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Research Article

Fixed Points for ψ-Graphic Contractions with Application to Integral Equations

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The aim of this paper is to define modified weak α-ψ-contractive mappings and to establish fixed point results for such mappings defined on partial metric spaces using the notion of triangular α-admissibility. As an application, we prove new fixed point results for graphic weak ψ-contractive mappings. Moreover, some examples and an application to integral equation are given here to illustrate the usability of the obtained results.

1. Introduction and Preliminaries

The concept of partial metric space was introduced by Matthews [1] in 1994. Partial metric space is a generalized metric space in which each object does not necessarily have to have a zero distance from itself. A motivation behind introducing the concept of a partial metric was to obtain appropriate mathematical models in the theory of computation and, in particular, to give a modified version of the Banach contraction principle [2, 3]. Subsequently, several authors studied the problem of existence and uniqueness of a fixed point for mappings satisfying different contractive conditions on partial metric spaces (e.g., [4–10]). For a recent survey on the existence of fixed points in different spaces with generalized distance functions, the reader may check [11–18]. We start by recalling some definitions and properties of partial metric spaces.

Definition 1. A *partial metric* on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$,

(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;

(p2) $p(x, x) \le p(x, y)$;

(p3) $p(x, y) = p(y, x)$;

(p4) $p(x, z) \le p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) such that X is nonempty set and p is a partial metric on X.

From the above definition, if $p(x, y) = 0$, then $x = y$. But if $x = y$, $p(x, y)$ may not be 0 in general. A trivial example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $p(x, y) = \max\{x, y\}$. For more examples of partial metric spaces, we refer to [6, 7].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. A sequence $\{x_n\}$ in X converges to a point $x \in X$, with respect to τ_p, if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$. A sequence $\{x_n\}$ in X is called Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite. A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

A sequence $\{x_n\}$ in X is called 0-Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. A partial metric space (X, p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

The partial metric space $(\mathbb{Q} \cap [0; \infty), p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x, y) = \max\{x, y\}$, provides an example of a 0-complete partial metric space which is not complete.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1}$$

is a metric on X. A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n, z) = 0$ if and only if $p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m)$. Some more properties of partial metric spaces can be found in [3, 7, 19].

Samet et al. [20] defined the notion of α-admissible mappings and proved the following result.

Definition 2. Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ a function. One says that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{2}$$

Denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

Theorem 3. *Let (X, d) be a complete metric space and T be α-admissible mapping. Assume that*

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \tag{3}$$

where $\psi \in \Psi$. Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, one has $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then, T has a fixed point.

Very recently, Salimi et al. [21] modified the notions of α-ψ-contractive mappings and α-admissible mappings as follows.

Definition 4 (see [21]). Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty)$ two functions. One says that T is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Longrightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty). \tag{4}$$

Note that if we take $\eta(x, y) = 1$, then this definition reduces to Definition 2. Also, if we take, $\alpha(x, y) = 1$, then we say that T is an η-subadmissible mapping.

The following result properly contains Theorem 3 and Theorems 2.3 and 2.4 of [22].

Theorem 5 (see [21]). *Let (X, d) be a complete metric space and T be α-admissible mapping with respect to η. Assume that*

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Longrightarrow d(Tx, Ty) \leq \psi(M(x, y)), \tag{5}$$

where $\psi \in \Psi$ and

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}. \tag{6}$$

Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, one has $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then, T has a fixed point.

In fact, Banach contraction principle and Theorem 5 hold for the following example, but Theorem 3 does not hold.

Example 6 (see [21]). Let $X = [0, \infty)$ be endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$, and let $T : X \to X$ be defined by $Tx = (1/4)x$. Also, define, $\alpha : X^2 \to [0, \infty)$ by $\alpha(x, y) = 3$ and $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = (1/2)t$.

In this paper, we define modified weak α-ψ-contractive mappings and establish fixed point results for such mappings defined on ordinary as well as ordered partial metric spaces using the notion of triangular α-admissibility. As an application, we prove new fixed point results for graphic weak ψ-contractive mappings. Moreover, some examples and an application to integral equation are given here to illustrate the usability of the obtained results.

2. Modified Weak α-ψ-Contractions

Recently, Karapınar et al. [23] introduced the notion of triangular α-admissible mapping as follows.

Definition 7 (see [23]). Let $T : X \to X$ and $\alpha : X \times X \to (-\infty, +\infty)$. One says that T is a triangular α-admissible mapping if

(T1) $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$, $x, y \in X$;

(T2) $\alpha(x, z) \geq 1, \alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$.

Lemma 8 (see [23]). *Let $T : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define sequence $\{x_n\}$ by $x_n = T^n x_0$. Then,*

$$\alpha(x_m, x_n) \geq 1 \quad \forall m, n \in \mathbb{N} \text{ with } m < n. \tag{7}$$

Motivated by Samet et al. [20] and Salimi et al. [21], we introduce the following mapping.

Definition 9. Let (X, p) be a partial metric space and $T : X \to X$, $\alpha : X \times X \to [0, \infty)$ two mappings. If there exists an upper semicontinuous from the right nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(t) < t$ for all $t > 0$ such that

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Longrightarrow p(Tx, Ty) \leq \psi\left(M_p(x, y)\right), \tag{8}$$

where

$$M_p(x, y) = \max\left\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\right\}. \tag{9}$$

Then, we say that T is a modified weak α-ψ-contractive mapping.

