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Research Article

Bifurcation of Limit Cycles by Perturbing a Piecewise Linear Hamiltonian System

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This paper concerns limit cycle bifurcations by perturbing a piecewise linear Hamiltonian system. We first obtain all phase portraits of the unperturbed system having at least one family of periodic orbits. By using the first-order Melnikov function of the piecewise near-Hamiltonian system, we investigate the maximal number of limit cycles that bifurcate from a global center up to first order of ε.

1. Introduction and Main Results

Recently, piecewise smooth dynamical systems have been well concerned, especially in the scientific problems and engineering applications. For example, see the works of Filippov [1], Kunze [2], di Bernardo et al. [3], and the references therein. Because of the variety of the nonsmoothness, there can appear many complicated phenomena in piecewise smooth dynamical systems such as stability (see [4, 5]), chaos (see [6]), and limit cycle bifurcation (see [7–10]). Here, we are more concerned with bifurcation of limit cycles in a perturbed piecewise linear Hamiltonian system:

$$\begin{aligned} \dot{x} &= y + \varepsilon p(x, y, \delta), \\ \dot{y} &= -g(x) + \varepsilon q(x, y, \delta), \end{aligned} \tag{1}$$

where $\varepsilon > 0$ is a sufficiently small real parameter,

$$g(x) = \begin{cases} a_1 x + a_0, & x \geq 0, \\ b_1 x + b_0, & x < 0 \end{cases} \tag{2}$$

with $a_1, a_0, b_1,$ and b_0 real numbers satisfying $a_1^2 + a_0^2 \neq 0, b_1^2 + b_0^2 \neq 0$,

$$p(x, y, \delta) = \begin{cases} p^+(x, y, \delta) = \sum_{i+j=0}^{n} a_{ij}^+ x^i y^j, & x \geq 0, \\ p^-(x, y, \delta) = \sum_{i+j=0}^{n} a_{ij}^- x^i y^j, & x < 0, \end{cases} \tag{3}$$

$$q(x, y, \delta) = \begin{cases} q^+(x, y, \delta) = \sum_{i+j=0}^{n} b_{ij}^+ x^i y^j, & x \geq 0, \\ q^-(x, y, \delta) = \sum_{i+j=0}^{n} b_{ij}^- x^i y^j, & x < 0, \end{cases} \tag{4}$$

and $\delta = (a_{ij}^+, a_{ij}^-, b_{ij}^+, b_{ij}^-) \in D \subset \mathbb{R}^{2(n+1)(n+2)}$ with D compact. Then system (1) has two subsystems

$$\begin{aligned} \dot{x} &= y + \varepsilon p^+(x, y, \delta), \\ \dot{y} &= -a_1 x - a_0 + \varepsilon q^+(x, y, \delta), \end{aligned} \tag{5a}$$

$$\begin{aligned} \dot{x} &= y + \varepsilon p^-(x, y, \delta), \\ \dot{y} &= -b_1 x - b_0 + \varepsilon q^-(x, y, \delta), \end{aligned} \tag{5b}$$

which are called the right subsystem and the left subsystem, respectively. For $\varepsilon = 0$, systems (5a) and (5b) are Hamiltonian with the Hamiltonian functions, respectively,

$$H^+(x,y) = \frac{1}{2}y^2 + \frac{1}{2}a_1x^2 + a_0x,$$
$$H^-(x,y) = \frac{1}{2}y^2 + \frac{1}{2}b_1x^2 + b_0x. \tag{6}$$

Note that the phase portrait of the linear system

$$\dot{x} = y,$$
$$\dot{y} = -ax - b, \tag{7}$$

with $a^2 + b^2 \neq 0$ has possibly the following four different phase portraits on the plane (see Figure 1).

Then, one can find that system (1) $|_{\varepsilon=0}$ can have 13 different phase portraits (see Figure 2) when at least one family of periodic orbits appears.

