides of (19) yields (16) and (17). By the definition, taking into account limit on both sides of the above inequality (16) as $t \to +\infty$ results in inequality (18).

Now, the characters of some of the chaotic systems known are analysed by condition (6). When $a_{11} = -\sigma$, $a_{12} = \sigma$, $a_{21} = \rho$, $a_{22} = -\gamma$, $a_{33} = -\beta$, $b_{213} = -1$, $b_{312} = 1$, else $a_{ij} = 0$, $b_{ijk} = 0$, $c_1 = c_2 = c_3 = 0$, and $\lambda_1 = \sqrt{\lambda}, \lambda_2 = \lambda_3 = 1$, $d_1 = d_2 = 0$, $d_3 = \lambda\sigma + \rho$, $\mu_1 = \sigma$, $\mu_2 = \gamma$, $\mu_3 = \min\{\sigma, \gamma\}$, $\eta = \eta_1 = \mu_3$, $\beta > \eta_1$, system (10) can be rewritten as system (7):

$$V(X(t)) = \frac{1}{2} \left[\lambda x_1^2 + x_2^2 + (x_3 - \lambda\sigma - \rho)^2 \right],$$

$$f(\mu, X) = -(\beta - \eta_1) x_3^2 + (\beta - 2\eta_1)(\lambda\sigma + \rho) x_3$$

(20)

$$+ \eta_1 (\lambda\sigma + \rho)^2.$$

We have $M = \beta^2 (\lambda\sigma + \rho)^2 / 4(\beta - \eta_1)$. Thus

$$\left[V(X(t)) - \frac{\beta^2 (\lambda\sigma + \rho)^2}{8(\beta - \eta_1)\eta_1} \right]$$

$$\le \left[V(X(t_0)) - \frac{\beta^2 (\lambda\sigma + \rho)^2}{8(\beta - \eta_1)\eta_1} \right] e^{-2\eta_1(t - t_0)},$$

$$\Omega_1 = \left\{ X \mid V(X) \le \frac{\beta^2 (\lambda\sigma + \rho)^2}{8(\beta - \eta_1)\eta_1} \right\}$$

$$= \left\{ X \mid \lambda x_1^2 + x_2^2 + (x_3 - \lambda\sigma - \rho)^2 \le \frac{\beta^2 (\lambda\sigma + \rho)^2}{4(\beta - \eta_1)\eta_1} \right\}$$

(21)

is the globally exponentially attractive set and positive invariant set of system (7). $\qquad\square$

Example 8. Further, taking ito accout $\mu_1 = \sigma$, $\mu_2 = \gamma$, $\mu_3 = \beta/2$, $\eta = \eta_2 = \min\{\sigma, \gamma, \beta/2\}$, the estimate

$$\left[V(X(t)) - \frac{\beta(\lambda\sigma + \rho)^2}{4\eta_2} \right]$$

(22)

$$\le \left[V(X(t_0)) - \frac{\beta(\lambda\sigma + \rho)^2}{4\eta_2} \right] e^{-2\eta_2(t - t_0)}$$

holds and that

$$\Omega_2 = \left\{ X \mid V(X) \le \frac{\beta(\lambda\sigma + \rho)^2}{4\eta_2} \right\}$$

$$= \left\{ X \mid \lambda x_1^2 + x_2^2 + (x_3 - \lambda\sigma - \rho)^2 \le \frac{\beta(\lambda\sigma + \rho)^2}{2\eta_2} \right\} \quad (23)$$

is the globally uniform exponentially attractive set and positive invariant set of system (7).

Proof. Again applying Lyapunov function given in (19) and evaluating the derivative of dV_1/dt along the trajectory of system (16) lead to

$$f(\mu, X) = -\frac{x_3^2 \beta}{2} + \frac{(\lambda\sigma + \rho)^2 \beta}{2},$$

$$M = \frac{\beta(\lambda\sigma + \rho)^2}{2}. \quad (24)$$

The conclusion of **Example 9** is obtained. $\qquad \square$

Example 9. Furthermore, choose $\mu_1 = \sigma$, $\mu_2 = \gamma$, $\mu_3 = \beta$, $0 < \xi_1 < \beta$, $\eta = \eta_3 = \min\{\sigma, \gamma, \xi_1\}$. Get

$$f(\mu, X) = -(\beta - \eta_3) x_3^2 + (\beta - 2\eta_3)(\lambda\sigma + \rho) x_3$$

$$+ \eta_2(\lambda\sigma + \rho)^2, \quad (25)$$

$$M = \frac{\beta^2(\lambda\sigma + \rho)^2}{4(\beta - \eta_3)}.$$

Then, the estimate

$$\left[V(X(t)) - \frac{\beta^2(\lambda\sigma + \rho)^2}{8(\beta - \eta_3)\eta_3} \right]$$

$$\le \left[V(X(t_0)) - \frac{\beta^2(\lambda\sigma + \rho)^2}{8(\beta - \eta_3)\eta_3} \right] e^{-2\eta_3(t - t_0)} \quad (26)$$

holds and that

$$\Omega_3 = \left\{ X \mid V(X) \le \frac{\beta^2(\lambda\sigma + \rho)^2}{8(\beta - \eta_3)\eta_3} \right\}$$

$$= \left\{ X \mid \lambda x_1^2 + x_2^2 + (x_3 - \lambda\sigma - \rho)^2 \le \frac{\beta^2(\lambda\sigma + \rho)^2}{4(\beta - \eta_3)\eta_3} \right\} \quad (27)$$

is the globally exponentially attractive set and positive invariant set of system (7).

Example 10. Taking $a_{11} = -a$, $a_{12} = b$, $a_{21} = c$, $a_{22} = -1$, $a_{32} = d$, $a_{33} = -1$, $b_{123} = b_{312} = 1$, $b_{213} = -1$ else $a_{ij} = 0$, $b_{ijk} = 0$, $c_1 = c_2 = c_3 = 0$, and $\lambda_1 = \lambda_3 = 1$, $\lambda_2 = \sqrt{2}$, $d_1 = d$,

