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Research Article

Dual Quaternion Functions and Its Applications

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A dual quaternion is associated with two quaternions that have basis elements e_0, e_1, e_2, e_3, and ε. Dual numbers are often written in the form $z = \zeta + \varepsilon\zeta^*$, where ε is the dual identity and has the properties $\varepsilon^2 = 0$ $(\varepsilon \neq 0)$. We research the properties of some regular functions with values in dual quaternion and give applications of the extension problem for dual quaternion functions.

1. Introduction

Let $\mathcal{T}$ be the quaternion algebra constructed over a real anti-Euclidean quadratic four-dimensional vector space. Brackx [1], Deavours [2], and Sudbery [3] researched properties of theories of a quaternion function. Naser [4] gave properties of hyperholomorphic functions, and Nôno [5, 6] gave properties of various hyperholomorphic functions. They obtained basic theorems such as Cauchy Theorem, Morera's Theorem, and Cauchy Integral Formula with respect to Clifford analysis. Also, we [7–10] have investigated certain properties of hyperholomorphic functions and some regular functions in Clifford analysis.

A dual quaternion algebra $\mathcal{DH}$ is an ordered pair of quaternions and is constructed from real eight-dimensional vector spaces. A dual quaternion can be represented in the form $z = \zeta + \varepsilon\zeta^*$, where ζ and ζ^* are ordinary quaternions and ε is the dual symbol. The quaternion can represent only rotation, while the dual quaternion can do both rotation and translation. So, the dual quaternion is used in applications to 3D computer graphics, robotics, and computer vision. Kenwright [11] gave characteristics of dual quaternions; Pennestrì and Stefanelli [12] researched some properties using dual, and Kula and Yayli [13] investigated dual split quaternions and screw motion in Minkowski 3-space.

Son [14–16] gave the extension problem for the solutions of partial differential equations in $\mathbf{R}^n$ and it is generalized for the solutions of the Riesz system. In this paper, we give some regular functions with values in dual quaternions and research the extension problem for regular functions with values in dual quaternions. Also, we give some applications for these problems.

2. Preliminaries

We consider associated Pauli matrices

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$
$$e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{1}$$

Then the associated Pauli matrices satisfy the triple rule as follows:

$$e_j^2 = -1, \quad e_j e_k + e_k e_j = -\delta_{jk} \quad (j, k = 1, 2, 3), \tag{2}$$

where δ_{jk} is Kronecker delta. And we let the dual symbol

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3}$$

be a nonzero and satisfy $0\varepsilon = \varepsilon 0 = 0$, $1\varepsilon = \varepsilon 1 = \varepsilon$, $\varepsilon^2 = 0$. The element e_0 is the identity and the element ε is the dual identity of $\mathcal{T}$.

The dual quaternion algebra $\mathcal{DH}$ is a noncommutative and associative one of the quaternion algebra. Then

$$\mathcal{DH} := \left\{ z = \sum_{j=0}^{3} \left(e_j x_j + e_j x_j^* \varepsilon \right) \mid x_j, x_j^* \in \mathbf{R} \quad (j = 0, 1, 2, 3) \right\} = \{ z = \zeta + \zeta^* \varepsilon \mid \zeta, \zeta^* \in \mathcal{T} \} \cong \mathcal{T} \times \mathcal{T}, \tag{4}$$

where $\zeta = \sum_{j=0}^{3} e_j x_j$, $\zeta^* = \sum_{j=0}^{3} e_j x_j^*$, and x_j^* is a dual quaternion component of x_j. We can identify $\mathcal{DH}$ with $\mathbf{C}^4$. The numbers of the skew field $\mathcal{DH}$ of dual quaternions are

$$\begin{aligned} z &= \sum_{j=0}^{3} \left(e_j x_j + e_j x_j^* \varepsilon \right) = \sum_{j=0}^{3} e_j \xi_j = \begin{pmatrix} \xi_0 + i\xi_1 & \xi_2 + i\xi_3 \\ -\xi_2 + i\xi_3 & \xi_0 - i\xi_1 \end{pmatrix}, \\ w &= \sum_{j=0}^{3} \left(e_j y_j + e_j y_j^* \varepsilon \right) = \sum_{j=0}^{3} e_j \eta_j = \begin{pmatrix} \eta_0 + i\eta_1 & \eta_2 + i\eta_3 \\ -\eta_2 + i\eta_3 & \eta_0 - i\eta_1 \end{pmatrix}, \end{aligned} \tag{5}$$

