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Research Article

C^*-Algebras from Groupoids on Self-Similar Groups

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We show that the Smale spaces from self-similar groups are topologically mixing and their stable algebra and stable Ruelle algebra are strongly Morita equivalent to groupoid algebras of Anantharaman-Delaroche and Deaconu. And we show that $C^*(R_\infty)$ associated to a postcritically finite hyperbolic rational function is an AT-algebra of real-rank zero with a unique trace state.

1. Introduction

Nekrashevych has developed a theory of dynamical systems and C^*-algebras for self-similar groups in [1, 2]. These groups include groups acting on rooted trees and finite automata and iterated monodromy groups of self-covering on topological spaces. From self-similar groups, Nekrashevych constructed Smale spaces of Ruelle and Putnam with their corresponding stable and unstable algebras and those of Ruelle algebras for various equivalence relations on the Smale spaces [3–7].

Main approach to C^*-algebras structures in [2] is based on Cuntz-Pimsner algebras generated by self-similar groups. However Smale spaces and their corresponding C^*-algebras have rich dynamical structures, and it is conceivable that dynamical systems associated with self-similar groups may give another way to study C^*-algebras from self-similar groups. Our intention is to elucidate self-similar groups from the perspective of dynamical systems.

This paper is concerned with groupoids and their groupoid C^*-algebras from the stable equivalence relation on the limit solenoid $(S_G, \overline{\sigma})$ of a self-similar group (G, X). Instead of using the groupoids G_s and $G_s \rtimes \mathbb{Z}$ on the Smale space $(S_G, \overline{\sigma})$ as Putnam [3, 4] and Nekrashevych [2] did, we consider the essentially principal groupoids R_∞ and $\Gamma(J_G, \sigma)$ of Anantharaman-Delaroche [8] and Deaconu [9] on a presentation (J_G, σ) of $(S_G, \overline{\sigma})$. While G_s and $G_s \rtimes \mathbb{Z}$ are not r-discrete groupoids, R_∞ and $\Gamma(J_G, \sigma)$ are r-discrete. And R_∞ and $\Gamma(J_G, \sigma)$ are defined on (J_G, σ) so that we do not need to entail the inverse limit structure of $(S_G, \overline{\sigma})$. Thus R_∞ and $\Gamma(J_G, \sigma)$ are more manageable than G_s and $G_s \rtimes \mathbb{Z}$ for the structures of their C^*-algebras.

In this paper, we prove that, for a self-similar group (G, X), its limit dynamical system (J_G, σ) is topologically mixing so that $(S_G, \overline{\sigma})$ is an irreducible Smale space. And we show that R_∞ is equivalent to G_s and $\Gamma(J_G, s)$ is equivalent to $G_s \rtimes \mathbb{Z}$ in the sense of Muhly et al. [10]. Consequently, the groupoid C^*-algebras $C^*(R_\infty)$ and $C^*(\Gamma(J_G, \sigma))$ are strongly Morita equivalent to the stable algebra S and the stable Ruelle algebra R_s, respectively, of $(S_G, \overline{\sigma})$. Then we use R_∞ and $\Gamma(J_G, \sigma)$ to study structures of C^*-algebras from a self-similar group (G, X). Finally we show that groupoid algebras of R_∞ from postcritically finite hyperbolic rational functions are AT-algebras of real-rank zero.

The outline of the paper is as follows. In Section 2, we review the notions of self-similar groups and their groupoids and show that the induced limit dynamical system and the limit solenoid of a self-similar group are topologically mixing. In Section 3, we observe that R_∞ is equivalent to G_s and $\Gamma(J_G, s)$ is equivalent to $G_s \rtimes \mathbb{Z}$. In Section 4, we give a proof that its groupoid algebra $C^*(\Gamma(J_G, \sigma))$ is simple, purely infinite, separable, stable, and nuclear and satisfies the Universal Coefficient Theorem. For R_∞, we show that $C^*(R_\infty)$ is simple and nuclear. And, when self-similar group is defined by a postcritically finite hyperbolic rational function and its Julia set, we show that $C^*(R_\infty)$ is an AT-algebra.

2. Self-Similar Groups

We review the properties of self-similar groups. As for general references for the notions of self-similar groups, we refer to [1, 2].