$d_2 = 0$, $d_3 = b + 2c$, system (6), $V(X(t))$, and $f(u, X)$ can be rewritten as system (8):

$$V(X(t)) = \frac{1}{2} \left[(x_1 - d)^2 + 2x_2^2 + (x_3 - b - 2c)^2 \right],$$

$$f(\mu, X) = -(a - \mu_1) x_1^2 + (a - 2\mu_1) dx_1 - 2(1 - \mu_2) x_2^2$$

$$- 2(b + c) dx_2 - (1 - \mu_3) x_3^2$$

$$+ (b + 2c)(1 - 2\mu_3) x_3 + \mu_1 d^2 + \mu_3 (b + 2c)^2. \quad (28)$$

Thus

$$M = \frac{(a - 2\mu_1)^2 d^2}{4(a - \mu_1)} + \frac{(b + c)^2 d^2}{2(1 - \mu_2)} + \frac{(b + 2c)^2 (1 - 2\mu_3)}{4(1 - \mu_3)}$$

$$+ \mu_1 d^2 + \mu_3 (b + 2c)^2. \quad (29)$$

We have

$$V(x(t)) - M \le [V(x(t_0)) - M] e^{-2\eta(t - t_0)}, \quad (30)$$

then

$$\Omega_4 = \{ X \mid V(x(t)) - M \}$$

$$= \left\{ X \mid (x_1 - d)^2 + 2x_2^2 + (x_3 - b - 2c)^2 \le 2M \right\} \quad (31)$$

is the estimation of the globally exponentially attractive and positive invariant sets of system (8).

If $b_{111} = b_{222} = b_{333} = 0$, $\exists b_{ijj} \ne 0$, $i, j = 1, 2, 3$, the dynamic system (1) is shown as

$$\dot{x}_1 = \sum_{j=1}^{3} a_{1j} x_j + \sum_{i=2,3} b_{1ii} x_i^2 + \sum_{i \ne j=1}^{3} (b_{1ij} + b_{1ji}) x_i x_j + c_1,$$

$$c\dot{x}_2 = \sum_{j=1}^{3} a_{2j} x_j + \sum_{i=1,3} b_{2ii} x_i^2 + \sum_{i \ne j=1}^{3} (b_{2ij} + b_{2ji}) x_i x_j + c_2,$$

$$\dot{x}_3 = \sum_{j=1}^{3} a_{3j} x_j + \sum_{i=1,2} b_{3ii} x_i^2 + \sum_{i \ne j=1}^{3} (b_{3ij} + b_{3ji}) x_i x_j + c_3. \quad (32)$$

In this case, we can take into account

$$f(\mu, X) = \sum_{i=1}^{3} \left[\lambda_i^2 (a_{ii} + \mu_i) + \lambda_{[i+1]_3} d_{[i+1]_3} b_{[i+1]_3, ii} \right.$$

$$\left. + \lambda_{[i+2]_3} d_{[i+2]_3} b_{[i+2]_3, ii} \right] x_i^2$$

$$+ \sum_{i<j,1}^{3} \left(\lambda_i^2 a_{ij} + \lambda_j^2 a_{ji} + \lambda_1 d_1 \left(b_{1ij} + b_{1ji} \right) \right.$$

$$+ \lambda_2 d_2 \left(b_{2ij} + b_{2ji} \right) + \lambda_3 d_3 \left(b_{3ij} + b_{3ji} \right) \right) x_i x_j$$

$$+ \sum_{i=1}^{3} \left(\lambda_i^2 \left(c_i - \mu_i d_i \right) + \mu_i \lambda_i d_i - a_{1i} \lambda_1 d_1 \right.$$

$$\left. - a_{2i} \lambda_2 d_2 - a_{3i} \lambda_3 d_3 \right) x_1$$

$$+ \sum_{i=1}^{3} \lambda_i d_i \left(c_i - d_i \right), \tag{33}$$

where $[\cdot]_3$ denotes modulo-3.

Theorem 11. *Suppose that $G_0 = (x_1^0, x_2^0, \ldots, x_n^0)$ is the stable point of the $f(\mu, X)$ defined by* (33). *If the Hesse matrix of the $f(\mu, X)$ is a negative definite matrix, the $f(\mu, X)$ has maximum M and the estimation*

$$\left[V(X(t)) - \frac{1}{2\eta} M^+ \right]$$

$$\leq \left[V(X(t_0)) - \frac{1}{2\eta} M^+ \right] e^{-2\eta(t-t_0)} \tag{34}$$

holds; that is,

$$\varlimsup_{t \to \infty} V(X(t)) \leq \frac{1}{2\eta} M^+, \tag{35}$$

and the set

$$\Omega = \left\{ X \mid V(X(t)) \leq \frac{1}{2\eta} M^+ \right\}$$

$$= \left\{ X \mid \left[(\lambda_1 x_1 - d_1)^2 + (\lambda_2 x_2 - d_2)^2 + (\lambda_3 x_3 - d_3)^2 \right] \right.$$

$$\left. \leq \frac{1}{\eta} M^+ \right\} \tag{36}$$

is the globally exponentially attractive set and positive invariant set of system (32).

Proof. If G_0 is the stable point of the $f(\mu, X)$, that is,

$$\nabla(f)_{G_0} = \left(f_{x_1}', f_{x_2}', f_{x_3}' \right) = 0, \tag{37}$$

and the Hesse matrix H_f of the $f(\mu, X)$ is a negative definite matrix, namely,

$$f_{x_1 x_1}'' < 0, \qquad \begin{vmatrix} f_{x_1 x_1}'' & f_{x_1 x_2}'' \\ f_{x_2 x_1}'' & f_{x_2 x_2}'' \end{vmatrix} > 0,$$

$$\begin{vmatrix} f_{x_1 x_1}'' & f_{x_1 x_2}'' & f_{x_1 x_3}'' \\ f_{x_2 x_1}'' & f_{x_2 x_2}'' & f_{x_2 x_3}'' \\ f_{x_3 x_1}'' & f_{x_3 x_2}'' & f_{x_3 x_3}'' \end{vmatrix} < 0. \tag{38}$$
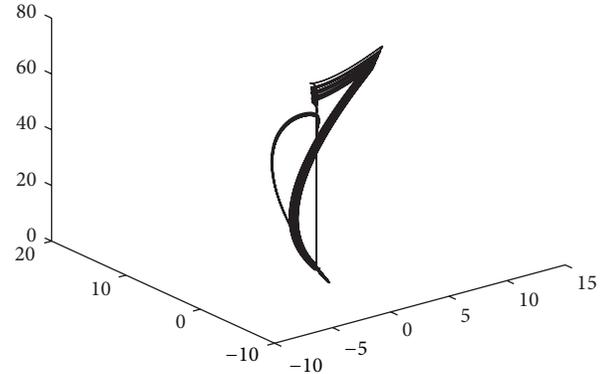

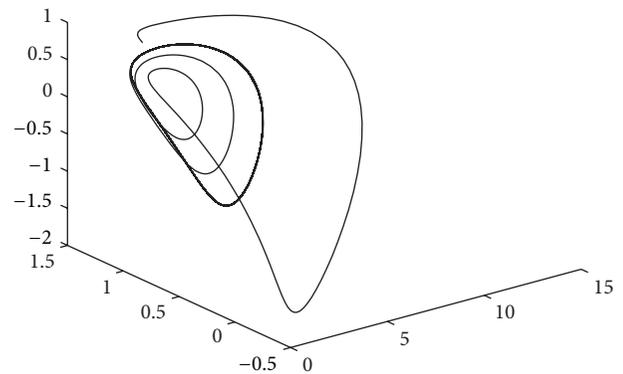FIGURE 1: Simulation of system (40).