where $\xi_j = x_j + \varepsilon x_j^*$ and $\eta_j = y_j + \varepsilon y_j^*$ $(j = 0, 1, 2, 3)$. The dual quaternion conjugate z^* of z is

$$z^* = \sum_{j=0}^{3} \left(\overline{e_j} x_j + \overline{e_j} x_j^* \varepsilon \right) = \sum_{j=0}^{3} \overline{e_j} \xi_j = \begin{pmatrix} \xi_0 - i\xi_1 & -\xi_2 - i\xi_3 \\ \xi_2 - i\xi_3 & \xi_0 + i\xi_1 \end{pmatrix}, \tag{6}$$

where $\overline{e_j} = -e_j$. The absolute value $|z|$ of z and the inverse z^{-1} of z are, respectively,

$$|z| = \sqrt{zz^*} = \sqrt{\sum_{j=0}^{3} \xi_j^2}, \qquad z^{-1} = \frac{z^*}{|z|^2} \quad (z \neq 0). \tag{7}$$

Let Ω be an open subset of $\mathbf{C}^2 \times \mathbf{C}^2$ and let the dual quaternion function

$$f : \Omega \longrightarrow \mathcal{DH} \tag{8}$$

satisfy

$$z \in \Omega \longmapsto f(z) = \sum_{j=0}^{3} e_j f_j(\zeta, \zeta^*) = \begin{pmatrix} f_0(\zeta, \zeta^*) + i f_1(\zeta, \zeta^*) & f_2(\zeta, \zeta^*) + i f_3(\zeta, \zeta^*) \\ -f_2(\zeta, \zeta^*) + i f_3(\zeta, \zeta^*) & f_0(\zeta, \zeta^*) - i f_1(\zeta, \zeta^*) \end{pmatrix} \in \mathcal{DH}, \tag{9}$$

where $f_j(\zeta, \zeta^*) = u_j(\zeta, \zeta^*) + \varepsilon u_j^*(\zeta, \zeta^*)$ and u_j, u_j^* $(j = 0, 1, 2, 3)$ are real-valued functions.

We use the following dual quaternion differential operators in $\mathcal{DH}$:

$$D = \sum_{j=0}^{3} \overline{e_j} \frac{\partial}{\partial q_j} = \begin{pmatrix} \frac{\partial}{\partial q_0} - i\frac{\partial}{\partial q_1} & -\frac{\partial}{\partial q_2} - i\frac{\partial}{\partial q_3} \\ \frac{\partial}{\partial q_2} - i\frac{\partial}{\partial q_3} & \frac{\partial}{\partial q_0} + i\frac{\partial}{\partial q_1} \end{pmatrix}, \tag{10}$$

and the dual quaternion conjugates differential operators

$$D^* = \sum_{j=0}^{3} e_j \frac{\partial}{\partial q_j} = \begin{pmatrix} \frac{\partial}{\partial q_0} + i\frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} + i\frac{\partial}{\partial q_3} \\ -\frac{\partial}{\partial q_2} + i\frac{\partial}{\partial q_3} & \frac{\partial}{\partial q_0} - i\frac{\partial}{\partial q_1} \end{pmatrix}, \tag{11}$$

where $\partial/\partial q_j = \partial/\partial x_j + \varepsilon(\partial/\partial x_j^*)$ $(j = 0, 1, 2, 3)$ and $q_j = x_j + (1/\varepsilon)x_j^*$. Then we have

$$DD^* = \sum_{j=0}^{3} \frac{\partial^2}{\partial q_j^2} = \Delta_q. \tag{12}$$

Definition 1. Let Ω be an open set in $\mathbf{C}^2 \times \mathbf{C}^2$. A function $f(z)$ is said to be ε-regular in Ω if the following two conditions are satisfied:

(a) f_j $(j = 0, 1, 2, 3)$ are continuously differential functions in Ω,

(b) $D^* f(z) = 0$ in Ω.

Definition 2. Let Ω be an open set in $\mathbf{C}^2\times\mathbf{C}^2$. A function $f(z)$ is said to be ε-biregular in Ω if the following two conditions are satisfied:

(a) f_j $(j = 0,1,2,3)$ are continuously differential functions in Ω,

(b) $D^* f(z) = 0$ and $f(z)D^* = 0$ in Ω.