Definition 10. Let (X, p) be a partial metric space. Let $\alpha : X \times X \to (-\infty, +\infty)$ and $T : X \to X$. One says that T is an α-continuous function on (X, p^s) if for given x and $\{x_n\}$ in X,

$$x_n \longrightarrow x \quad \text{as } n \longrightarrow \infty, \qquad \alpha(x_n, x_{n+1}) \geq 1 \quad \forall n \in \mathbb{N} \text{ imply } Tx_n \longrightarrow Tx. \tag{10}$$

Example 11. Let $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$. Assume that $T : X \to X$ and $\alpha : X^2 \to (-\infty, +\infty)$ defined by

$$Tx = \begin{cases} 3x^2, & \text{if } x \in [0, 1] \\ 5x + 1, & \text{if } (1, \infty), \end{cases} \qquad \alpha(x, y) = \begin{cases} 1, & \text{if } x \in [0, 1] \\ -1, & \text{otherwise.} \end{cases} \tag{11}$$

Clearly, T is not continuous, but T is an α-continuous on (X, p^s). Indeed, if $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$, then $x_n \in [0, 1]$ and so $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} 3x_n^2 = 3x^2 = Tx$.

Theorem 12. *Let (X, p) be a 0-complete partial metric space and T a modified weak α-ψ-contractive and triangular α-admissible mapping. Suppose that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(ii) *T is an α-continuous function on (X, p^s).*

Then, T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Then, by Lemma 8, we have

$$\alpha(x_m, x_n) \geq 1 \quad \forall m, n \in \mathbb{N} \text{ with } m < n. \tag{12}$$

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for T, and the result is proved. Hence, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Now, by (8) and (12) with $x = x_{n-1}$, $y = x_n$, we get

$$p(Tx_{n-1}, Tx_n) \leq \psi(M(x_{n-1}, x_n)) \tag{13}$$

On the other hand,

$$\begin{aligned} \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} &\leq M_p(x_{n-1}, x_n) \\ &= \max\left\{p(x_{n-1}, x_n), p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n), \frac{p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1})}{2}\right\} \\ &= \max\left\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{p(x_{n-1}, x_{n+1}) + p(x_n, x_n)}{2}\right\} \\ &\leq \max\left\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}\right\} \\ &\leq \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} \end{aligned} \tag{14}$$

and then, $M_p(x_{n-1}, x_n) = \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}$. So, from (13), we have

$$p(x_n, x_{n+1}) \leq \psi(\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}). \tag{15}$$

Now, if $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$ for some $n \in \mathbb{N}$, then

$$\begin{aligned} p(x_n, x_{n+1}) &\leq \psi(\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}) \\ &= \psi(p(x_n, x_{n+1})) < p(x_n, x_{n+1}), \end{aligned} \tag{16}$$

which is a contradiction. Hence, for all $n \in \mathbb{N}$, we have

$$p(x_n, x_{n+1}) \leq \psi(p(x_{n-1}, x_n)) < p(x_{n-1}, x_n). \tag{17}$$

This implies that the sequence $\{p(x_{n-1}, x_n)\}$ is decreasing, and so, by (17), there is $s \geq 0$ such that

$$\lim_{n\to+\infty} p(x_n, x_{n+1}) = \lim_{n\to+\infty} \psi(p(x_{n-1}, x_n)) = s. \tag{18}$$

Now, we show that s must be equal to 0.

In fact, if $s > 0$, then we get

$$s = \limsup_{n\to+\infty} \psi(p(x_{n-1}, x_n)) \leq \psi(s) < s, \tag{19}$$

which is a contradiction. Hence,

$$\lim_{n\to+\infty} p(x_n, x_{n+1}) = 0. \tag{20}$$

We prove that $\{x_n\}$ is a 0-Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a 0-Cauchy sequence. Then, there is $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that, for all positive integers k,

$$\begin{aligned} &n(k) > m(k) > k, \\ &p(x_{n(k)}, x_{m(k)}) \geq \varepsilon, \\ &p(x_{n(k)}, x_{m(k)-1}) < \varepsilon. \end{aligned} \tag{21}$$

Now, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon &\leq p(x_{n(k)}, x_{m(k)}) \\ &\leq p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\ &< \varepsilon + p(x_{m(k)-1}, x_{m(k)}). \end{aligned} \tag{22}$$

Taking the limit as $k \to +\infty$ in the above inequality and using (20), we get

$$\lim_{k \to +\infty} p(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{23}$$

Again, from

$$\begin{aligned} p(x_{n(k)}, x_{m(k)}) \leq\ & p(x_{m(k)}, x_{m(k)+1}) \\ &+ p(x_{m(k)+1}, x_{n(k)+1}) \\ &+ p(x_{n(k)+1}, x_{n(k)}) \\ p(x_{n(k)+1}, x_{m(k)+1}) \leq\ & p(x_{m(k)}, x_{m(k)+1}) \\ &+ p(x_{m(k)}, x_{n(k)}) \\ &+ p(x_{n(k)+1}, x_{n(k)}), \end{aligned} \tag{24}$$

taking the limit as $k \to +\infty$ and by (20) and (23), we deduce

$$\lim_{k \to +\infty} p(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \tag{25}$$