FIGURE 2: Simulation of system (43).

The $f(\mu, X)$ has the maximum M. Differentiating the Lyapunov function $V(X(t))$ in (3) with respect to time t along the trajectory of system (32) yields

$$\dot{V}(X(t))\big|_{(32)} = \lambda_1 \left(\lambda_1 x_1 - d_1 \right) \dot{x}_1 + \lambda_2 \left(\lambda_2 x_2 - d_2 \right) \dot{x}_2$$

$$+ \lambda_3 \left(\lambda_3 x_3 - d_3 \right) \dot{x}_3$$

$$\leq -\eta \sum_{j=1}^{3} \left(\lambda_j x_j - d_j \right)^2 + f(\mu, X)$$

$$\leq -\eta \sum_{j=1}^{3} \left(\lambda_j x_j - d_j \right)^2 + M^+$$

$$\leq -2\eta \left[V(X(t)) - \frac{1}{2\eta} M^+ \right] \leq 0, \tag{39}$$

$$\text{when } V(X(t)) > \frac{1}{2\eta} M^+.$$

The proof is complete. □

4. Switched Chaotic Systems

Condition (6) has helpfully provided us with instructions on how to find the new chaotic systems. We construct a series
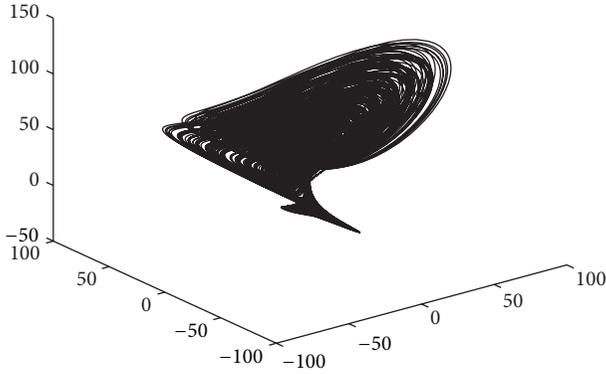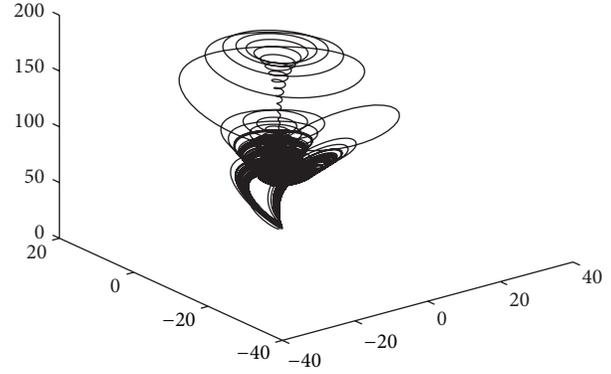
Figure 3: Simulation of system (44).
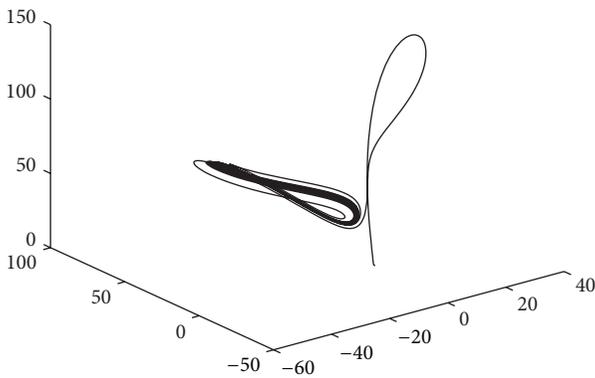

Figure 5: Simulation of system (46).

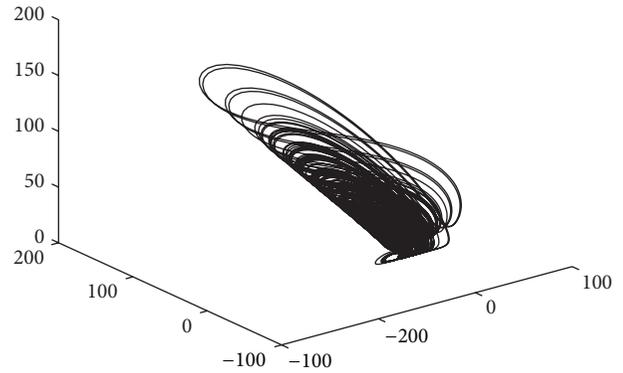Figure 4: Simulation of system (45).

Figure 6: Simulation of system (47).

of new chaotic systems that the condition (6) is fulfilled and study the switching system between them.

Example 12. Consider a Lorenz system shown in Figure 1:

$$
\begin{aligned}
\dot{x}_1 &= -12x_1 + 5x_2 - 0.8x_1x_3 + x_2x_3, \\
\dot{x}_2 &= 28x_1 - x_2 - x_1x_3, \\
\dot{x}_3 &= -3x_2 - x_3 + 10x_1^2 + x_1x_2.
\end{aligned}
\tag{40}
$$