Suppose that X is a finite set. We denote by X^n the set of words of length n in X with $X^0 = \{\emptyset\}$, and define $X^* = \cup_{n=0}^{\infty} X^n$. A *self-similar group* (G, X) consists of an X and a faithful action of a group G on X such that, for all $g \in G$ and $x \in X$, there exist unique $y \in X$ and $h \in G$ such that

$$g(xw) = yh(w) \quad \text{for every } w \in X^*. \tag{1}$$

The above equality is written formally as

$$g \cdot x = y \cdot h. \tag{2}$$

We observe that for any $g \in G$ and $v \in X^*$, there exists a unique element $h \in G$ such that $g(vw) = g(v)h(w)$ for every $w \in X^*$. The unique element h is called the *restriction* of g at v and is denoted by $g|_v$. For $u = g(v)$ and $h = g|_v$, we write

$$g \cdot v = u \cdot h. \tag{3}$$

A self-similar group (G, X) is called *recurrent* if, for all $x, y \in X$, there is a $g \in G$ such that $g \cdot x = y \cdot 1$; that is, $g(xw) = yw$ for every $w \in X^*$. We say that (G, X) is *contracting* if there is a finite subset N of G satisfying the following: for every $g \in G$, there is $n \geq 0$ such that $g|_v \in N$ for every $v \in X^*$ of length $|v| \geq n$. If the group is contracting, the smallest set N satisfying this condition is called the *nucleus* of the group.

Standing Assumption. We assume that our self-similar group (G, X) is a contracting, recurrent, and regular group and that the group G is finitely generated.

Path Spaces. For a self-similar group (G, X), the set X^* has a natural structure of a rooted tree: the root is $\emptyset$, the vertices are words in X^*, and the edges are of the form v to vx, where $v \in X^*$ and $x \in X$. Then the boundary of the tree X^* is identified with the space X^ω of right-infinite paths of the form $x_1 x_2 \cdots$, where $x_i \in X$. The product topology of discrete set X is given on X^ω.

We say that a self-similar group (G, X) is *regular* if, for every $g \in G$ and every $w \in X^\omega$, either $g(w) \neq w$ or there is a neighborhood of w such that every point in the neighborhood is fixed by g.

We also consider the space $X^{-\omega}$ of left-infinite paths $\cdots x_{-2} x_{-1}$ over X with the product topology. Two paths $\cdots x_{-2} x_{-1}$ and $\cdots y_{-2} y_{-1}$ in $X^{-\omega}$ are said to be *asymptotically equivalent* if there is a finite set $I \subset G$ and a sequence $g_n \in I$ such that

$$g_n (x_{-n} \cdots x_{-1}) = y_{-n} \cdots y_{-1}, \tag{4}$$

for every $n \in \mathbb{N}$. The quotient of the space $X^{-\omega}$ by the asymptotic equivalence relation is called the *limit space* of (G, X) and is denoted by $\mathcal{J}_G$. Since the asymptotic equivalence relation is invariant under the shift map $\cdots x_{-2} x_{-1} \mapsto \cdots x_{-3} x_{-2}$, the shift map induces a continuous map $\sigma : \mathcal{J}_G \to \mathcal{J}_G$. We call the induced dynamical system $(\mathcal{J}_G, \sigma)$ the *limit dynamical system* of (G, X) (see [1, 2] for details).

Remark 1. Recurrent and finitely generated conditions imply that $\mathcal{J}_G$ is a compact, connected, locally connected, metrizable space of a finite dimension by Corollary 2.8.5 and Theorem 3.6.4 of [1]. And regular condition implies that σ is an $|X|$-fold self-covering map by Proposition 6.1 of [2].

A cylinder set $Z(u)$ for each $u \in X^* = \cup_{n\geq 0} X^n$ is defined as follows:

$$\begin{aligned} Z(u) = \{\xi \in X^{-\omega} : \xi = \cdots x_{-n-1} x_{-n} \cdots x_{-1} \\ \text{such that } x_{-n} \cdots x_{-1} = u\}. \end{aligned} \tag{5}$$

Then the collection of all such cylinder sets forms a basis for the product topology on $X^{-\omega}$. And we recall that a dynamical system (Y, f) is called *topologically mixing* if, for every pair of nonempty open sets A, B in Y, there is an $n \in \mathbb{N}$ such that $f^k(A) \cap B \neq \emptyset$ for every $k \geq n$.

Theorem 2. *$(\mathcal{J}_G, \sigma)$ is a topologically mixing system.*

Proof. As $X^{-\omega}$ has the product topology and $\mathcal{J}_G$ has the quotient topology induced from asymptotic equivalence relation, it is sufficient to show that, for arbitrary cylinder sets $Z(u)$ and $Z(v)$ of $X^{-\omega}$, there are infinite paths $\xi = \cdots x_{-2} x_{-1} \in Z(u)$ and $\eta = \cdots y_{-2} y_{-1} \in Z(v)$ such that ξ is asymptotically equivalent to η. Moreover we can assume that $u, v \in X^n$ for some $n \in \mathbb{N}$ so that $u = x_{-n} \cdots x_{-1}$ and $v = y_{-n} \cdots y_{-1}$.

We choose sufficiently large m and let $a, b \in X^{m-n}$ so that au and bv are elements of X^m. Then by recurrent condition and [1, Corollary 2.8.5], for au and bv in X^m, there is a $g \in G$ such that $g(au) = g(a) g|_a(u) = bv$. Since we chose large m, by contracting condition, $g|_a$ is an element of the nucleus of (G, X).