Also, since

$$\begin{aligned} \varepsilon &\leq p(x_{n(k)}, x_{m(k)}) \\ &\leq p(x_{n(k)}, x_{m(k)+1}) + p(x_{m(k)+1}, x_{m(k)}), \end{aligned} \tag{26}$$

then by taking limit as $n \to \infty$ in the last inequality and applying (20) and (23), we deduce

$$\lim_{k \to +\infty} p(x_{n(k)}, x_{m(k)+1}) = \varepsilon. \tag{27}$$

Similarly,

$$\lim_{k \to +\infty} p(x_{m(k)}, x_{n(k)+1}) = \varepsilon. \tag{28}$$

Then, by (8) and (12), we obtain

$$\begin{aligned} p(x_{n(k)+1}, x_{m(k)+1}) &= p(Tx_{n(k)}, Tx_{m(k)}) \\ &\leq \psi(M_p(x_{n(k)}, x_{m(k)})), \end{aligned} \tag{29}$$

where

$$\begin{aligned} &M_p(x_{n(k)}, x_{m(k)}) \\ &= \max\Bigg\{p(x_{n(k)}, x_{m(k)}), p(x_{n(k)}, Tx_{n(k)}), \\ &\quad p(x_{m(k)}, Tx_{m(k)}), \\ &\quad \frac{p(x_{n(k)}, Tx_{m(k)}) + p(x_{m(k)}, Tx_{n(k)})}{2}\Bigg\} \\ &= \max\Bigg\{p(x_{n(k)}, x_{m(k)}), p(x_{n(k)}, x_{n(k)+1}), \\ &\quad p(x_{m(k)}, x_{m(k)+1}), \\ &\quad \frac{p(x_{n(k)}, x_{m(k)+1}) + p(x_{m(k)}, x_{n(k)+1})}{2}\Bigg\}. \end{aligned} \tag{30}$$

Taking limit supremum as $n \to \infty$ in the above inequality and applying (20), (23), (25), (27), and (28), we get

$$\varepsilon \leq \psi(\varepsilon) < \varepsilon, \tag{31}$$

which is a contradiction. Hence, $\{x_n\}$ is a 0-Cauchy sequence.

Since T is orbitally α-continuous on (X, p^s), $x_n \to z$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$, then we have

$$Tz = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = z. \tag{32}$$

So, z is a fixed point of T. □

Theorem 13. *Let (X, p) be a 0-complete partial metric space and $T : X \to X$ a modified weak α-ψ-contractive and α-admissible mapping. Suppose that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(ii) *$\alpha(x, x) \geq 1$ for all $x \in X$ and if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then, T has a fixed point.

Proof. Condition (ii) implies property (T2) in definition of triangular α-admissible map. Indeed, if $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$, then applying (ii) to (x_n) defined by

$$x_1 := x, \qquad x_2 := y, \qquad x_n := z \quad \text{for } n \geq 3, \tag{33}$$

we get $\alpha(x_n, z) \geq 1$ for $n \in \mathbb{N}$, and hence, $\alpha(x, z) \geq 1$. Thus, as in Theorem 12, we obtain a 0-Cauchy sequence $\{x_n\}$ such that $x_n \to z$ as $n \to \infty$. Since,

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{34}$$

for all $n \in \mathbb{N}$ and $x_n \to z$ as $n \to \infty$, then from (ii), we have

$$\alpha(x_n, z) \geq 1 \tag{35}$$

for all $n \in \mathbb{N}$. Then, from (8), we get

$$p(x_{n+1}, Tz) \leq \psi(M_p(x_n, z)), \tag{36}$$

where

$$M_p(x_n, z) = \max\left\{p(x_n, z), p(x_n, x_{n+1}), p(z, Tz), \frac{p(x_n, Tz) + p(z, x_{n+1})}{2}\right\}. \tag{37}$$

By taking limit supremum as $n \to \infty$ in the above inequality, we have

$$\begin{aligned} p(z, Tz) &= \limsup_{n\to\infty} p(x_{n+1}, Tz) \\ &\le \limsup_{n\to\infty} \psi\left(M_p(x_n, z)\right) \\ &\le \psi\left(\limsup_{n\to\infty} M_p(x_n, z)\right) \\ &= \psi(p(z, Tz)) < p(z, Tz), \end{aligned} \tag{38}$$

which is a contradiction. Hence, $p(z, Tz) = 0$. That is, $z = Tz$. □

Example 14. Let $X = [0, \infty)$ be endowed with the partial metric $p(x, y) = \max\{x, y\}$ for all $x, y \in X$, and let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{2}x^3 & \text{if } x \in [0, 1] \\ x^3 + 2\ln x & \text{if } x \in (1, \infty). \end{cases} \tag{39}$$

Define $\alpha : X \times X \to (-\infty, +\infty)$ and $\psi : [0, \infty) \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x, y \in [0, 1] \text{ or } x = y \\ -2 & \text{otherwise} \end{cases} \qquad \psi(t) = \frac{1}{2}t. \tag{40}$$

Clearly, (X, p) is a 0-complete partial metric space. We show that T is a triangular α-admissible mapping. Let $x, y \in X$, if $\alpha(x, y) \ge 1$, then $x, y \in [0, 1]$ or $x = y$. On the other hand, for all $x \in [0, 1]$, we have $Tx \le 1$. It follows that $\alpha(Tx, Ty) \ge 1$. Also, if $\alpha(x, z) \ge 1$, and $\alpha(z, y) \ge 1$ then $x, y, z \in [0, 1]$. That is, $\alpha(x, y) \ge 1$. Hence, the assertion holds. In reason of the above arguments, $\alpha(0, T0) \ge 1$.

Now, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subset [0, 1]$, and hence, $x \in [0, 1]$. This implies that $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}$.

Let $\alpha(x, y) \ge 1$. Then, $x, y \in [0, 1]$. We get

$$\begin{aligned} p(Tx, Ty) &= \frac{1}{2}\max\left\{x^3, y^3\right\} \\ &\le \frac{1}{2}\max\{x, y\} \le \frac{1}{2}M_p(x, y) \\ &= \psi\left(M_p(x, y)\right). \end{aligned} \tag{41}$$

That is,

$$\alpha(x, y) \ge 1 \Longrightarrow p(Tx, Ty) \le \psi\left(M_p(x, y)\right). \tag{42}$$

Hence, all conditions of Theorem 13 hold and 0 is a fixed point of T.