Solution. Here

$$a_{11} = -12, \quad a_{12} = 5, \quad a_{21} = 28, \quad a_{22} = -1, \quad a_{32} = -3,$$

$$a_{33} = -1, \quad \text{els } a_{ij} = 0, \quad b_{113} = b_{131} = -0.4,$$

$$b_{123} = b_{132} = 0.5, \quad b_{213} = b_{231} = -0.5, \quad b_{311} = 10,$$

$$b_{312} = b_{321} = 0.5, \quad \text{els } b_{ijk} = 0, \quad \lambda_1 = \sqrt{12.5},$$

$$\lambda_2 = \sqrt{13.5}, \quad \lambda_3 = 1, \quad d_1 = -\frac{6}{25}, \quad d_2 = \frac{4}{45},$$

$$d_3 = 440.5, \quad \mu_1 = 2, \quad \mu_2 = \mu_3 = \frac{1}{2}, \quad \eta = \frac{1}{2},$$

$$f_2(\mu, X) = -125x_1^2 - 6.75x_2^2 - 0.5x_3^2$$

$$+ \left(-\frac{48}{25}\sqrt{12.5} - \frac{112}{45}\sqrt{13.5} \right) x_1$$

$$+ \left(\frac{168}{25}\sqrt{12.5} + \frac{2643}{2} \right) x_2 + \frac{39293196661}{405000},$$

$$f'_{2x_1}(\mu, X) = -250x_1 - \left(\frac{48}{25}\sqrt{12.5} + \frac{112}{45}\sqrt{13.5} \right),$$

$$f'_{2x_1}(\mu, X) = 0, \quad x_1 = -\frac{1}{1250}\left(\frac{48}{5}\sqrt{12.5} + \frac{112}{9}\sqrt{13.5} \right)$$

$$\approx 0.064,$$

$$f'_{2x_2}(\mu, X) = -13.5x_2 + \left(\frac{168}{25}\sqrt{12.5} + \frac{2643}{2} \right),$$

$$f'_{2x_2}(\mu, X) = 0, \quad x_2 = \frac{30}{135}\left(\frac{56}{25}\sqrt{12.5} + \frac{881}{2} \right) \approx 99.65,$$

$$f'_{2x_3}(\mu, X) = -x_3, \quad f'_{2x_3}(\mu, X) = 0, \quad x_3 = 0,$$

$$f''_{2x_1^2}(\mu, X) = -250, \quad f''_{2x_2^2}(\mu, X) = -13.5,$$

$$f''_{2x_3^2}(\mu, X) = -1,$$

$$f''_{2x_1x_2}(\mu, X) = f''_{2x_2x_1}(\mu, X) = f''_{2x_1x_3}(\mu, X) = 0,$$

$$f''_{2x_3x_1}(\mu, X) = f''_{2x_2x_3}(\mu, X) = f''_{2x_3x_2}(\mu, X) = 0.$$

$$\tag{41}$$

(a) System (40)–(43)

(b) System (40)–(44)

(c) System (40)–(45)
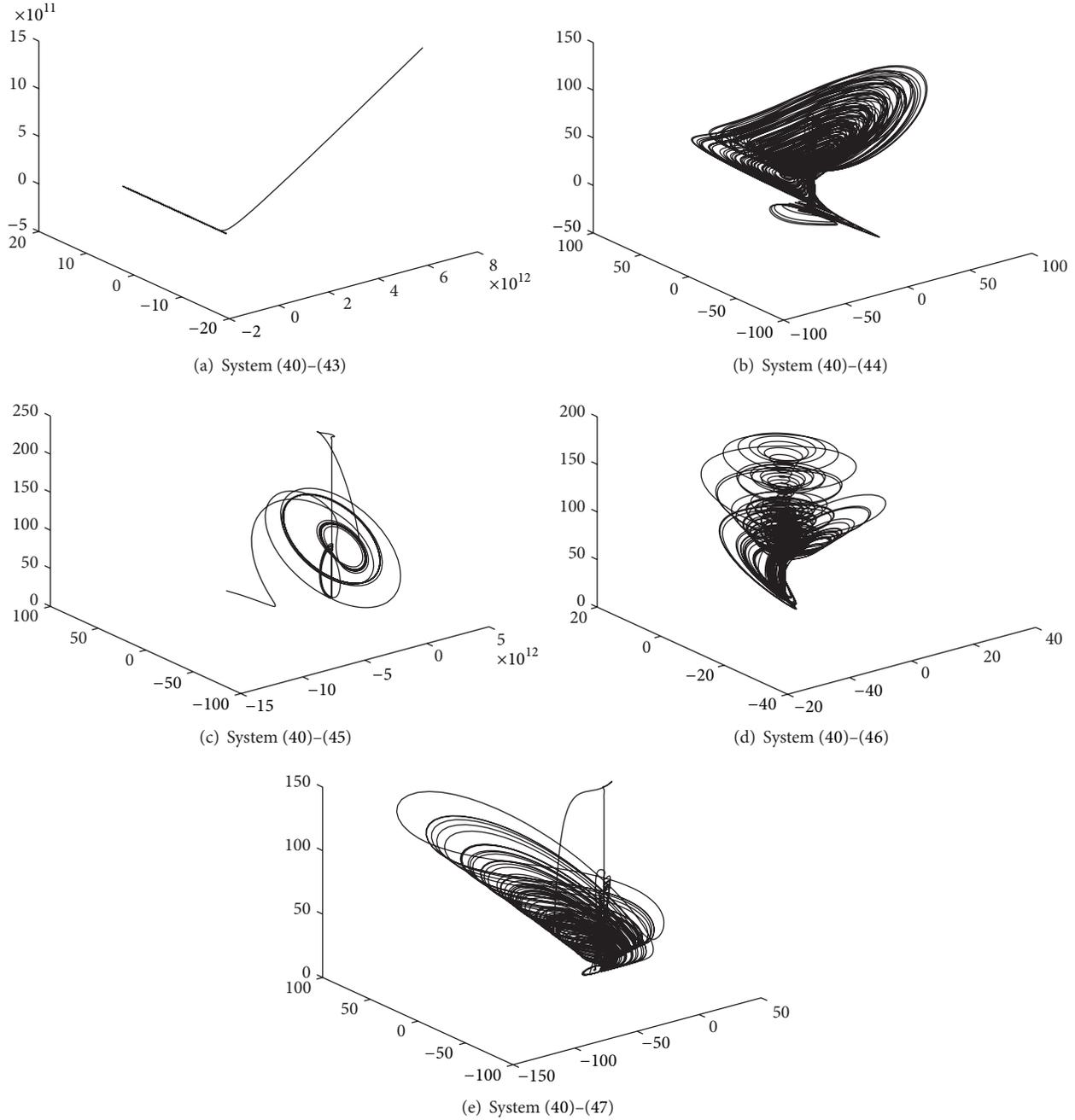

(d) System (40)–(46)

(e) System (40)–(47)

FIGURE 7: Switched system between system (40) and others.

The Hesse matrix of the $f(\mu, X)$ is a negative definite matrix, max $f(\mu, X) \approx 164045.42$. The set

$$\Omega = \left\{ X \mid \left(\sqrt{12.5}x_1 + \frac{5}{26} \right)^2 + \left(\sqrt{13.5}x_2 - \frac{4}{45} \right)^2 \right.$$
$$\left. + (x_3 - 440.5)^2 \leq 328090.84 \right\} \tag{42}$$

is the globally exponentially attractive set and positive invariant set of system (40).