The operators act for a function $f(z)$ on $\mathscr{DH}$ as follows:

$$D^* f(z) = \left(\sum_{j=0}^{3} e_j \frac{\partial}{\partial q_j}\right)\left(\sum_{j=0}^{3} e_j f_j\right) = \begin{pmatrix} D_1^* & D_2^* \\ D_3^* & D_4^* \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned}
D_1^* &= \left(\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \frac{\partial f_2}{\partial q_2} - \frac{\partial f_3}{\partial q_3}\right) + i\left(\frac{\partial f_1}{\partial q_0} + \frac{\partial f_0}{\partial q_1} + \frac{\partial f_3}{\partial q_2} - \frac{\partial f_2}{\partial q_3}\right),\\
D_2^* &= \left(\frac{\partial f_2}{\partial q_0} - \frac{\partial f_3}{\partial q_1} + \frac{\partial f_0}{\partial q_2} + \frac{\partial f_1}{\partial q_3}\right) + i\left(\frac{\partial f_3}{\partial q_0} + \frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} + \frac{\partial f_0}{\partial q_3}\right),\\
D_3^* &= \left(-\frac{\partial f_2}{\partial q_0} + \frac{\partial f_3}{\partial q_1} - \frac{\partial f_0}{\partial q_2} - \frac{\partial f_1}{\partial q_3}\right) + i\left(\frac{\partial f_3}{\partial q_0} + \frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} + \frac{\partial f_0}{\partial q_3}\right),\\
D_4^* &= \left(\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \frac{\partial f_2}{\partial q_2} - \frac{\partial f_3}{\partial q_3}\right) + i\left(-\frac{\partial f_1}{\partial q_0} - \frac{\partial f_0}{\partial q_1} - \frac{\partial f_3}{\partial q_2} + \frac{\partial f_2}{\partial q_3}\right),\\
f(z)D^* &= \left(\sum_{j=0}^{3} e_j f_j\right)\left(\sum_{j=0}^{3} e_j \frac{\partial}{\partial q_j}\right) = \begin{pmatrix} D_5^* & D_6^* \\ D_7^* & D_8^* \end{pmatrix},
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
D_5^* &= \left(\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \frac{\partial f_2}{\partial q_2} - \frac{\partial f_3}{\partial q_3}\right) + i\left(\frac{\partial f_1}{\partial q_0} + \frac{\partial f_0}{\partial q_1} - \frac{\partial f_3}{\partial q_2} + \frac{\partial f_2}{\partial q_3}\right),\\
D_6^* &= \left(\frac{\partial f_2}{\partial q_0} + \frac{\partial f_3}{\partial q_1} + \frac{\partial f_0}{\partial q_2} - \frac{\partial f_1}{\partial q_3}\right) + i\left(\frac{\partial f_3}{\partial q_0} - \frac{\partial f_2}{\partial q_1} + \frac{\partial f_1}{\partial q_2} + \frac{\partial f_0}{\partial q_3}\right),\\
D_7^* &= \left(-\frac{\partial f_2}{\partial q_0} - \frac{\partial f_3}{\partial q_1} - \frac{\partial f_0}{\partial q_2} + \frac{\partial f_1}{\partial q_3}\right) + i\left(\frac{\partial f_3}{\partial q_0} - \frac{\partial f_2}{\partial q_1} + \frac{\partial f_1}{\partial q_2} + \frac{\partial f_0}{\partial q_3}\right),\\
D_8^* &= \left(\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \frac{\partial f_2}{\partial q_2} - \frac{\partial f_3}{\partial q_3}\right) + i\left(-\frac{\partial f_1}{\partial q_0} - \frac{\partial f_0}{\partial q_1} + \frac{\partial f_3}{\partial q_2} - \frac{\partial f_2}{\partial q_3}\right).
\end{aligned} \tag{15}$$