Corollary 15. *Let (X, p) be a 0-complete partial metric space and $T : X \to X$ be such that $\alpha(x_0, Tx_0) \ge 1$ for some $x_0 \in X$. Assume that*

$$\left|\alpha(x, y)\right| p(Tx, Ty) \le \psi\left(M_p(x, y)\right) \tag{43}$$

hold for all $x, y \in X$. Also, suppose that one of the following assertions holds:

(i) *T is triangular α-admissible and α-continuous on (X, p^s);*
(ii) *T is α-admissible, $\alpha(x, x) \ge 1$ for all $x \in X$ and if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}$.*

Then, T has a fixed point.

Corollary 16. *Let (X, p) be a 0-complete partial metric space and $T : X \to X$ be such that $\alpha(x_0, Tx_0) \ge 1$ for some $x_0 \in X$. Assume that*

$$\left(\left|\alpha(x, y)\right| + \ell\right)^{p(Tx, Ty)} \le (1 + \ell)^{\psi(M_p(x, y))} \tag{44}$$

hold for all $x, y \in X$ where $\ell > 0$. Also, suppose that one of the following assertions holds:

(i) *T is triangular α-admissible and α-continuous on (X, p^s);*
(ii) *T is α-admissible, $\alpha(x, x) \ge 1$ for all $x \in X$ and if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N}$.*

Then, T has a fixed point.

Proof. Let $\alpha(x, y) \ge 1$. Then, $|\alpha(x, y)| \ge 1$. Hence, by (44), we have

$$\begin{aligned} (1 + \ell)^{p(Tx, Ty)} &\le \left(\left|\alpha(x, y)\right| + \ell\right)^{p(Tx, Ty)} \\ &\le (1 + \ell)^{\psi(M_p(x, y))}. \end{aligned} \tag{45}$$

Thus, $p(Tx, Ty) \le \psi(M_p(x, y))$. Hence, conditions of Corollary 15 hold, and T has a fixed point. □

Similarly, we have the following corollary.

Corollary 17. *Let (X, p) be a 0-complete partial metric space and $T : X \to X$ be such that $\alpha(x_0, Tx_0) \ge 1$ for some $x_0 \in X$. Assume that*

$$\left(p(Tx, Ty) + \ell\right)^{|\alpha(x, y)|} \le \psi\left(M_p(x, y)\right) + \ell \tag{46}$$

hold for all $x, y \in X$ where $\ell > 0$. Also, suppose that one of the following assertions holds:

(i) *T is triangular α-admissible and α-continuous on (X, p^s);*

(ii) *T is α-admissible, $\alpha(x,x) \geq 1$ for all $x \in X$ and if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then, T has a fixed point.

Corollary 18 (Matthews [1]). *Let (X, p) be a 0-complete partial metric space and $T : X \to X$ a given mapping satisfying*

$$p(Tx, Ty) \leq kp(x, y) \tag{47}$$

for all $x, y \in X$, where $k \in [0, 1)$. Then, T has a unique fixed point.

3. Fixed Point Results in Partially Ordered Partial Metric Spaces

Fixed point theorems for monotone operators in partially ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [24, 25] and references therein).

Theorem 19 (see [24]). *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $T : X \to X$ a continuous increasing self-mapping such that $x_0 \preceq Tx_0$ for some $x_0 \in X$. Assume that*

$$d(Tx, Ty) \leq rd(x, y) \tag{48}$$

hold for all $x, y \in X$ with $x \preceq y$, where $0 \leq r < 1$. Then, T has a fixed point.

Theorem 20. *Let $(X, p, \preceq)$ be a partially ordered 0-complete partial metric space and $T : X \to X$ an increasing self-mapping such that $x_0 \preceq Tx_0$ for some $x_0 \in X$. Assume that*

$$p(Tx, Ty) \leq \psi\left(M_p(x, y)\right) \tag{49}$$

hold for all $x, y \in X$ with $x \preceq y$. Now, if T is a continuous mapping on (X, p^s), then T has a fixed point.

Proof. Define, $\alpha : X^2 \to (-\infty, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \preceq y \\ 0, & \text{otherwise.} \end{cases} \tag{50}$$

At first we prove that T is a triangular α-admissible mapping. Let $\alpha(x, y) \geq 1$, then $x \preceq y$. As T is an increasing mapping, we have $Tx \preceq Ty$. That is, $\alpha(Tx, Ty) \geq 1$. Also, let $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $x \preceq z$ and $z \preceq y$. So, from transitivity, we have $x \preceq y$. That is, $\alpha(x, y) \geq 1$. Thus, T is a triangular α-admissible mapping. Also, there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ which implies $\alpha(x_0, Tx_0) \geq 1$. Let $\alpha(x, y) \geq 1$, then $x \preceq y$. Now, from (49), we have $p(Tx, Ty) \leq \psi(M_p(x, y))$. That is,

$$\alpha(x, y) \geq 1 \Longrightarrow p(Tx, Ty) \leq \psi\left(M_p(x, y)\right). \tag{51}$$

Hence, all conditions of Theorem 12 are satisfied, and T has a fixed point. □

Theorem 21 (see [25]). *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $T : X \to X$ an increasing mapping such that*

$$d(Tx, Ty) \leq rd(x, y) \tag{52}$$

for all $x, y \in X$ with $x \preceq y$, where $0 \leq r < 1$. Suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then, T has a fixed point.