Note. (a) If the Hesse matrix of the $f(\mu, X)$ is not a negative definite matrix, the $f(\mu, X)$ has no maximum M.

(b) If $\exists a_{i_0 i_0} \geq 0$, $\lim_{x_{i_0} \to \infty} f(\mu, X) = +\infty$, this type of chaotic system needs further research.

(c) We call the dynamic system (1) the second class three-dimensional chaotic system with cross-product nonlinearities, if it does not satisfy condition (6). For this class of chaotic systems, $f(\mu, X)$ is a cubic polynomial and there is not maximum if we choose energy function (3) differentiating this Lyapunov function with respect to t along the trajectory of system (1). It is very useful to research these problems.
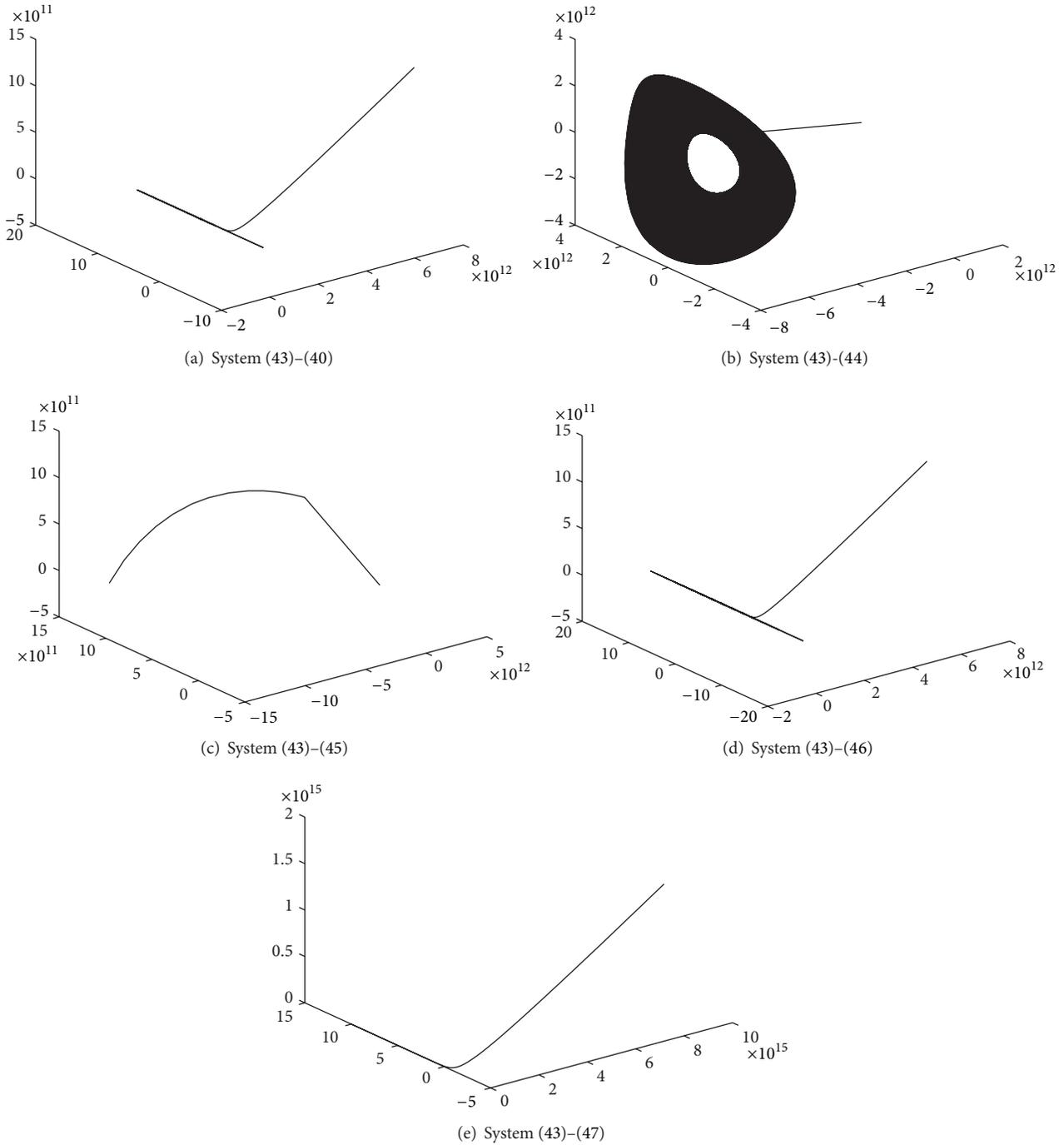
(a) System (43)–(40)

(b) System (43)-(44)

(c) System (43)–(45)

(d) System (43)–(46)

(e) System (43)–(47)

Figure 8: Switched system between system (43) and others.

Example 13. The new chaotic system shown in Figure 2 is

$$\dot{x}_1 = -2x_1 + 5x_3 + 5.7x_1x_3 + 4.7x_2x_3 + 4,$$
$$\dot{x}_2 = -x_1 - 2x_2 + 3x_3 + 5, \tag{43}$$
$$\dot{x}_3 = -6x_1 - 2x_2 - 4x_3 + 0.2x_1^2 + 3.4x_1x_2 + 7.$$

Example 14. The chaotic system shown in Figure 3 is

$$\dot{x}_1 = -11x_1 + 0.15x_1x_2 + 1.38x_2x_3 + 1,$$
$$\dot{x}_2 = 30x_1 - x_2 - x_1x_3 + 0.1x_2x_3, \tag{44}$$
$$\dot{x}_3 = 15x_2 - 2.5x_3 + x_1x_2.$$

(a) System (44)–(40)

(b) System (44)-(43)