We remind that the nucleus of (G, X) is a finite set and equal to

$$N = \cup_{g\in G} \cap_{n\geq 0} \{g|_v : v \in X^*, |v| \geq n\}. \tag{6}$$

So an element of the nucleus is a restriction of another element of the nucleus. Hence $g|_a \in N$ implies that there exist a letter x_{-n-1} and a $g_{-n-1} \in N$ such that $g_{-n-1}|_{x_{-n-1}} = g|_a$. Then, for $g_{-n-1}(x_{-n-1}) = y_{-n-1}$, we have

$$g_{-n-1} (x_{-n-1} u) = y_{-n-1} v. \tag{7}$$

So by induction there are a letter x_{-m} and a $g_{-m} \in N$ for every $m \geq n$ such that

$$g_{-m} (x_{-m} \cdots x_{-n-1} u) = y_{-m} \cdots y_{-n-1} v. \tag{8}$$

Let $\xi = \cdots x_{-2} x_{-1}$ and let $\eta = \cdots y_{-2} y_{-1}$. Then it is trivial that $\xi \in Z(u)$ and $\eta \in Z(v)$. And ξ is asymptotically equivalent to η. Therefore the limit dynamical system $(\mathcal{J}_G, \sigma)$ is topologically mixing. □

Let $X^{\mathbb{Z}}$ be the space of bi-infinite paths $\cdots x_{-1} x_0 \cdot x_1 x_2 \cdots$ over the alphabet X. The direct product topology of the discrete set X is given on $X^{\mathbb{Z}}$. We say that two paths $\cdots x_{-1} x_0 \cdot x_1 x_2 \cdots$ and $\cdots y_{-1} y_0 \cdots y_1 y_2 \cdots$ in $X^{\mathbb{Z}}$ are *asymptotically* *equivalent* if there is a finite set $I \subset G$ and a sequence $g_n \in I$ such that

$$g_n (x_n x_{n+1} \cdots) = y_n y_{n+1} \ldots, \tag{9}$$

for every $n \in \mathbb{Z}$. The quotient of $X^{\mathbb{Z}}$ by the asymptotic equivalence relation is called the *limit solenoid* of (G, X) and is denoted by S_G. As in the case of J_G, the shift map on $X^{\mathbb{Z}}$ is transferred to an induced homeomorphism on S_G, which we will denote by $\overline{\sigma}$.

Theorem 3 (see [1, 2]). *The limit solenoid S_G is homeomorphic to the inverse limit space of (J_G, σ)*

$$\begin{aligned} J_G \xleftarrow{\sigma} J_G \xleftarrow{\sigma} \cdots = \Big\{ (\xi_0, \xi_1, \xi_2, \ldots) \in \prod_{n \geq 0}^{\infty} J_G : \sigma(\xi_{n+1}) \\ = \xi_n \text{ for every } n \geq 0 \Big\}, \end{aligned} \tag{10}$$

and $\overline{\sigma} : S_G \to S_G$ is the induced homeomorphism defined by

$$\begin{aligned} (\xi_0, \xi_1, \xi_2, \ldots) &\longmapsto (\sigma(\xi_0), \sigma(\xi_1), \sigma(\xi_2), \ldots) \\ &= (\sigma(\xi_0), \xi_0, \xi_1, \ldots). \end{aligned} \tag{11}$$

Moreover, the limit solenoid system $(S_G, \overline{\sigma})$ is a Smale space.

Then we have the following from Theorem 2.

Corollary 4. *$(S_G, \overline{\sigma})$ is topologically mixing.*

We have a natural projection $\pi : S_G \to J_G$ induced from the map

$$\cdots x_{-1} x_0 \cdot x_1 x_2 \cdots \longmapsto \cdots x_{-1} x_0, \tag{12}$$

and the relation that $\cdots x_{n-1} x_n \in X^{-\omega}$ represents $\xi_n \in J_G$. Then it is easy to check $\pi \circ \overline{\sigma} = \sigma \circ \pi$. The *stable equivalence relation* on $(S_G, \overline{\sigma})$ is defined as follows [2, Proposition 6.8]:

Definition 5. One says that two elements α and β in S_G are stably equivalent and write $\alpha \sim_s \beta$ if there is a $k \in \mathbb{Z}$ such that $\pi \overline{\sigma}^k(\alpha) = \pi \overline{\sigma}^k(\beta)$.

In other words, when α and β are represented by infinite paths $(x_n)_{n \in \mathbb{Z}}$ and $(y_n)_{n \in \mathbb{Z}}$ in $X^{\mathbb{Z}}$, $\alpha \sim_s \beta$ if and only if the corresponding left-infinite paths $\cdots x_{k-1} x_k$ and $\cdots y_{k-1} y_k$ in $X^{-\omega}$ are asymptotically equivalent for some $k \in \mathbb{Z}$.