Theorem 22. *Let $(X, p, \preceq)$ be a partially ordered 0-complete partial metric space and $T : X \to X$ an increasing mapping such that*

$$p(Tx, Ty) \leq \psi\left(M_p(x, y)\right) \tag{53}$$

for all $x, y \in X$ with $x \preceq y$. Suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then, T has a fixed point.

Proof. Define $\alpha : A \times A \to (-\infty, +\infty)$ as in proof of Theorem 20. Assume $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ such that $x_n \to x$ as $n \to \infty$. Then, $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Hence, by (ii), we get $x_n \preceq x$ for all $n \in \mathbb{N}$ and so $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Proceeding as in proof of Theorem 20, we can deduce that T is a modified weak α-ψ-contractive and α-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Hence, all conditions of Theorem 13 hold, and T has a fixed point. □

Remark 23. Similarly, we may obtain more fixed point results on ordered partial metric spaces as immediate consequences of Corollaries 15–17.

4. Fixed Point Results for Graphic Contractions

Consistent with Jachymski [26], let (X, p) be a partial metric space, and let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [26]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \ldots, N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected (see for details [11, 26–28]).

Recently, some results have appeared providing sufficient conditions for a mapping to be a Picard operator if (X, d) is endowed with a graph. The first result in this direction was given by Jachymski [26].

Definition 24 (see [26]). One says that a mapping $T : X \to X$ is a Banach G-contraction or simply G-contraction if T preserves edges of G, that is,

$$\forall x, y \in X \quad (x, y) \in E(G) \Longrightarrow (T(x), T(y)) \in E(G), \tag{54}$$

and T decreases weights of edges of G in the following way:

$$\exists \alpha \in (0, 1), \forall x, y \in X \quad (x, y) \in E(G) \Longrightarrow d(T(x), T(y)) \leq \alpha d(x, y). \tag{55}$$

Definition 25 (see [26]). A mapping $T : X \to X$ is called G-continuous if given $x \in X$ and sequence $\{x_n\}$

$$x_n \longrightarrow x \quad \text{as } n \longrightarrow \infty, \quad (x_n, x_{n+1}) \in E(G) \quad \forall n \in \mathbb{N} \text{ imply } Tx_n \longrightarrow Tx. \tag{56}$$

Definition 26. Let (X, p) be a partial metric space endowed with a graph G and $T : X \to X$ a self-mapping. If there exists an upper semicontinuous from the right function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(t) < t$ for all $t > 0$ such that

$$\begin{aligned} &\forall x, y \in X, \quad (x, y) \in E(G) \Longrightarrow (T(x), T(y)) \in E(G) \\ &\forall x, y \in X, \quad (x, y) \in E(G), \Longrightarrow p(Tx, Ty) \leq \psi(M_p(x, y)), \end{aligned} \tag{57}$$

where

$$M_p(x, y) = \max\left\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\right\}. \tag{58}$$

Then, T is called a weak ψ-graphic contractive mapping.

Theorem 27. *Let (X, p) be a 0-complete partial metric space endowed with a graph G and $T : X \to X$ a weak ψ-graphic contractive mapping. Suppose that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*

(ii) *T is G-continuous on (X, p^s);*

(iii) *$(x, z) \in E(G)$ and $(z, y) \in E(G)$ imply $(x, y) \in E(G)$ for all $x, y, z \in X$, that is, $E(G)$ is a quasi-order [26].*

Then, T has a fixed point.

Proof. Define $\alpha : X^2 \to (-\infty, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G) \\ 0, & \text{otherwise.} \end{cases} \tag{59}$$

At first we prove that T is a triangular α-admissible mapping. Let $\alpha(x, y) \geq 1$, then $(x, y) \in E(G)$. As T is a weak ψ-graphic contractive mapping, we have $(Tx, Ty) \in E(G)$. That is, $\alpha(Tx, Ty) \geq 1$. Also, let $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $(x, z) \in E(G)$ and $(z, y) \in E(G)$. So, from (iii), we have $(x, y) \in E(G)$. That is, $\alpha(x, y) \geq 1$. Thus, T is a triangular α-admissible mapping. Let T be G-continuous on (X, p^s). Then,

$$x_n \longrightarrow x \quad \text{as } n \longrightarrow \infty, \quad (x_n, x_{n+1}) \in E(G) \quad \forall n \in \mathbb{N} \text{ imply } Tx_n \longrightarrow Tx. \tag{60}$$

That is,

$$x_n \longrightarrow x \quad \text{as } n \longrightarrow \infty, \quad \alpha(x_n, x_{n+1}) \geq 1 \quad \forall n \in \mathbb{N} \text{ imply } Tx_n \longrightarrow Tx \tag{61}$$

which implies T is α-continuous on (X, p^s). From (i), there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. That is, $\alpha(x_0, Tx_0) \geq 1$.

If $\alpha(x, y) \geq 1$, then $(x, y) \in E(G)$. Now, since T is a weak ψ-graphic contractive mapping, so $p(Tx, Ty) \leq \psi(M_p(x, y))$. That is,

$$\alpha(x, y) \geq 1 \Longrightarrow p(Tx, Ty) \leq \psi(M_p(x, y)). \tag{62}$$

Hence, all conditions of Theorem 12 are satisfied, and T has a fixed point. □

If G is a connected graph, then condition (iii) of Theorem 27 is automatically satisfied. Thus, we have the following result.

Corollary 28. *Let (X, p) be a 0-complete partial metric space endowed with a graph G and T a weak ψ-graphic contractive mapping. Suppose that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*

(ii) *T is G-continuous on (X, p^s);*

(iii) *G is a connected graph.*

Then, T has a fixed point.