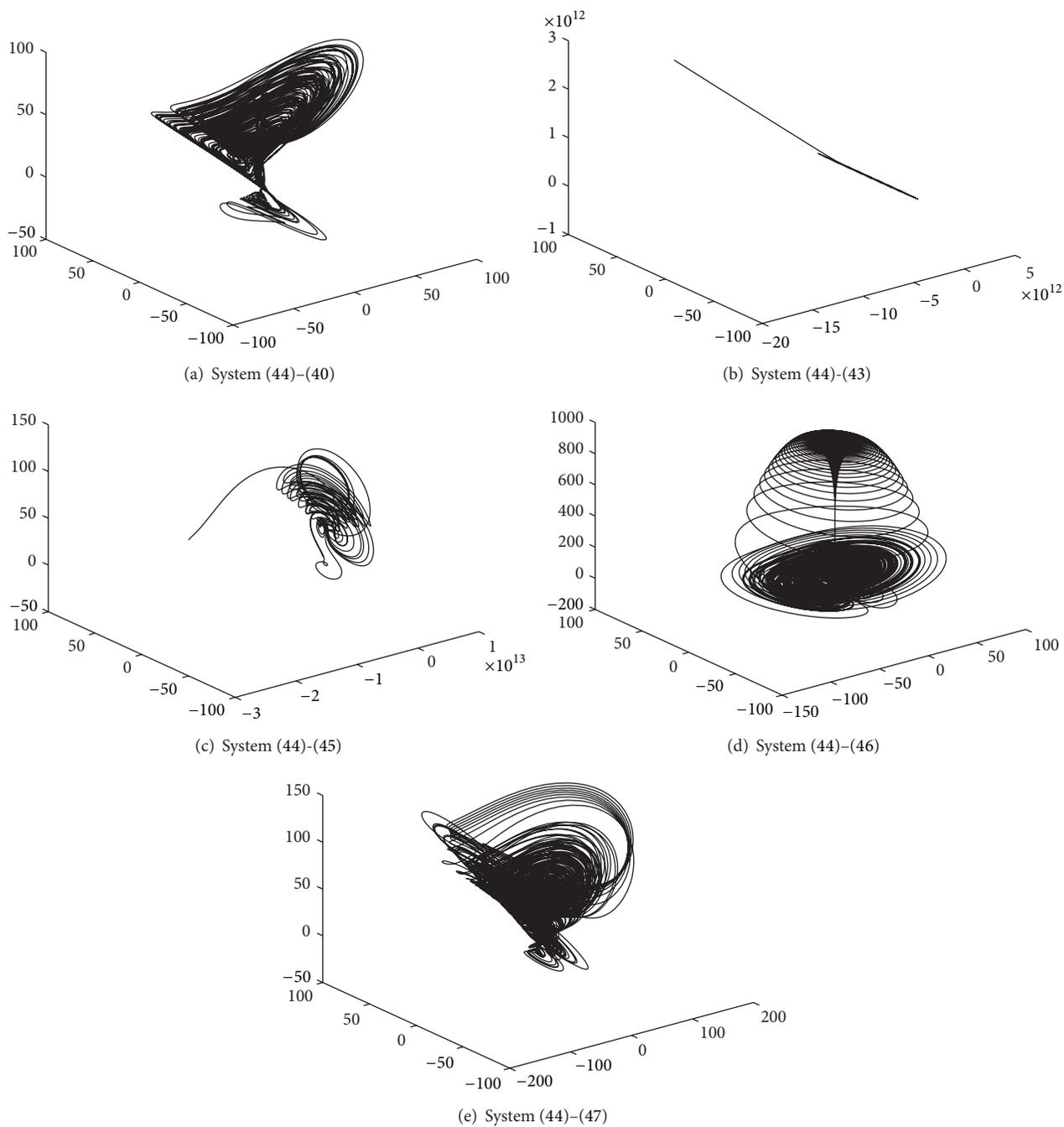(c) System (44)-(45)

(d) System (44)–(46)

(e) System (44)–(47)

Figure 9: Switched system between Example (44) and others.

Example 15. The chaotic system shown in Figure 4 is

$$
\begin{aligned}
\dot{x}_1 &= 30\left(x_2 - x_1\right) - 0.48x_1^2, \\
\dot{x}_2 &= 80x_1 - 6x_2 - x_1x_3, \\
\dot{x}_3 &= x_1x_2 - 5x_3.
\end{aligned}
\tag{45}
$$

Example 16. The chaotic system shown in Figure 5 is

$$
\begin{aligned}
\dot{x}_1 &= -12x_1 + 5x_2 + x_2x_3, \\
\dot{x}_2 &= 28x_1 - x_2 - x_1x_3, \\
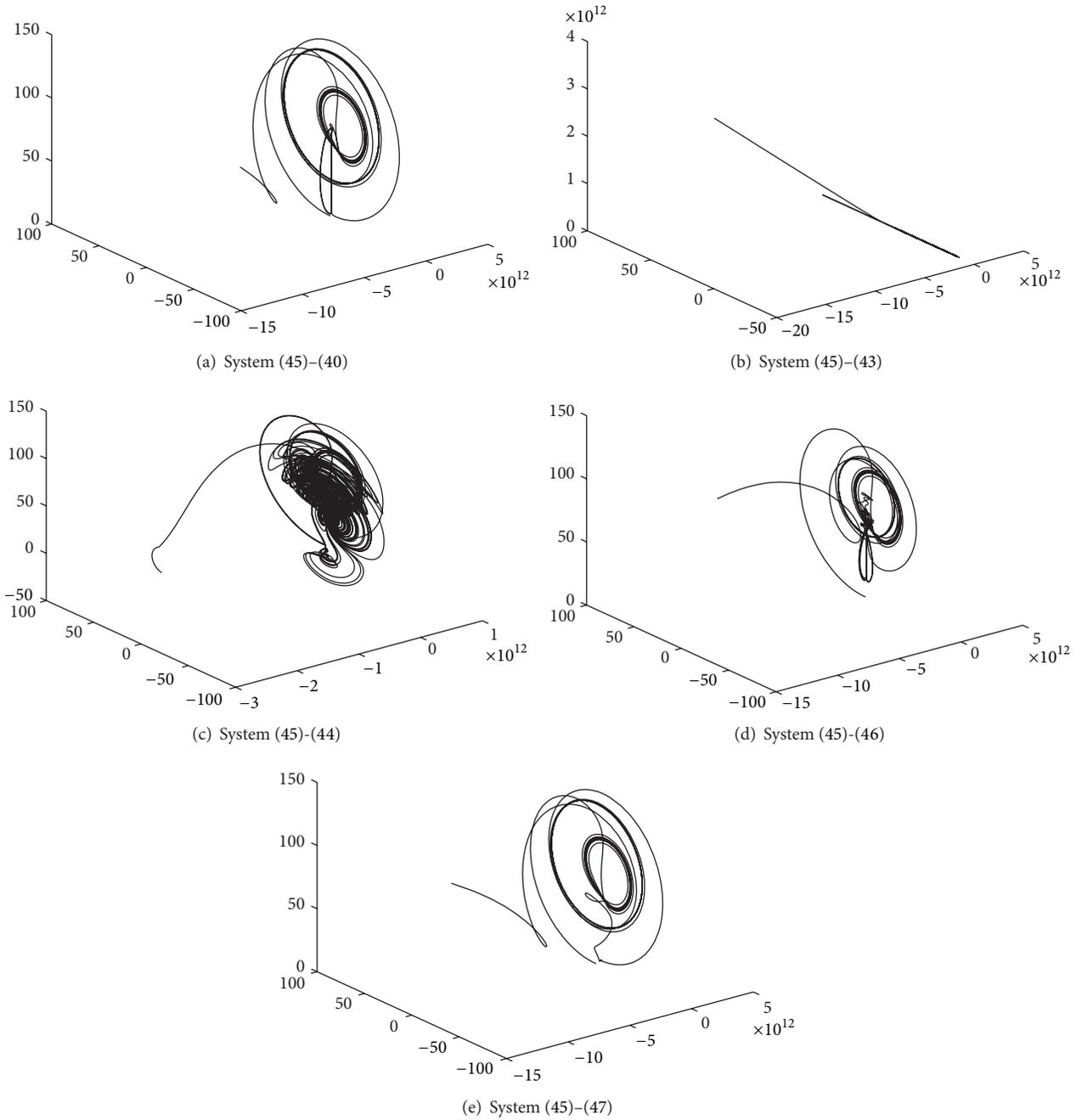\dot{x}_3 &= -3x_2 - x_3 + 4x_1^2 + x_1x_2.
\end{aligned}
\tag{46}
$$