Groupoids on (J_G, σ) and $(S_G, \overline{\sigma})$. Suppose that (G, X) is a self-similar group and $(S_G, \overline{\sigma})$ is its corresponding limit solenoid. We recall from [3] that the stable equivalence groupoid G_s on S_G and its semidirect product by $\mathbb{Z}$ are defined to be

$$\begin{aligned} G_s &= \{ (\alpha, \beta) \in S_G \times S_G : \alpha \sim_s \beta \}, \\ G_s \rtimes \mathbb{Z} &= \{ (\alpha, n, \beta) \in S_G \times \mathbb{Z} \times S_G : n \in \mathbb{Z}, \\ &\qquad (\overline{\sigma}^n(\alpha), \beta) \in G_s \}. \end{aligned} \tag{13}$$

Then G_s and $G_s \rtimes \mathbb{Z}$ are groupoids with the natural structure maps. The unit spaces of G_s and $G_s \rtimes \mathbb{Z}$ are identified with S_G via the maps $\alpha \in S_G \mapsto (\alpha, \alpha) \in G_s$ and $\alpha \mapsto (\alpha, 0, \alpha) \in G_s \rtimes \mathbb{Z}$, respectively.

To give topologies on these groupoids, we consider subgroupoids of G_s. For each $n \geq 0$, set

$$G_{s,n} = \{ (\alpha, \beta) \in S_G \times S_G : \pi \overline{\sigma}^n(\alpha) = \pi \overline{\sigma}^n(\beta) \}. \tag{14}$$

Then $G_{s,n}$ is a subgroupoid of G_s. Note that if μ and ν in S_G are stably equivalent with $\pi \overline{\sigma}^l(\mu) = \pi \overline{\sigma}^l(\nu)$ for some negative integer l, then

$$\pi(\mu) = \sigma^{-l} \pi \overline{\sigma}^l(\mu) = \sigma^{-l} \pi \overline{\sigma}^l(\nu) = \pi(\nu) \tag{15}$$

implies that $(\mu, \nu) \in G_{s,0}$. So we obtain the stable equivalence groupoid

$$G_s = \bigcup_{n \geq 0} G_{s,n}. \tag{16}$$

Each $G_{s,n}$ is given the relative topology from $S_G \times S_G$, and G_s is given the inductive limit topology. Under this topology, it is not difficult to check that G_s is a locally compact Hausdorff principal groupoid with the natural structure maps. For $G_s \rtimes \mathbb{Z}$, we transfer the product topology of $G_s \times \mathbb{Z}$ to $G_s \rtimes \mathbb{Z}$ via the map $((\alpha, \beta), n) \mapsto (\alpha, n, \overline{\sigma}(\beta))$. Amenability and Haar systems on G_s and $G_s \times \mathbb{Z}$ are explained in [2–4]. We denote the groupoid C^*-algebra of G_s by S and that of $G_s \rtimes \mathbb{Z}$ by R_s and call it *stable Ruelle algebra* on $(S_G, \overline{\sigma})$.

For the limit dynamical system $(J_G \cdot \sigma)$ of a self-similar group (G, X), we construct groupoids R_∞ and $\Gamma(J_G, \sigma)$ of Anantharaman-Delaroche [8] and Deaconu [9]. Let $R_n = \{ (\xi, \eta) \in J_G \times J_G : \sigma^n(\xi) = \sigma^n(\eta) \}$ for $n \geq 0$ and define

$$\begin{aligned} R_\infty &= \bigcup_{n \geq 0} R_n, \\ \Gamma(J_G, \sigma) &= \{ (\xi, n, \eta) \in J_G \times \mathbb{Z} \times J_G : \exists k, l \geq 0, \\ &\qquad n = k - l, \sigma^k(\xi) = \sigma^l(\eta) \} \end{aligned} \tag{17}$$

with the natural structure maps. The unit spaces of R_∞ and $\Gamma(J_G, \sigma)$ are identified with J_G via $\xi \mapsto (\xi, \xi)$ and $\xi \mapsto (\xi, 0, \xi)$.

We give the relative topology from $J_G \times J_G$ on R_n and the inductive limit topology on R_∞. Then R_∞ is a second countable, locally compact, Hausdorff, r-discrete groupoid with the Haar system given by the counting measures. A topology on $\Gamma(J_G, \sigma)$ is given by basis of the form

$$\Lambda(U, V, k \cdot l) = \left\{ \left(\xi, k - l, \left(\sigma^l|_V \right)^{-1} \circ \sigma^k(\xi) \right) : \xi \in U \right\}, \tag{18}$$

where U and V are open sets in J_G and $k, l \geq 0$ such that $\sigma^k|_U$ and $\sigma^l|_V$ are homeomorphisms with the same range. Then $\Gamma(J_G, \sigma)$ is a second countable, locally compact, Hausdorff, r-discrete groupoid, and the counting measure is a Haar system [9, 11]. Amenability of R_∞ and $\Gamma(J_G, \sigma)$ is explained in Proposition 2.4 of [12]. We denote the groupoid C^*-algebras of R_∞ and $\Gamma(J_G, \sigma)$ by $C^*(R_\infty)$ and $C^*(\Gamma(J_G, \sigma))$, respectively.