Theorem 29. *Let (X, p) be a 0-complete partial metric space endowed with a graph G and T a weak ψ-graphic contractive mapping. Suppose that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*

(ii) *if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.*

Then, T has a fixed point.

Proof. Define, $\alpha : X^2 \to (-\infty, +\infty)$ as in proof of Theorem 27. Condition (ii) implies that $E(G)$ is a quasi-order, that is, $(x, z) \in E(G)$ and $(z, y) \in E(G)$ imply $(x, y) \in E(G)$ for all $x, y, z \in X$ (see Remark 3.1 [26]). Let, $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$. So, by (ii), we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n, x) \geq 1$. All other conditions of Theorem 13, follow similarly as in proof of Theorem 27 and consequently T has a fixed point. □

Theorem 3.2(2°) in [26] and Theorem 2.3(2) in [29] are extended to weak ψ-graphic contractive maps defined on a 0-complete partial metric space as follows.

Corollary 30. *Let (X, p) be a 0-complete partial metric space endowed with a graph G and T a weak ψ-graphic contractive mapping. Suppose that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*

(iis) *$E(G)$ is a quasi-order and if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then there is a subsequence $\{x_{k_n}\}$ with $(x_{k_n}, x) \in E(G)$ for all $n \in \mathbb{N}$.*

Then, T has a fixed point.

Proof. Condition (iis) implies that of (ii) in Theorem 29 (see Remark 3.1 [26]). Now, the conclusion follows from Theorem 29. □

Corollary 31. *Let (X, p) be a 0-complete partial metric space and ϵ-chainable for some $\epsilon > 0$, that is, given $x, y \in X$, there is $N \in \mathbb{N}$ and a sequence $\{x_i\}_{i=0}^{N}$ such that $x_0 = x$, $x_N = y$ and $p(x_{i-1}, x_i) < \epsilon$ for $i = 1, \ldots, N$. Suppose that $T : X \to X$ is a mapping satisfying*

$$\forall x, y \in X, \quad p(x, y) < \epsilon, \quad \Longrightarrow p(Tx, Ty) \leq \psi(p(x, y)). \tag{63}$$

Then, T has a fixed point.

Proof. Consider the graph G with $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : p(x, y) < \epsilon\}$. Then, ϵ-chainability of (X, p) means G is connected, and hence, $E(G)$ is quasi-order. If $(x, y) \in E(G)$, then

$$p(Tx, Ty) \leq \psi(p(x, y)) < p(x, y) < \epsilon, \tag{64}$$

so $(Tx, Ty) \in E(G)$, hence T is a (G, ψ)-contraction. Let $\{x_n\}$ be in X with $x_n \to x$, then $p(x_n, x) < \epsilon$ for sufficiently large n, so there is $\{x_{k_n}\}$ such that $(x_{k_n}, x) \in E(G)$. Thus, by Corollary 30, T has a fixed point. □

Definition 32. Let $(X, p, \preceq)$ be a partially ordered partial metric space endowed with a graph G and $T : X \to X$ a self-mapping. If there exists an upper semicontinuous from the right function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(t) < t$ for all $t > 0$ such that, for all $x, y \in X$, $(x, y) \in E(G)$ with $x \preceq y \Rightarrow (T(x), T(y)) \in E(G)$, where $Tx \preceq Ty$

$$(x, y) \in E(G) \text{ with } x \preceq y, \quad \Longrightarrow p(Tx, Ty) \leq \psi(M_p(x, y)), \tag{65}$$

where

$$M_p(x, y) = \max\left\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\right\}. \tag{66}$$

Then, we say T is an ordered weak ψ-graphic contractive mapping.

Theorem 33. *Let $(X, p, \preceq)$ be a partially ordered 0-complete partial metric space endowed with a graph G and T an ordered weak ψ-graphic contractive mapping. Suppose that the following assertions hold:*

(i) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ with $x_0 \preceq Tx_0$,*

(ii) *either T is G-continuous in (X, p^s) and $(x, z) \in E(G)$ and $(z, y) \in E(G)$ imply $(x, y) \in E(G)$ or;*

(iii) *if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ with $x_n \preceq x$ for all $n \in \mathbb{N} \cup 0$.*

Then, T has a fixed point.

Proof. Define $\alpha : X^2 \to (-\infty, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G) \text{ with } x \preceq y \\ 0, & \text{otherwise.} \end{cases} \tag{67}$$

At first, we prove that T is a triangular α-admissible mapping. Let $\alpha(x, y) \geq 1$, then $(x, y) \in E(G)$ with $x \preceq y$. As T is an ordered weak ψ-graphic contractive mapping, we have $(Tx, Ty) \in E(G)$ where $Tx \preceq Ty$. That is, $\alpha(Tx, Ty) \geq 1$. Also, let $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $(x, z) \in E(G)$ with $x \preceq z$ and $(z, y) \in E(G)$ with $z \preceq y$. So from (ii), we have $(x, y) \in E(G)$. Also, $x \preceq z$ and $z \preceq y$ implies $x \preceq y$. Hence, $\alpha(x, y) \geq 1$. Thus, T is a triangular α-admissible mapping. Let T be G-continuous on (X, p^s). Then,

$$x_n \longrightarrow x \quad \text{as } n \longrightarrow \infty, \quad (x_n, x_{n+1}) \in E(G) \quad \forall n \in \mathbb{N} \text{ imply } Tx_n \longrightarrow Tx. \tag{68}$$

That is,

$$x_n \longrightarrow x \quad \text{as } n \longrightarrow \infty, \quad \alpha(x_n, x_{n+1}) \geq 1 \quad \forall n \in \mathbb{N} \text{ imply } Tx_n \longrightarrow Tx \tag{69}$$

which implies T is α-continuous on (X, p^s). From (i), there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. That is, $\alpha(x_0, Tx_0) \geq 1$.