(a) System (45)–(40)

(b) System (45)–(43)

(c) System (45)-(44)

(d) System (45)-(46)

(e) System (45)–(47)

FIGURE 10: Switched system between Example (45) and others.

Example 17. The chaotic system shown in Figure 6 is

$$\dot{x} = \left(\frac{20}{7}\right)x - yz + 9,$$

$$\dot{y} = -10y + xz + 0.5z^2, \tag{47}$$

$$\dot{z} = -4z + xy + yz.$$

Note. When we analyse Examples 13 to 17 by the previous means, for $\sup_{X \in R^3} f(\mu, X) = +\infty$, the globally exponentially

attractive set and positive invariant set of them have not been obtained. The globally exponentially attractive set and positive invariant set really exist by their trajectories. Particularly, by Lü et al. chaotic system [11] and Example 17 we conjecture that they have globally conditional exponentially attractive set and positive invariant set, according to preliminary study. These are waiting for us to do further research. Meanwhile, we can compute that the maximum Lyapunov exponents of Examples 12–17 are 1.06, 0.02, 1.84, 0.01, 0.92, and 0.95, respectively.
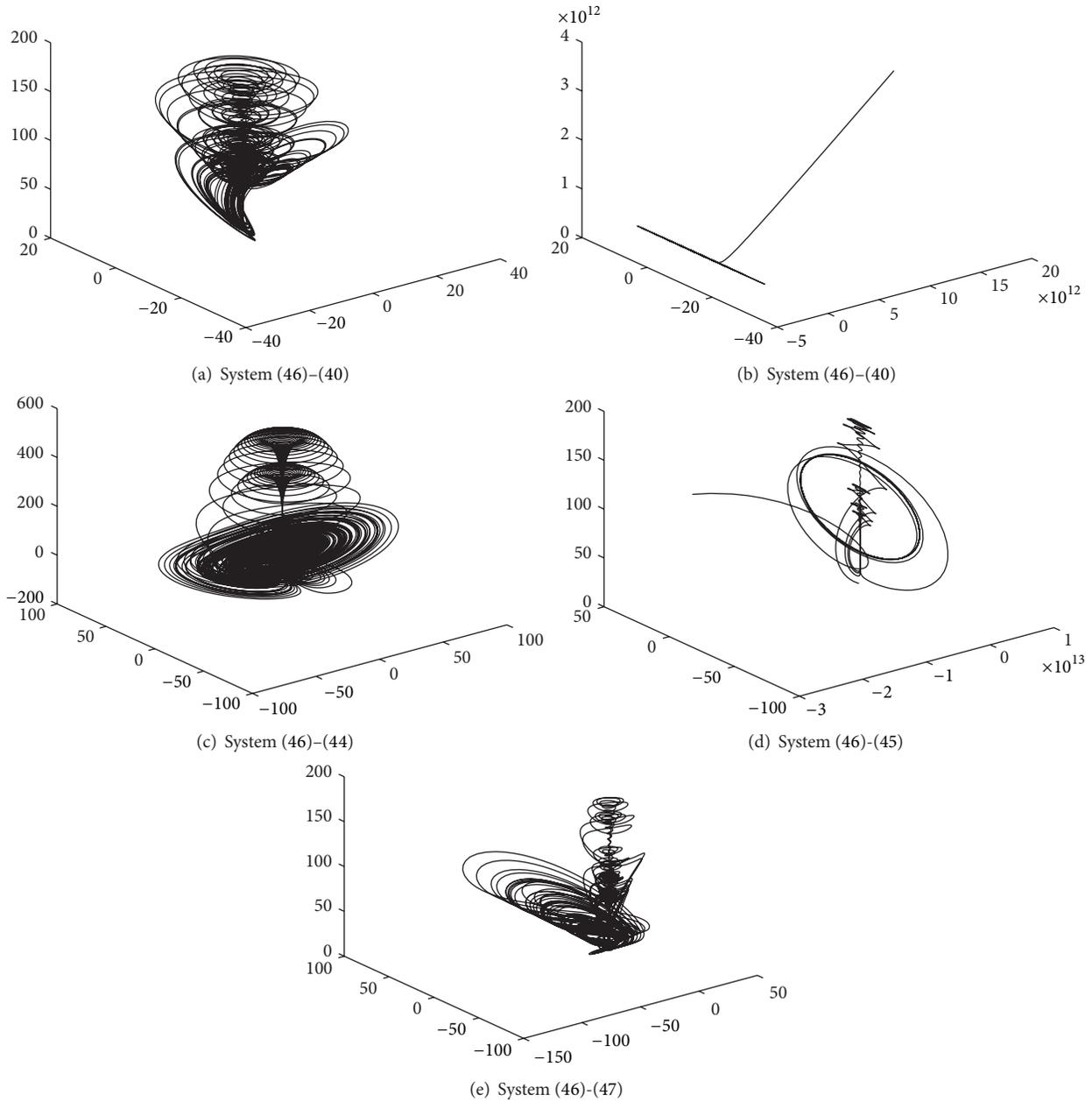
(a) System (46)–(40)

(b) System (46)–(40)

(c) System (46)–(44)

(d) System (46)-(45)

(e) System (46)-(47)

Figure 11: Switched system between Example (46) and others.