Remark 3. Equations (b) of Definition 2 are equivalent to the following system:

$$\begin{gathered}
\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \frac{\partial f_2}{\partial q_2} - \frac{\partial f_3}{\partial q_3} = 0,\\
\frac{\partial f_1}{\partial q_0} = -\frac{\partial f_0}{\partial q_1}, \qquad \frac{\partial f_3}{\partial q_2} = \frac{\partial f_2}{\partial q_3},\\
\frac{\partial f_2}{\partial q_0} = -\frac{\partial f_0}{\partial q_2}, \qquad \frac{\partial f_3}{\partial q_1} = \frac{\partial f_1}{\partial q_3},\\
\frac{\partial f_3}{\partial q_0} = -\frac{\partial f_0}{\partial q_3}, \qquad \frac{\partial f_2}{\partial q_1} = \frac{\partial f_1}{\partial q_2}.
\end{gathered} \tag{16}$$

3. Extension Problem for the Dual Quaternion Functions

Definition 4. Let Ω be a domain in $\mathbf{C}^n \times \mathbf{C}^n$ $(n \geq 1)$. A function $f(z) = \sum_{j=0}^{n-1} e_j f_j(z)$ is said to be regular in Ω if

$$\widetilde{D} f(z) = 0, \tag{17}$$

where $\widetilde{D} = \sum_{j=0}^{n-1} e_j(\partial/\partial q_j)$ on Ω.

Theorem 5 (uniqueness theorem for regular functions). *If two regular functions f and g in a domain $\Omega \subset \mathbf{C}^n \times \mathbf{C}^n$ $(n \geq 1)$ and coincide on a nonempty open set $G \subset \Omega$, then $f \equiv g$ in Ω.*

Rocha-Chávez et al. [17] obtained the following remark.

Remark 6. For a regular function f in the domain $\Omega \subset \mathbf{C}^n \times \mathbf{C}^n$ $(n \geq 1)$ and a bounded domain G with smooth boundary bG, such that $\overline{G} \subset \Omega$, one has

$$f(z) = \frac{1}{a_n}\int_{bG} \frac{\overline{\zeta} - \overline{z}}{|\zeta - z|^n} d\sigma_\zeta f(\zeta), \quad z \in G, \tag{18}$$

with a_n the area of the unit sphere in $\mathbf{C}^n$ and $d\sigma_\zeta$ a Clifford algebra valued differential form of order $n-1$.

Let $\Omega = \Omega_1 \times \Omega_2$ be a domain in $\mathbf{C}^4 \times \mathbf{C}^{n-4}$ $(n \geq 5)$ where Ω_1 is a domain in $\mathbf{C}^4(\xi_0,\xi_1,\xi_2,\xi_3)$ and Ω_2 is a domain in $\mathbf{C}^{n-4}(\xi_4,\xi_5,\ldots,\xi_{n-1})$. Let U be an open connected neighborhood of $b\Omega$.

Proposition 7. *If $f(z)$ is a regular function in $U \subset \mathbf{C}^4 \times \mathbf{C}^{n-4}$ $(n \geq 5)$ which satisfies the condition*

$$D^* f(z) = 0, \tag{19}$$

then $f(z)$ can be extended continuously to a regular function in the whole domain of Ω. That is, there exists a regular function $\tilde{f}(z)$ in Ω such that $\tilde{f}(z) = f(z)$ in U.

Proof. By Remark 6 and the proof of the main extension theorem of Son [15], it is proved. □

We consider the system of an extension of the system (16)

$$\begin{gathered} \frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \cdots - \frac{\partial f_{n-1}}{\partial q_{n-1}} = 0, \\ \frac{\partial f_j}{\partial q_0} = -\frac{\partial f_0}{\partial q_j}, \qquad \frac{\partial f_l}{\partial q_k} = \frac{\partial f_k}{\partial q_l} \\ (j,k,l = 1,\dots,n-1), \end{gathered} \tag{20}$$

where $f(z) = \{f_0(z), f_1(z), \dots, f_{n-1}(z)\}$ are the unknown functions.

By using the same technique as in Son [15], we have the following theorem.