3. Groupoid Equivalence

We follow Kumjian and Pask [13, Section 5] to obtain equivalence of groupoids between G_s and R_∞ and between $G_s \rtimes \mathbb{Z}$ and $\Gamma(J_G, \sigma)$, respectively, in the sense of Muhly et al. [10].

We repeat Kumjian and Pask's observation [13]. Suppose that Y is a locally compact Hausdorff space and that Γ is a locally compact Hausdorff groupoid. For a continuous open surjection $\phi : Y \to \Gamma^0$, we set a topological space

$$Z = Y * \Gamma = \{(y,\gamma) : y \in Y,\ \gamma \in \Gamma,\ \phi(y) = s(\gamma)\} \quad (19)$$

with the relative topology in $Y \times \Gamma$ and a locally compact Hausdorff groupoid

$$\begin{aligned}\Gamma^\phi = \{(y_1,\gamma,y_2) : y_1,\ y_2 \in Y,\ \gamma \in \Gamma,\\ \phi(y_1) = s(\gamma),\ r(\gamma) = \phi(y_2)\}\end{aligned} \quad (20)$$

with the relative topology.

Theorem 6 (see [13, Lemma 5.1]). *Suppose that Y, Γ, ϕ, Z, and Γ^ϕ are as previous. Then Z implements an equivalence between Γ and Γ^ϕ in the sense of Muhly-Renault-Williams.*

Now we consider $\phi : S_G \to R_\infty^0$ defined by $\alpha \mapsto (\pi(\alpha), \pi(\alpha))$. Since ϕ is the composition of the projection map $\pi : S_G \to J_G$ and the identity map from J_G to R_∞^0, ϕ is a continuous open surjection. Then we have

$$\begin{aligned}R_\infty^\phi = \{(\alpha,(\pi(\alpha),\pi(\beta)),\beta) : \alpha,\beta \in S_G,\\ (\pi(\alpha),\pi(\beta)) \in R_\infty\}.\end{aligned} \quad (21)$$

It is not difficult to check that $R_\infty^\phi = \cup_{n\geq 0} R_n^\phi$, where

$$\begin{aligned}R_n^\phi = \{(\alpha,(\pi(\alpha),\pi(\beta)),\beta) : \alpha,\beta \in S_G,\\ \sigma^n(\pi(\alpha)) = \sigma^n(\pi(\beta))\},\end{aligned} \quad (22)$$

and that the relative topology on R_∞^ϕ is equivalent to the inductive limit topology.

Lemma 7. *Suppose that $(S_G, \overline{\sigma})$ is the limit solenoid system induced from a self-similar group (G, X) and that G_s is the stable equivalence groupoid associated with $(S_G, \overline{\sigma})$. Then $\widetilde{\phi} : G_s \to R_\infty^\phi$ defined by $(\alpha, \beta) \mapsto (\alpha, (\pi(\alpha), \pi(\beta)), \beta)$ is a groupoid isomorphism.*

Proof. Remember that $G_s = \cup_{n\geq 0} G_{s,n}$ and $R_\infty^\alpha = \cup_{n\geq 0} R_n^\alpha$. From the commutative relation $\sigma\pi = \pi\overline{\sigma}$, we observe

$$\begin{aligned}(\alpha,\beta) \in G_{s,n} \Longleftrightarrow \pi\overline{\sigma}^n(\alpha) = \pi\overline{\sigma}^n(\beta) \Longleftrightarrow \sigma^n\pi(\alpha)\\ = \sigma^n\pi(\beta).\end{aligned} \quad (23)$$

Hence $\widetilde{\phi}|_{G_{s,n}}$ is a well-defined bijective map between $G_{s,n}$ and R_n^ϕ.

Since topologies on $G_{s,n}$ and R_n^ϕ are relative topologies from $S_G \times S_G$, $\widetilde{\phi}|_{G_{s,n}}$ is a homeomorphism. Then $\widetilde{\phi}$ is a homeomorphism as the inductive limit topologies are given on G_s and R_∞^ϕ. It is routine to check that $\widetilde{\phi}$ is a groupoid morphism. □

The groupoid equivalence between R_∞ and G_s follows from Theorem 6 and Lemma 7. Strong Morita equivalence is from [10, Proposition 2.8] as both groupoids have Haar systems.