5. Simulation of Switched System

In this section, we will show some simulation results of the following switching system

$$\dot{x} = A_\sigma x + f_\sigma(x) + C_\sigma, \tag{48}$$

where $x^T = (x_1, x_2, x_3)$, σ is the switching law, and

$$A_\sigma = \begin{bmatrix} a_{11}^\sigma & a_{12}^\sigma & a_{13}^\sigma \\ a_{21}^\sigma & a_{22}^\sigma & a_{23}^\sigma \\ a_{31}^\sigma & a_{32}^\sigma & a_{33}^\sigma \end{bmatrix}, \qquad C_\sigma = \begin{bmatrix} c_1^\sigma \\ c_2^\sigma \\ c_3^\sigma \end{bmatrix},$$

$$f_\sigma(x) = \begin{bmatrix} x^T B_1^\sigma x \\ x^T B_2^\sigma x \\ x^T B_3^\sigma x \end{bmatrix} \tag{49}$$

with $a_{ij}^\sigma, b_{ijk}^\sigma, c_i^\sigma \in R$, $i, j, k = 1, 2, 3$. Each pair of $(A_\sigma, C_\sigma, B_1^\sigma, B_2^\sigma, B_3^\sigma)$ takes the form from Example 8 to Example 14. The switching law is that the system will stay in each subsystem for a constant time. In the following, we assume that (a, b) denotes a switched system which switches between system (a) and system (b). It can be seen from Figures 7 to 12 that the switched systems (18,20), (18,22), (18,23), (20,18), (20,22), (20,23), (22,18), (22,20), (22,23), (23,18), (23,20), and (23,22) can also yield chaotic systems.

6. Conclusion

In this paper, the methods in [19–21] have been extended to study the globally exponentially or globally conditional

(a) System (47)–(40)

(b) System (47)–(43)

(c) System (47)–(44)

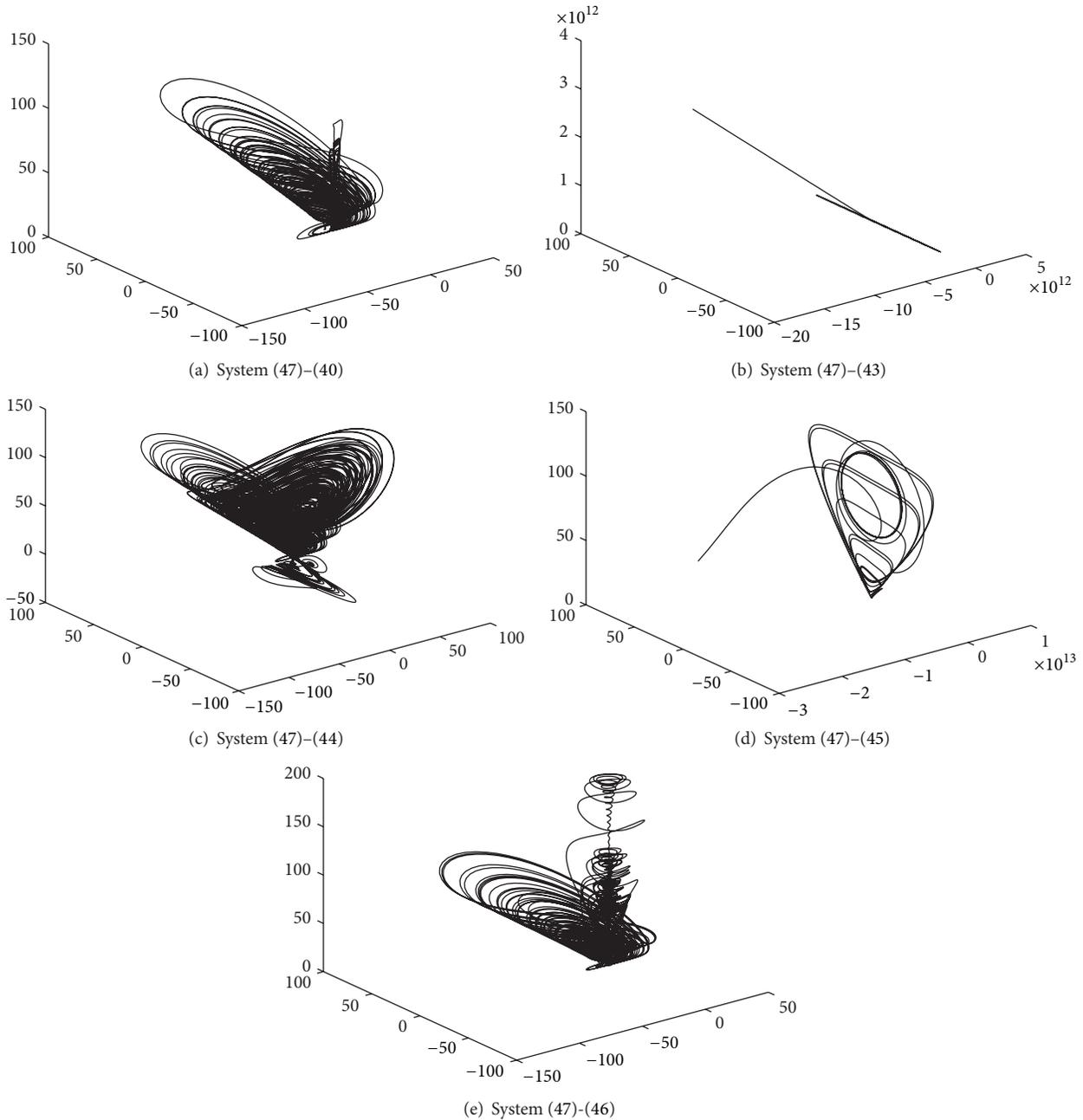(d) System (47)–(45)

(e) System (47)-(46)

FIGURE 12: Switched system between Example (47) and others.

exponentially attractive set and positive invariant set of the three-dimensional chaotic system family with cross-product nonlinearities. We have given two theorems for studying this question and given some examples to show that such system indeed has the globally exponentially or globally conditional exponentially attractive set and positive invariant set, and the exponential estimation is explicitly derived. We have also suggested an idea to construct the chaotic systems, and some new chaotic systems have been illustrated. The simulation results are given for switched system between these new chaotic systems. It is very interesting to further research that the Hesse matrix of the $f(\mu, X)$ is not a negative definite matrix, and the dynamic system (1) is a second class three-dimensional chaotic system with cross-product nonlinearities.

Acknowledgments

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