We remark that in Figure 2,

GC: global center,

Ho: homoclinic,

He: heteroclinic,

C^+: center in the region $\{(x,y) \mid x > 0\}$,

C^-: center in the region $\{(x,y) \mid x < 0\}$,

S^+: saddle in the region $\{(x,y) \mid x > 0\}$,

S^-: saddle in the region $\{(x,y) \mid x < 0\}$,

L^+: curvilinear or straightline in the region $\{(x,y) \mid x > 0\}$,

L^-: curvilinear or straightline in the region $\{(x,y) \mid x < 0\}$.

It is easy to obtain the following Table 1 which shows conditions for each possible phase portrait appearing above. Also, cases (3), (5), (7), (9), and (13) in Figure 2 are equivalent to cases (2), (6), (8), (10), and (12), respectively, by making the transformation

$$(x,y) \longrightarrow (-x,y), \tag{8}$$

together with time rescaling $dt = -d\tau$.

The authors Liu and Han [7] studied system (1) in a subcase of the case (1) of Figure 2 by taking $a_1 = b_1 = 1$, $a_0 = b_0 = 0$. By using the first order Melnikov function, they proved that the maximal number of limit cycles on Poincaré bifurcations is n up to first-order in ε. The authors Liang et al. [8] considered system (1) in the case (5) of Figure 2 by taking $a_1 = -1$, $a_0 = 1$, $b_1 = 1$, and $b_0 = 0$. By using the same method, they gave lower bounds of the maximal number of limit cycles in Hopf, and Homoclinic bifurcations, and derived an upper bound of the maximal number of limit cycles bifurcating from the periodic annulus between the center and the Homoclinic loop up to the first-order in ε. Clearly, the maximal number of limit cycles in the case (7) or (8) of Figure 2 is $[(n-1)/2]$ on Poincaré, Hopf and Homoclinic bifurcations up to first-order in ε, by using the first order Melnikov function.

This paper focuses on studying the limit cycle bifurcations of system (1) in the case (1) of Figure 2 by using the first order Melnikov function. That is, system (1) satisfies

$$a_1 \geq 0, \quad a_0 \geq 0, \quad a_0 + a_1 > 0,$$
$$b_1 \geq 0, \quad b_0 \leq 0, \quad b_0 < b_1. \tag{9}$$

Clearly, system (1) $|_{\varepsilon=0}$ satisfying (9) has a family of periodic orbits

$$\begin{aligned} L_h &= L_h^+ \cup L_h^- \\ &= \{(x,y) \mid H^+(x,y) = h\} \\ &\quad \cup \{(x,y) \mid H^-(x,y) = h\}, \quad h > 0, \end{aligned} \tag{10}$$

such that the limit of L_h as $h \to 0^+$ is the origin. The intersection points of the closed curve L_h with the positive y-axis and the negative y-axis are denoted by $A(h) = (0, \sqrt{2h})$ and $A_1(h) = (0, -\sqrt{2h})$, respectively. Let

$$M^+(h,\delta) = \int_{\widehat{AA_1}} q^+dx - p^+dy,$$
$$M^-(h,\delta) = \int_{\widehat{A_1A}} q^-dx - p^-dy, \quad h > 0. \tag{11}$$

Then, from Liu and Han [7], the first-order Melnikove function corresponding to system (1) is

$$M(h,\delta) = M^+(h,\delta) + M^-(h,\delta), \quad h \in (0, +\infty). \tag{12}$$

Let $Z(n)$ denote the maximal number of zeros of $M(h,\delta)$ for $h > 0$ and $N(n)$ the cyclicity of system (1) at the origin. Then, we can obtain the following.

Theorem 1. *Let* (9) *be satisfied. For any given* $n \geq 1$, *one has Table 2.*

This paper is organized as follows. In Section 2, we will provide some preliminary lemmas, which will be used to prove the main results. In Section 3, we present the proof of Theorem 1.

2. Preliminary Lemmas

In this section, we will derive expressions of $M^+(h,\delta)$, $M^-(h,\delta)$ in (11). First, we have the following.