Let $\alpha(x, y) \geq 1$, then $(x, y) \in E(G)$ with $x \preceq y$. Now, since T is an ordered weak ψ-graphic contractive mapping, then $p(Tx, Ty) \leq \psi(M_p(x, y))$. That is,

$$\alpha(x, y) \geq 1 \Longrightarrow p(Tx, Ty) \leq \psi(M_p(x, y)). \tag{70}$$

Hence, all conditions of Theorem 12 (or 13) are satisfied, and T has a fixed point. □

Remark 34. All our results established above are new even in the setting of complete metric spaces.

5. Application to Existence of Solutions of Integral Equations

Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [30, 31] and references therein). In this section, we apply our result to the existence of a solution of an integral equation. Let $X = C([0, T], \mathbb{R})$ be the set of real continuous functions defined on $[0, T]$, and let $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x, y) = \|x - y\|_\infty \tag{71}$$

for all $x, y \in X$. Then, (X, d) is a complete metric space. Also, assume this metric space endowed with a graph G.

Consider the integral equation

$$x(t) = p(t) + \int_0^T S(t, s) f(s, x(s)) \, ds \tag{72}$$

and let $F : X \to X$ be defined by

$$F(x)(t) = p(t) + \int_0^T S(t, s) f(s, x(s)) \, ds. \tag{73}$$

We assume that

(A) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous;

(B) $p : [0, T] \to \mathbb{R}$ is continuous;

(C) $S : [0, T] \times \mathbb{R} \to [0, +\infty)$ is continuous;

(D) there exists an upper semicontinuous from the right nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(t) < t$ for all $t > 0$ such that for all $s \in [0, T]$,

$$\forall x, y \in X \; (x, y) \in E(G) \Longrightarrow (F(x), F(y)) \in E(G)$$

$$\begin{aligned} \forall x, y \in X \; (x, y) \in E(G) \Longrightarrow 0 &\leq f(s, x(s)) - f(s, y(s)) \\ &\leq \psi\Big(\max\Big\{ |x(s) - y(s)|, \\ &\qquad |x(s) - F(x(s))|, \\ &\qquad |y(s) - F(y(s))|, \\ &\qquad \frac{1}{2}\left[|x(s) - F(y(s))| \right. \\ &\qquad \left. + |y(s) - F(x(s))|\right] \Big\}\Big); \end{aligned} \tag{74}$$

(F) there exist $x_0 \in X$ such that $(x_0, F(x_0)) \in E(G)$;

(G) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$;

(H) assume

$$\sup_{t \in [0, T]} \int_0^T S(t, s) \, ds \leq 1. \tag{75}$$

Theorem 35. *Under assumptions (A)–(H), the integral equation (72) has a solution in $X = C([0, T], \mathbb{R})$.*

Proof. Consider the mapping $F : X \to X$ defined by (73). Let $(x, y) \in E(G)$. Then, from (D), we deduce

$$\begin{aligned} &|F(x)(t) - F(y)(t)| \\ &= \left| \int_0^T S(t, s) [f(s, x(s)) - f(s, y(s))] \, ds \right| \\ &\leq \int_0^T S(t, s) |f(s, x(s)) - f(s, y(s))| \, ds \\ &\leq \psi\Big(\max\Big\{ |x(s) - y(s)|, |x(s) - F(x(s))|, \\ &\qquad |y(s) - F(y(s))|, \\ &\qquad \frac{1}{2}\left[|x(s) - F(y(s))| \right. \\ &\qquad \left. + |y(s) - F(x(s))|\right] \Big\}\Big) \\ &\leq \psi\Big(\max\Big\{ \|x(s) - y(s)\|, \|x(s) - F(x(s))\|, \\ &\qquad \|y(s) - F(y(s))\|, \\ &\qquad \frac{1}{2}\left[\|x(s) - F(y(s))\| \right. \\ &\qquad \left. + \|y(s) - F(x(s))\|\right] \Big\}\Big). \end{aligned} \tag{76}$$

Then,

$$\begin{aligned}
&\|Fx - Fy\|_\infty \\
&\le \psi\left(\max\left\{\|x(s) - y(s)\|, \|x(s) - F(x(s))\|,\right.\right. \\
&\qquad \|y(s) - F(y(s))\|, \\
&\qquad \frac{1}{2}\left[\|x(s) - F(y(s))\|\right. \\
&\qquad\quad \left.\left.\left.+ \|y(s) - F(x(s))\|\right]\right\}\right).
\end{aligned} \tag{77}$$

That is, $(x, y) \in E(G)$ implies

$$\begin{aligned}
&\|Fx - Fy\|_\infty \\
&\le \psi\left(\max\left\{\|x - y\|_\infty, \|x - F(x)\|_\infty, \|y - F(y)\|_\infty,\right.\right. \\
&\qquad \left.\left.\frac{1}{2}\left[\|x - F(y)\|_\infty + \|y - F(x)\|_\infty\right]\right\}\right).
\end{aligned} \tag{78}$$

Thus, all the hypotheses of Theorem 29 are satisfied, and hence, the mapping F has a fixed point, that is, a solution in $X = C([0, T], \mathbb{R})$ of the integral equation (72). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] S. G. Matthews, "Partial metric topology," in *Proceedings of the 8th Summer Conference on General Topology and Applications*, vol. 728, pp. 183–197, Annals of the New York Academy of Sciences, 1994.