Theorem 8. *Suppose that (G, X) is a self-similar group, that R_∞ is the groupoid associated with (J_G, σ), and that G_s is the stable equivalence groupoid associated with $(S_G, \overline{\sigma})$. Then R_∞ and G_s are equivalent in the sense of Muhly-Renault-Williams. Therefore $C^*(R_\infty)$ is strongly Morita equivalent to the stable algebra S on the limit solenoid system $(S_G, \overline{\sigma})$.*

Analogous assertions hold for $\Gamma(J_G, \sigma)$ and $G_s \rtimes \mathbb{Z}$. For $\psi : S_G \to \Gamma(J_G, \sigma)^0$ defined by $\alpha \mapsto (\pi(\alpha), 0, \pi(\alpha))$, we observe

$$\begin{aligned}\Gamma(J_G,\sigma)^\psi = \{(\alpha,(\pi(\alpha),n,\pi(\beta)),\beta) : \alpha,\beta \in S_G,\\ (\pi(\alpha),n,\pi(\beta)) \in \Gamma(J_G,\sigma)\}.\end{aligned} \quad (24)$$

Lemma 9. *Suppose that G_s is the stable equivalence groupoid of $(S_G, \overline{\sigma})$ and that $G_s \rtimes \mathbb{Z}$ is the semidirect product groupoid. Then $\widetilde{\psi} : G_s \rtimes \mathbb{Z} \to \Gamma(J_G, \sigma)^\psi$ defined by $(\alpha, n, \beta) \mapsto (\alpha, (\pi(\alpha), n, \pi(\beta)), \beta)$ is a groupoid isomorphism.*

Proof. Recall that $(\alpha, n, \beta) \in G_s \rtimes \mathbb{Z} \Leftrightarrow (\overline{\sigma}^n(\alpha), \beta) \in G_s$. Then $G_s = \cup_{n\geq 0} G_{s,n}$ implies that $(\overline{\sigma}^n(\alpha), \beta) \in G_{s,l}$ for some $l \geq 0$. So from the proof of Lemma 7, we obtain that

$$\begin{aligned}(\overline{\sigma}^n(\alpha),\beta) \in G_{s,l} \Longleftrightarrow \sigma^{n+l}(\pi(\alpha))\\ = \sigma^l(\pi(\beta)) \Longleftrightarrow (\pi(\alpha),n,\pi(\beta)) \in \Gamma(J_G,\sigma).\end{aligned} \quad (25)$$

Thus $\widetilde{\psi}$ is a well-defined bijective map. As $G_s \rtimes \mathbb{Z}$ has the product topology, we notice that $\widetilde{\psi}|_{G_s \rtimes \{0\}}$ is the homeomorphism $\widetilde{\phi}$ defined in Lemma 7 and that $\widetilde{\psi}|_{G_s \rtimes \{n\}}$ is homeomorphism onto

$$\begin{aligned}\{(\alpha,(\pi(\alpha),n,\pi(\beta)),\beta) : \alpha,\beta \in S_G,\\ \sigma^{n+l}(\pi(\alpha)) = \sigma^l(\pi(\beta))\}.\end{aligned} \quad (26)$$

It is trivial that $\widetilde{\psi}$ is a groupoid morphism. □

Theorem 10. *Suppose that (G, X) is a self-similar group. Then $\Gamma(J_G, \sigma)$ and $G_s \rtimes \mathbb{Z}$ are equivalent in the sense of Muhly-Renault-Williams. Therefore $C^*(\Gamma(J_G, \sigma))$ is strongly Morita equivalent to the stable Ruelle algebra R_s on $(S_G, \overline{\sigma})$.*

Remark 11. In [11], Chen and Hou showed similar result under an extra condition that a Smale space is the inverse limit of an expanding surjection on a compact metric space.

4. Groupoid Algebras

Suppose that (G, X) is a self-similar group. We use its corresponding R_∞ and $\Gamma(J_G, \sigma)$ to study C^*-algebraic structures of stable algebra and stable Ruelle algebra from (G, X).

Following Renault [15], we say that a topological groupoid Γ with an open range map is *essentially principal* if Γ is locally compact and, for every closed invariant subset E of its unit space Γ^0, $\{u \in E : r^{-1}(u) \cap s^{-1}(u) = \{u\}\}$ is dense in E. A subset E of Γ^0 is called invariant if $r \circ s^{-1}(E) = E$. And Γ is called *minimal* if the only open invariant subsets of Γ^0 are the empty set $\emptyset$ and Γ^0 itself. We refer [15] for details.

Proposition 12. *The groupoid $\Gamma(J_G, \sigma)$ is essentially principal.*

Proof. Let

$$A = \left\{\xi \in J_G : \text{for } k, l \geq 0,\ \sigma^k(\xi) = \sigma^l(\xi) \text{ implies } k = l\right\},$$
$$B = \left\{b \in \Gamma\left(J_f, f\right)^0 : r^{-1}(b) \cap s^{-1}(b) = \{b\}\right\}. \tag{27}$$

Then we observe $\xi \in A \Leftrightarrow (\xi, 0, \xi) \in B$. Hence A is dense in X implying that B is dense in $\Gamma(J_f, f)^0$ so that $\Gamma(J_G, \sigma)$ is essentially principal.