Lemma 2. *Suppose system* (1) *satisfies* (9). *Then,*

(i) $M^+(h,\delta)$ *in* (11) *can be written as*

$$M^+(h,\delta) = M_1^+(h,\delta) + \sum_{2k=0}^{n} \frac{2^{k+1+1/2}}{2k+1} a_{0,2k}^+ h^{k+1/2}, \tag{13}$$

where

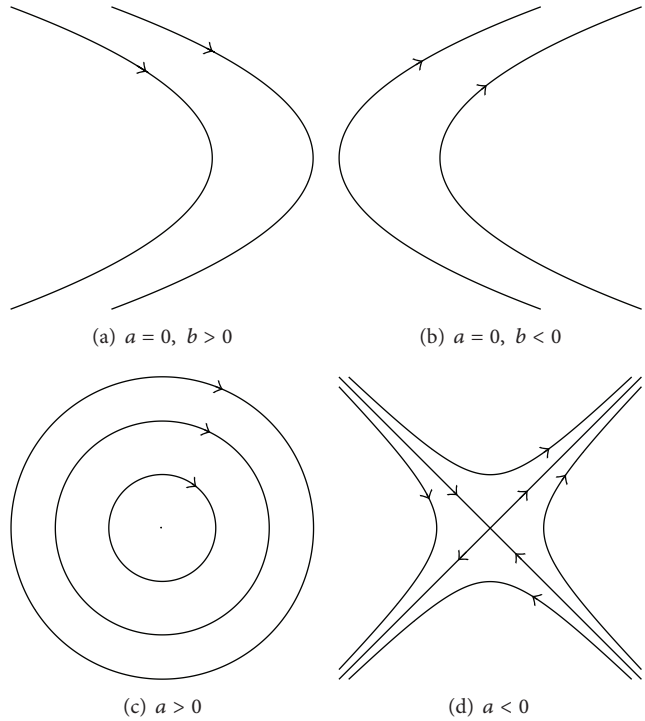
$$M_1^+(h,\delta) = \sum_{i+j=0}^{n-1} \overline{p}_{ij}^+ \int_{\widehat{A_1A}} x^{i+1}y^j dy = \sum_{i+j=0}^{n-1} \overline{q}_{ij}^+ \int_{\widehat{A_1A}} x^i y^{j+1} dx, \tag{14}$$

Table 1: Coefficient conditions for phase portraits (1)–(13).

Coefficient conditions	$a_1 \geq 0,\ a_0 \geq 0$ $a_1 + a_0 > 0$	$a_1 > 0$ $a_0 < 0$	$a_1 \leq 0,\ a_0 \leq 0$ $a_1 + a_0 < 0$	$a_1 < 0$ $a_0 > 0$
$b_1 \geq 0,\ b_0 \leq 0,\ b_1 > b_0$	(1)	(2)		(5)
$b_1 > 0,\ b_0 > 0$	(3)	(4)	(7)	(9)
$b_1 \leq 0,\ b_0 \geq 0,\ b_0 > b_1$		(8)		
				(11) $(a_0^2 b_1 = a_1 b_0^2)$,
$b_1 < 0,\ b_0 < 0$	(6)	(10)		(12) $(a_0^2 b_1 > a_1 b_0^2)$,
				(13) $(a_0^2 b_1 < a_1 b_0^2)$

Table 2

	$a_1 > 0,\ a_0 = 0$	$a_1 = 0,\ a_0 > 0$	$a_1 > 0,\ a_0 > 0$
$b_1 > 0,$	$Z(n) = n,$	$Z(n) = n + \left[\frac{n+1}{2}\right],$	$Z(n) = n + \left[\frac{n+1}{2}\right],$
$b_0 = 0$	$N(n) \geq n$	$N(n) \geq n + \left[\frac{n+1}{2}\right]$	$N(n) \geq n + \left[\frac{n+1}{2}\right]$
$b_1 = 0,$	$Z(n) = n + \left[\frac{n+1}{2}\right],$	$Z(n) = n,$	$n + \left[\frac{n+1}{2}\right] \leq Z(n) \leq n + 2\left[\frac{n+1}{2}\right]$
$b_0 < 0$	$N(n) \geq n + \left[\frac{n+1}{2}\right]$	$N(n) \geq n$	$N(n) \geq n + \left[\frac{n+1}{2}\right],$
$b_1 > 0,$	$Z(n) = n + \left[\frac{n+1}{2}\right],$	$n + [\frac{n+1}{2}] \leq Z(n) \leq n + 2\left[\frac{n+1}{2}\right]$	$n \leq Z(n) \leq n + \left[\frac{n+1}{2}\right],\ \frac{a_0^2}{a_1} = \frac{b_0^2}{b_1},$
$b_0 < 0$	$N(n) \geq n + \left[\frac{n+1}{2}\right]$	$N(n) \geq n + \left[\frac{n+1}{2}\right],$	$n \leq Z(n) \leq n + 2\left[\frac{n+1}{2}\right],\ \frac{a_0^2}{a_1} \neq \frac{b_0^2}{b_1}$
			$N(n) \geq n,$