Theorem 8. *Let $f(z) = \{f_0(z), f_1(z), \dots, f_{n-1}(z)\}$ be a given $\mathscr{C}^2$-solution of the system (20) in $U \subset \mathbf{C}^4 \times \mathbf{C}^{n-4}$ $(n \geq 5)$, which satisfies the system (16) in Remark 3. If the functions $f_4(z), f_5(z), \dots, f_{n-1}(z)$ depend only on ξ_0, ξ_1, ξ_2, ξ_3, and U is an open neighborhood of the boundary of the domain $\Omega \subset \mathbf{C}^4 \times \mathbf{C}^{n-4}$ $(n \geq 5)$, then $f(z)$ can be extended to a solution of the system (20) in the whole domain of Ω.*

Proof. Let the function $\phi(z)$ with values in Clifford algebra be defined by

$$\phi(z) = \sum_{j=0}^{n-1} f_j e_j. \tag{21}$$

Then we have $\widetilde{D}\phi(z) = 0$ and $D^*\phi(z) = 0$. By Proposition 7, the result follows. □

We consider the following system:

$$\sum_{j=0}^{n-1}\sum_{k=0}^{m-1} H_{jk}^{(P)}(z) \frac{\partial f_j}{\partial q_k} = 0 \quad (P = 1,\dots,p), \tag{22}$$

where $H_{jk}^{(P)}(z)$ are holomorphic functions and $f = \{f_0, f_1, \dots, f_{n-1}\}$ are the unknown functions of the system (22).

Let Ω be an open set in $\mathbf{C}^n$ $(n \geq 2)$ and let K be a compact subset of Ω such that $\Omega \setminus K$ is simply connected. We consider the system

$$\sum_{j=0}^{n-1}\sum_{k=0}^{m-1} H_{jk}^{(P)}(z) \frac{\partial \omega_j}{\partial q_k} = \varphi_{(z)}^{(P)} \quad (P = 1,\dots,p), \tag{23}$$

where $\varphi_{(z)}^{(P)} \in \mathscr{C}^\infty(\Omega)$.

By using the same technique as in Son [16], we have the following theorem and example.

Theorem 9. *If every $\varphi = \{\varphi^{(1)}, \dots, \varphi^{(p)}\} \in \mathscr{C}^\infty(\Omega)$ in the inhomogeneous system (23) has a solution*

$$\omega = \{\omega_0, \dots, \omega_{n-1}\} \in \mathscr{C}^\infty(\Omega), \tag{24}$$

then every solution f of (22) given in $\Omega \setminus K$ can be extended to a solution of this system (23) in the whole domain of Ω.

Proof. This result follows from the theorem in [18, page 30]. □

Example 10. We give an application of Theorem 9 to the system (20) and recall the system (16) as follows:

$$\begin{gathered} \frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \cdots - \frac{\partial f_{n-1}}{\partial q_{n-1}} = 0, \\ \frac{\partial f_j}{\partial q_0} + \frac{\partial f_0}{\partial q_j} = 0, \qquad \frac{\partial f_l}{\partial q_k} - \frac{\partial f_k}{\partial q_l} = 0 \\ (j,k,l = 1,\dots,n-1). \end{gathered} \tag{25}$$

Assume that

$$\frac{\partial f_{n-1}}{\partial q_{n-1}} = 0. \tag{26}$$

Then we have the following form:

$$\begin{gathered} \frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \cdots - \frac{\partial f_{n-2}}{\partial q_{n-2}} = 0, \\ \frac{\partial f_j}{\partial q_0} + \frac{\partial f_0}{\partial q_j} = 0, \qquad \frac{\partial f_l}{\partial q_k} - \frac{\partial f_k}{\partial q_l} = 0 \\ (j,k,l = 1,\dots,n-2). \end{gathered} \tag{27}$$

The corresponding inhomogeneous system of (27) has the following form:

$$\begin{gathered} \frac{\partial \omega_0}{\partial q_0} - \frac{\partial \omega_1}{\partial q_1} - \cdots - \frac{\partial \omega_{n-2}}{\partial q_{n-2}} = \varphi, \\ \frac{\partial \omega_j}{\partial q_0} + \frac{\partial \omega_0}{\partial q_j} = \varphi_{j,0}, \\ \frac{\partial \omega_j}{\partial q_k} - \frac{\partial \omega_k}{\partial q_j} = \varphi_{j,k}, \\ \frac{\partial \omega_k}{\partial q_0} + \frac{\partial \omega_0}{\partial q_k} = \varphi_{k,0}, \\ \frac{\partial \omega_j}{\partial q_l} - \frac{\partial \omega_l}{\partial q_j} = \varphi_{j,l}, \\ \frac{\partial \omega_k}{\partial q_l} - \frac{\partial \omega_l}{\partial q_k} = \varphi_{k,l} \\ (j,k,l = 1,\dots,n-2), \end{gathered} \tag{28}$$