[2] L. Ćirić, B. Samet, H. Aydi, and C. Vetro, "Common fixed points of generalized contractions on partial metric spaces and an application," *Applied Mathematics and Computation*, vol. 218, no. 6, pp. 2398–2406, 2011.

[3] N. Hussain, Z. Kadelburg, S. Radenović, and F. R. Al-Solamy, "Comparison functions and fixed point results in partial metric spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 605781, 15 pages, 2012.

[4] A. Amini-Harandi, "Metric-like spaces, partial metric spaces and fixed points," *Fixed Point Theory and Applications*, vol. 2012, article 204, 2012.

[5] D. Ilić, V. Pavlović, and V. Rakočević, "Some new extensions of Banach's contraction principle to partial metric space," *Applied Mathematics Letters*, vol. 24, no. 8, pp. 1326–1330, 2011.

[6] Z. Golubović, Z. Kadelburg, and S. Radenović, "Coupled coincidence points of mappings in ordered partial metric spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 192581, 18 pages, 2012.

[7] M. A. Kutbi, J. Ahmad, N. Hussain, and M. Arshad, "Common fixed point results for mappings with rational expressions," *Abstract and Applied Analysis*, vol. 2013, Article ID 549518, 11 pages, 2013.

[8] H. K. Nashine, Z. Kadelburg, S. Radenović, and J. K. Kim, "Fixed point theorems under Hardy-Rogers contractive conditions on 0-complete ordered partial metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 180, 2012.

[9] Z. Kadelburg, H. K. Nashine, and S. Radenović, "Fixed point results under various contractive conditions in partial metric spaces," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales A*.

[10] W. Long, S. Shukla, S. Radenović, and S. Radojević, "Some coupled coincidence and common fixed point results for hybrid pair of mappings in 0-complete partial metric spaces," *Fixed Point Theory and Applications*, vol. 2013, article 145, 2013.

[11] M. Abbas and T. Nazir, "Common fixed point of a power graphic contraction pair in partial metric spaces endowed with a graph," *Fixed Point Theory and Applications*, vol. 2013, article 20, 2013.

[12] Lj. Ćirić, B. Samet, C. Vetro, and M. Abbas, "Fixed point results for weak contractive mappings in ordered -metric spaces," *Fixed Point Theory*, vol. 13, no. 1, pp. 59–72, 2012.

[13] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, John Wiley & Sons, New York, NY, USA, 2001.

[14] P. Kumam, C. Vetro, and F. Vetro, "Fixed points for weak α-ψ-contactions in partial metric spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 986028, 9 pages, 2013.

[15] A. Shoaib, M. Arshad, and J. Ahmad, "Fixed point results of locally cotractive mappings in ordered quasi partial metric spaces," *The Scientific World Journal*, vol. 2013, Article ID 194897, 8 pages, 2013.

[16] S. Shukla, S. Radenović, and V. C. Rajić, "Some common fixed point theorems in 0–σ–complete metric-like spaces," *Vietnam Journal of Mathematics*, 2013.

[17] H. K. Nashine, Z. Kadelburg, and S. Radenović, "Fixed point theorems via various cyclic contractive conditions in partial metric spaces," *Publications de LfInstitut Mathematique, Nouvelle Serie Tome*, vol. 93, no. 107, pp. 69–93, 2013.

[18] A. G. Bin Ahmad, Z. M. Fadail, V. C. Rajić, and S. Radenović, "Nonlinear contractions in 0-complete partial metric spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 451239, 12 pages, 2012.

[19] S. M. Alsulami, N. Hussain, and A. Alotaibi, "Coupled fixed and coincidence points for monotone operators in partial metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 173, 2012.

[20] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for $\alpha-\psi$-contractive type mappings," *Nonlinear Analysis: Theory and Methods*, vol. 75, no. 4, pp. 2154–2165, 2012.

[21] P. Salimi, A. Latif, and N. Hussain, "Modified $\alpha-\psi$-contractive mappings with applications," *Fixed Point Theory and Applications*, vol. 2013, article 151, 2013.

[22] E. Karapınar and B. Samet, "Generalized $(\alpha-\psi)$ contractive type mappings and related fixed point theorems with applications," *Abstract and Applied Analysis*, vol. 2012, Article ID 793486, 17 pages, 2012.

[23] E. Karapınar, P. Kumam, and P. Salimi, "On $\alpha-\psi$-Meir-Keeler contractive mappings," *Fixed Point Theory and Applications*, vol. 2013, article 94, 2013.

[24] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2003.

[25] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[26] J. Jachymski, "The contraction principle for mappings on a metric space with a graph," *Proceedings of the American Mathematical Society*, vol. 136, no. 4, pp. 1359–1373, 2008.

[27] I. Beg, A. R. Butt, and S. Radojević, "The contraction principle for set valued mappings on a metric space with a graph," *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1214–1219, 2010.

[28] F. Bojor, "Fixed point theorems for Reich type contractions on metric spaces with a graph," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 9, pp. 3895–3901, 2012.

[29] F. Bojor, "Fixed point of ϕ-contraction in metric spaces endowed with a graph," *Annals of the University of Craiova. Mathematics and Computer Science Series*, vol. 37, no. 4, pp. 85–92, 2010.

[30] R. P. Agarwal, N. Hussain, and M. A. Taoudi, "Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations," *Abstract and Applied Analysis*, vol. 2012, Article ID 245872, 15 pages, 2012.

[31] N. Hussain, A. R. Khan, and R. P. Agarwal, "Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces," *Journal of Nonlinear and Convex Analysis*, vol. 11, no. 3, pp. 475–489, 2010.