(a) $a = 0,\ b > 0$ (b) $a = 0,\ b < 0$

(c) $a > 0$ (d) $a < 0$

Figure 1: The possible phase portraits of system (7).

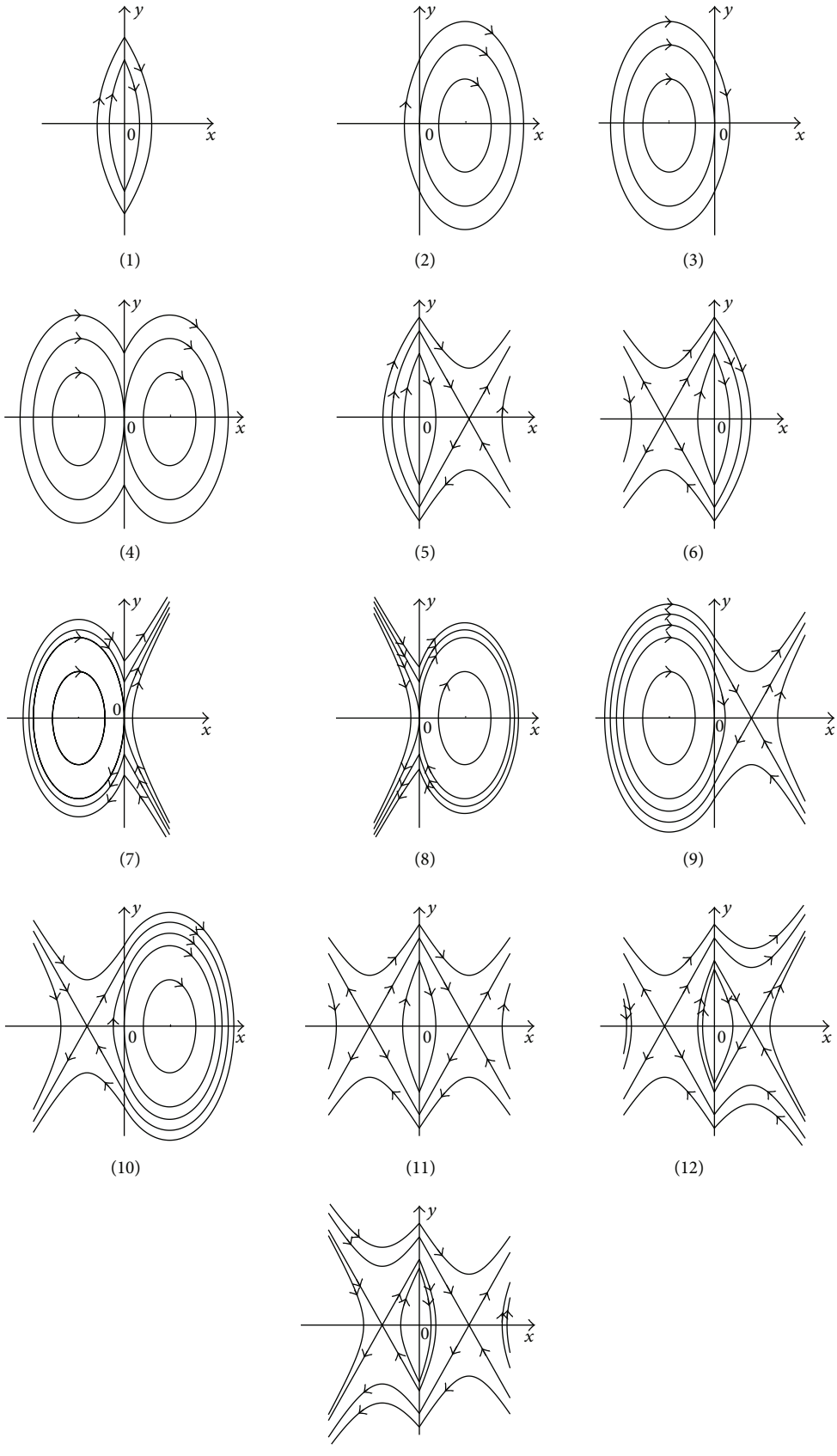


Figure 2: The possible phase portraits of system (1) $|_{\varepsilon=0}$. (1) GC. (2) HoC^{+}. (3) HoC^{-}. (4) $\mathrm{HoC}^{-}\mathrm{C}^{+}$. (5) HoS^{+}. (6) HoS^{-}. (7) $\mathrm{HoC}^{-}\mathrm{L}^{+}$. (8) $\mathrm{HoC}^{+}\mathrm{L}^{-}$. (9) $\mathrm{HoC}^{-}\mathrm{HoS}^{+}$. (10) $\mathrm{HoC}^{+}\mathrm{HoS}^{-}$. (11) $\mathrm{HeS}^{-}\mathrm{S}^{+}$. (12) HeS^{+}. (13) HeS^{-}.

with

$$\overline{p}_{ij}^{+} = a_{i+1,j}^{+} + \frac{j+1}{i+1} b_{i,j+1}^{+}, \qquad \overline{q}_{ij}^{+} = b_{i,j+1}^{+} + \frac{i+1}{j+1} a_{i+1,j}^{+}. \tag{15}$$

(ii) (h,δ) *in* (11) *can be expressed as*

$$M^{-}(h,\delta) = M_1^{-}(h,\delta) - \sum_{2k=0}^{n} \frac{2^{k+1+1/2}}{2k+1} a_{0,2k}^{-} h^{k+1/2}, \tag{16}$$

where

$$M_1^{-}(h,\delta) = \sum_{i+j=0}^{n-1} \overline{p}_{ij}^{-} \int_{\widehat{A_1 A}} x^{i+1} y^{j} dy = \sum_{i+j=0}^{n-1} \overline{q}_{ij}^{-} \int_{\widehat{A_1 A}} x^{i} y^{j+1} dx, \tag{17}$$