where $\varphi, \varphi_{j,0}, \varphi_{j,k}, \varphi_{k,0}, \varphi_{j,l}, \varphi_{k,l} \in \mathscr{C}^{\infty}(\Omega)$. Then we can get the system from (28) as

$$\frac{\partial \omega_k}{\partial q_j} = \frac{\partial \omega_j}{\partial q_k} - \varphi_{j,k}, \qquad \frac{\partial \omega_k}{\partial q_0} = -\frac{\partial \omega_0}{\partial q_k} + \varphi_{k,0},$$
$$\frac{\partial \omega_l}{\partial q_j} = \frac{\partial \omega_j}{\partial q_l} - \varphi_{j,l}, \qquad \frac{\partial \omega_l}{\partial q_k} = \frac{\partial \omega_k}{\partial q_l} - \varphi_{k,l}, \tag{29}$$

$$\frac{\partial^2 \omega_k}{\partial q_j \partial q_0} + \frac{\partial \varphi_{j,k}}{\partial q_0} = -\frac{\partial^2 \omega_0}{\partial q_j \partial q_k} + \frac{\partial \varphi_{j,0}}{\partial q_k},$$
$$\frac{\partial^2 \omega_j}{\partial q_l \partial q_k} - \frac{\partial \varphi_{j,l}}{\partial q_k} = \frac{\partial^2 \omega_k}{\partial q_j \partial q_l} - \frac{\partial \varphi_{k,l}}{\partial q_j}. \tag{30}$$

From (29), we have

$$\frac{\partial}{\partial q_j}\left(-\frac{\partial \omega_0}{\partial q_k} + \varphi_{k,0}\right) + \frac{\partial \varphi_{j,k}}{\partial q_0} = -\frac{\partial^2 \omega_0}{\partial q_j \partial q_k} + \frac{\partial \varphi_{j,0}}{\partial q_k},$$
$$\frac{\partial}{\partial q_l}\left(\frac{\partial \omega_k}{\partial q_j} + \varphi_{j,k}\right) - \frac{\partial \varphi_{j,l}}{\partial q_k} = \frac{\partial^2 \omega_k}{\partial q_j \partial q_l} - \frac{\partial \varphi_{k,l}}{\partial q_j}. \tag{31}$$

Thus, we can have the system

$$\frac{\partial \varphi_{k,0}}{\partial q_j} + \frac{\partial \varphi_{j,k}}{\partial q_0} = \frac{\partial \varphi_{j,0}}{\partial q_k}, \qquad \frac{\partial \varphi_{j,k}}{\partial q_l} - \frac{\partial \varphi_{j,l}}{\partial q_k} = -\frac{\partial \varphi_{k,l}}{\partial q_j}. \tag{32}$$

We let

$$\widetilde{\omega} = \sum_{j=0}^{n-2} e_j \omega_j, \qquad \widetilde{z} = \sum_{j=0}^{n-2} \overline{e_j} \xi_j. \tag{33}$$

The system (29) has the form

$$\widetilde{D}\widetilde{\omega} = F(\widetilde{z}, \xi_{n-1}), \qquad \frac{\partial \omega_0}{\partial q_k} + \frac{\partial \omega_k}{\partial q_0} = \varphi_{k,0}, \tag{34}$$

$$\frac{\partial \omega_j}{\partial q_l} - \frac{\partial \omega_l}{\partial q_j} = \varphi_{j,l}, \qquad \frac{\partial \omega_k}{\partial q_l} - \frac{\partial \omega_l}{\partial q_k} = \varphi_{k,l}, \tag{35}$$

where

$$F(\widetilde{z}, \xi_{n-1}) = \varphi + \sum_{j=1}^{n-2} e_j \varphi_{j,0} + \left(\frac{n}{2} - 1\right) \sum_{\alpha=1}^{n-2} \sum_{\substack{j,k=1 \\ j,k \neq \alpha}}^{n-2} e_\alpha \varphi_{j,k}. \tag{36}$$

We put

$$F_1 := \varphi + \sum_{j=1}^{n-2} e_j \varphi_{j,0},$$
$$F_2 := \left(\frac{n}{2} - 1\right) \sum_{\alpha=1}^{n-2} \sum_{\substack{j,k=1 \\ j,k \neq \alpha}}^{n-2} e_\alpha \varphi_{j,k}. \tag{37}$$