To show that A is dense in J_G, we assume A is not dense in J_G. Then we can find an open set $U \subset J_G$ such that $\overline{U} \cap A = \emptyset$ as J_G is a compact Hausdorff space. Since

$$J_G - A = \bigcup_{n=1}^{\infty}\bigcup_{k=0}^{\infty}\sigma^{-k}(\mathrm{Per}_n), \tag{28}$$

where $\mathrm{Per}_n = \{\xi \in J_G : \sigma^n(\xi) = \xi\}$, we have

$$\overline{U} = \overline{U} \cap \left(J_f - A\right) = \bigcup_{n=1}^{\infty}\bigcup_{k=0}^{\infty}\overline{U} \cap \sigma^{-k}(\mathrm{Per}_n). \tag{29}$$

Then by Baire category theorem, there exist some integers $n \geq 1$ and $k \geq 0$ such that $\overline{U} \cap \sigma^{-k}(\mathrm{Per}_n)$ has nonempty interior. But $\mathrm{Per}_n = \{\xi \in J_G : \sigma^n(\xi) = \xi\}$ is a finite set because X is a finite set, and $\sigma^{-k}(\mathrm{Per}_n)$ is a finite set as σ is an $|X|$-fold covering map, a contradiction. Therefore A is dense in J_G, and $\Gamma(J_G, \sigma)$ is an essentially principal groupoid. □

There are excellent criteria for groupoid algebras from dynamical systems to be simple and purely infinite developed by Renault [12].

Lemma 13 (see [12]). *For a topological space X and a local homeomorphism $T : X \to X$, let $\Gamma(X, T)$ be the groupoid of Anantharaman-Delaroche and Deaconu. Suppose that $\Gamma(X, T)$ is an essentially principal groupoid and $C^*(X, T)$ is its groupoid algebra.*

(1) *Assume that for every nonempty open set $U \subset X$ and every $x \in X$, there exist $m, n \in \mathbb{N}$ such that $T^n(x) \in T^m(U)$. Then $C^*(X, T)$ is simple.*

(2) *Assume that for every nonempty open set $U \subset X$, there exist an open set $V \subset U$ and $m, n \in \mathbb{N}$ such that $T^m(V)$ is strictly contained in $T^n(V)$. Then $C^*(X, T)$ is purely infinite.*

As $\Gamma(J_G, \sigma)$ is an essentially principal groupoid, we have an alternative proof for Theorem 6.5 of [2].

Theorem 14. *The algebra $C^*(\Gamma(J_G, \sigma))$ is simple, purely infinite, separable, stable, and nuclear and satisfies the Universal Coefficient Theorem of Rosenberg-Schochet.*

Proof. Suppose that U is an open set in J_G. Then the inverse image of U in $X^{-\omega}$, say U', is open, and there is a cylinder set $Z(u)$ defined by some $u \in X^n$ such that $Z(u) \subset U'$. By definition of cylinder sets, we have $\sigma^n(Z(u)) = X^{-\omega} \subseteq \sigma^n(U')$, which implies that $\sigma^n(U) = J_G$ on the quotient space. Thus for every $\xi \in J_G$, $\xi \in \sigma^n(U)$ and $C^*(\Gamma(J_G, \sigma))$ is simple.

For an open set U of J_G, let V be an open subset of U such that the inverse image of V in $X^{-\omega}$ is equal to the cylinder set $Z(v)$, where $v \in X^n$ for some $n \geq 2$. Then we obtain $\sigma^n(V) = J_G$ as in the previous, and $\sigma^m(V)$ is a proper subset of $\sigma^n(V)$ for every $1 \leq m \lneq n$. Hence $C^*(\Gamma(J_G, \sigma))$ is purely infinite.

Since $\Gamma(J_G, \sigma)$ is locally compact and second countable, $C^*(\Gamma(J_G, \sigma))$ is σ-unital, nonunital, and separable. So Zhang's dichotomy [16, Theorem 1.2] implies that $C^*(\Gamma(J_G, \sigma))$ is stable. By Proposition 2.4 of [12], nuclear is an easy consequence from amenability of $\Gamma(J_G, \sigma)$. Because $\Gamma(J_G, \sigma)$ is a locally compact amenable groupoid with Haar system, $C^*(\Gamma(J_G, \sigma))$ satisfied the Universal Coefficient Theorem by Lemma 3.5 and Proposition 10.7 of [17]. □

Corollary 15. *$C^*(\Gamma(J_G, \sigma))$ is $*$-isomorphic to the stable Ruelle algebra R_s.*

Proof. Because $C^*(\Gamma(J_G, \sigma))$ and R_s are stable, this is trivial from Theorem 10. □

For $C^*(R_\infty)$, we use the fact that $R_\infty = \cup R_n$ is a principal groupoid representing an AP equivalence relation [18].