with

$$\overline{p}_{ij}^{-} = a_{i+1,j}^{-} + \frac{j+1}{i+1} b_{i,j+1}^{-}, \qquad \overline{q}_{ij}^{-} = b_{i,j+1}^{-} + \frac{i+1}{j+1} a_{i+1,j}^{-}. \tag{18}$$

Proof. We only prove (i) since (ii) can be verified in a similar way. By (11), we obtain

$$\begin{aligned} M^{+}(h,\delta) = & \int_{\widehat{AA_1}} q^{+}(x,y,\delta)\,dx - p^{+}(x,y,\delta)\,dy \\ & + \int_{\overrightarrow{A_1 A}} q^{+}(x,y,\delta)\,dx - p^{+}(x,y,\delta)\,dy \\ & - \int_{\overrightarrow{A_1 A}} q^{+}(x,y,\delta)\,dx - p^{+}(x,y,\delta)\,dy \\ = & \oint_{\widehat{AA_1} \cup \overrightarrow{A_1 A}} q^{+}(x,y,\delta)\,dx - p^{+}(x,y,\delta)\,dy \\ & + \int_{\overrightarrow{A_1 A}} p^{+}(0,y,\delta)\,dy, \end{aligned} \tag{19}$$

which follows that by Green formula and (3)

$$M^{+}(h,\delta) = M_1^{+}(h,\delta) + \sum_{2k=0}^{n} \frac{2^{k+1+1/2}}{2k+1} a_{0,2k}^{+} h^{k+1/2}, \tag{20}$$

where

$$M_1^{+}(h,\delta) = \iint_{\operatorname{int} \widehat{AA_1} \cup \overrightarrow{A_1 A}} \left(p_x^{+} + q_y^{+}\right) dx\,dy. \tag{21}$$