From the system (28), we have

$$\omega_0 = \int \left(\varphi + \frac{\partial \omega_1}{\partial q_1} + \frac{\partial \omega_2}{\partial q_2} + \cdots + \frac{\partial \omega_{n-2}}{\partial q_{n-2}}\right) dq_0,$$
$$\omega_k = \int \left(\varphi_{k,0} - \frac{\partial \omega_0}{\partial q_k}\right) dq_0 \quad (k = 1, \ldots, n-2). \tag{38}$$

By the systems (29) and (38), we get

$$\begin{aligned}\widetilde{\omega} &= \int F\, dq_0 + \int \sum_{j=1}^{n-2} \frac{\partial \omega_j}{\partial q_j}\, dq_0 \\ &\quad - \int \sum_{j=1}^{n-2} e_j \frac{\partial \omega_0}{\partial q_j}\, dq_0 - \int F_2 dq_0 \\ &= \int F\, dq_0 - \int \sum_{j=1}^{n-2} e_j \frac{\partial \widetilde{\omega}}{\partial q_j}\, dq_0.\end{aligned} \tag{39}$$

By Cauchy Integral Formula,

$$F(Z, \xi_{n-1}) = \frac{1}{(2\pi i)^n} \int \frac{F(Z, \xi_{n-1})}{Z - \zeta} dZ, \tag{40}$$

where $Z = \sum_{k=0}^{n-2} e_k Z_k$. Thus we have

$$\widetilde{\omega} = \frac{1}{(2\pi i)^n} \iint \frac{F(Z, \xi_{n-1})}{Z - \zeta} dZ_0 dZ, \tag{41}$$

and $\widetilde{\omega} = 0$ when $|\xi_{n-1}|$ is large enough. Also, $\{\widetilde{\omega}, 0\} = \{\omega_0, \omega_1, \ldots, \omega_{n-2}, 0\}$ is a solution of the system (22) outside a compact set K of Ω. From Theorem 5, it follows that $\widetilde{\omega} = 0$ is outside the compact set K of Ω or $\omega \in \mathscr{C}^{\infty}(\Omega)$. That is, $\omega_0, \omega_1, \ldots, \omega_{n-2} \in \mathscr{C}^{\infty}(\Omega)$. It follows from the system (35) that

$$\omega_{n-1} = \int \left(\frac{\partial \omega_{m-1}}{\partial q_{n-1}} - \varphi_{m-1,n-1}\right) dq_{m-1}. \tag{42}$$

Since $\omega_{m-1} \in \mathscr{C}^{\infty}(\Omega)$, we get $\partial \omega_{m-1}/\partial q_{n-1} \in \mathscr{C}^{\infty}(\Omega)$. We can choose $\omega_{n-1} \in \mathscr{C}^{\infty}(\Omega)$ which satisfy the system (35). From (42), we find that

$$\begin{aligned}\frac{\partial \omega_{n-1}}{\partial q_P} &= \int \left(\frac{\partial^2 \omega_{m-1}}{\partial q_P \partial q_{n-1}} - \frac{\partial \varphi_{m-1,n-1}}{\partial q_P}\right) dq_{m-1} \\ &= \int \left(\frac{\partial}{\partial q_{n-1}}\left(\frac{\partial \omega_P}{\partial q_{m-1}}\right) + \frac{\partial \varphi_{m-1,P}}{\partial q_{n-1}} - \frac{\partial \varphi_{m-1,n-1}}{\partial q_P}\right) dq_{m-1} \\ &= \int \left(\frac{\partial}{\partial q_{n-1}}\left(\frac{\partial \omega_P}{\partial q_{m-1}}\right) - \frac{\partial \varphi_{P,n-1}}{\partial q_{m-1}}\right) dq_{m-1} \\ &= \frac{\partial \omega_P}{\partial q_{n-1}} - \varphi_{P,n-1}.\end{aligned} \tag{43}$$

Hence, ω_{n-1} satisfies the system (29). This means that the function $\omega = (\widetilde{\omega}, \omega_{n-1}) = (\omega_0, \ldots, \omega_{n-1})$ is a solution of the system (29) and $\omega \in \mathscr{C}^{\infty}(\Omega)$.

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