Proposition 16. *The groupoid R_∞ is minimal, and its groupoid algebra $C^*(R_\infty)$ is simple.*

Proof. In the proof of Theorem 14, we observed that for every cylinder set $Z(u)$ of $X^{-\omega}$, there is an $n > 0$ such that $\sigma^n(Z(u)) = X^{-\omega}$. Since the inverse image of a nonempty open set U in J_G contains a cylinder set $Z(u)$, this observation induces that $\sigma^n(U) = J_G$ on the quotient space. Then R_∞ is a minimal groupoid by [19, Proposition 2.1]. And simplicity of $C^*(R_\infty)$ follows from [15, Proposition II.4.6] as R_∞ is an r-discrete principal groupoid. □

Proposition 17. *$C^*(R_\infty)$ is the inductive limit of $C^*(R_n)$. And each $C^*(R_n)$ is strongly Morita equivalent to $C(R_n^0/R_n) = C(J_G/R_n)$.*

Proof. Note that $R_\infty = \cup_{n\geq 0} R_n$ is the groupoid representing an AP equivalence relation on stationary sequence $J_G \xrightarrow{\sigma} J_G \xrightarrow{\sigma} \cdots$. Thus it is easy to check that Corollary 2.2 of [18] implies the inductive limit structure.

Clearly $R_n = \{(u, v) \in J_G \times J_G : \sigma^n(u) = \sigma^n(v)\}$ is the groupoid representing an equivalence relation on J_G defined by $u \sim_n v$ if and only if $\sigma^n(u) = \sigma^n(v)$. And $(s \times r)(R_n) = (\sigma^{-n} \times \sigma^{-n})(\Delta)$, where $\Delta = \{(u, u) \in J_G \times J_G\}$ implies that $(s \times r)(R_n)$ is a closed subset of $J_G \times J_G$. Thus we have strong Morita equivalence of $C^*(R_n)$ and $C(J_G/R_n)$ by [20, Proposition 2.2]. □

Corollary 18. *$C^*(R_\infty)$ is a nuclear algebra.*

Proof. Since $C(J_G/R_n)$ is nuclear, $C^*(R_n)$ is also nuclear by [21, Theorem 15]. And it is a well-known fact that the class of nuclear C^*-algebras is closed under inductive limit. So $C^*(R_\infty)$ is nuclear.

Postcritically Finite Rational Maps. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a postcritically finite hyperbolic rational function of degree more than one, that is, a rational function of degree more than one such that the orbit of every critical point of f eventually belongs to a cycle containing a critical point. Then f is expanding on a neighborhood of its Julia set J_f, the group IMG(f) is contracting, recurrent, regular, and finitely generated, and the limit dynamical system $\sigma : J_{\mathrm{IMG}(f)} \to J_{\mathrm{IMG}(f)}$ is topologically conjugate with the action of f on its Julia set J_f (see [2, Sections 2 and 6] for details). □

We borrowed the following theorem from Theorem 3.16 and Remark 4.23 of [22].

Theorem 19. *Let $f : \mathbb{C} \to \mathbb{C}$ be a postcritically finite hyperbolic rational function of degree more than one and let R_∞ be the groupoid on its limit dynamical system as in Section 2. Then $C^*(R_\infty)$ is an AT-algebra of real-rank zero with a unique trace state.*

Proof. To show that $C^*(R_\infty)$ is an AT-algebra, we use the work of Gong [23, Corollary 6.7]. By Propositions 16 and 17, $C^*(R_\infty)$ is a simple algebra which is an inductive limit of an AH system with uniformly bounded dimensions of local spectra. And Nekrashevych showed that K-groups of $C^*(R_\infty)$ for postcritically finite hyperbolic rational functions are torsion free in [2, Theorem 6.6]. Hence $C^*(R_\infty)$ is an AT-algebra.

As $f : J_f \to J_f$ is an expanding local homeomorphism (see [2, Section 6.4]) and exact by Proposition 16 and [19, Proposition 2.1], $C^*(R_\infty)$ has a unique trace state by Remark 3.6 of [19]. Simplicity and uniformly bounded dimension conditions imply that $C^*(R_\infty)$ is approximately divisible in the sense of Blackadar et al. [24] as shown by Elliot et al. [14]. Therefore $C^*(R_\infty)$ has real-rank zero by Theorem 1.4 of [24]. □

Corollary 20. *$C^*(R_\infty)$ associated with postcritically finite hyperbolic rational functions of degree more than one belongs to the class of C^*-algebras covered by Elliot classification program.*

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