Then, by Green formula again

$$\begin{aligned} M_1^{+}(h,\delta) = & -\oint_{\widehat{AA_1} \cup \overrightarrow{A_1 A}} \overline{p}^{+}(x,y,\delta)\,dy \\ = & \oint_{\widehat{AA_1} \cup \overrightarrow{A_1 A}} \overline{q}^{+}(x,y,\delta)\,dx, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \overline{p}^{+}(x,y,\delta) &= p^{+}(x,y,\delta) - p^{+}(0,y,\delta) + \int_0^x q_y^{+}(u,y,\delta)\,du, \\ \overline{q}^{+}(x,y,\delta) &= q^{+}(x,y,\delta) - q^{+}(x,0,\delta) + \int_0^y p_x^{+}(x,v,\delta)\,dv. \end{aligned} \tag{23}$$

By (3), (4), and the above formulas, we have

$$\begin{aligned} \overline{p}^{+}(x,y,\delta) &= \sum_{i+j=0}^{n} a_{ij}^{+} x^{i} y^{j} - \sum_{j=0}^{n} a_{0j}^{+} y^{j} + \sum_{i+j=1}^{n} \frac{j}{i+1} b_{ij}^{+} x^{i+1} y^{j-1} \\ &= x \sum_{i+j=0}^{n-1} \left(a_{i+1,j}^{+} + \frac{j+1}{i+1} b_{i,j+1}^{+}\right) x^{i} y^{j} \\ &= x \sum_{i+j=0}^{n-1} \overline{p}_{ij}^{+} x^{i} y^{j}, \end{aligned} \tag{24}$$

$$\begin{aligned} \overline{q}^{+}(x,y,\delta) &= \sum_{i+j=0}^{n} b_{ij}^{+} x^{i} y^{j} - \sum_{j=0}^{n} b_{i0}^{+} x^{i} + \sum_{i+j=1}^{n} \frac{i}{j+1} a_{ij}^{+} x^{i-1} y^{j+1} \\ &= y \sum_{i+j=0}^{n-1} \left(b_{i,j+1}^{+} + \frac{i+1}{j+1} a_{i+1,j}^{+}\right) x^{i} y^{j} \\ &= y \sum_{i+j=0}^{n-1} \overline{q}_{ij}^{+} x^{i} y^{j}. \end{aligned} \tag{25}$$

Combining (20)–(25) gives (13) and (14). Thus, the proof is ended. □

Then, using Lemma 2 and (6) we can obtain the following three lemmas.

Lemma 3. (i) *If* $a_1 = 0,\ a_0 > 0$, *then* $M^{+}(h,\delta)$ *in* (11) *has form*

$$M^{+}(h,\delta) = h^{1/2} \sum_{i+2k=0}^{n} B_{i,2k}^{+} h^{i+k}, \tag{26}$$

where

$$\begin{aligned} B_{0,2k}^{+} &= \frac{2^{k+1+1/2} a_{0,2k}^{+}}{2k+1}, \\ B_{i,2k}^{+} &= \frac{2^{k+1+1/2}}{a_0^{i}} \left(a_{i,2k}^{+} + \frac{2k+1}{i} b_{i-1,2k+1}^{+}\right) \\ &\quad \times \int_0^{\pi/2} \sin^{2k}\theta \cos^{i+1}\theta\,d\theta, \quad 1 \le i \le n. \end{aligned} \tag{27}$$

(ii) *If* $b_1 = 0,\ b_0 < 0$, *then we have*

$$M^{-}(h,\delta) = h^{1/2} \sum_{i+2k=0}^{n} B_{i,2k}^{-} h^{i+k}, \tag{28}$$

where

$$B_{0,2k}^{-} = -\frac{2^{k+1+1/2}a_{0,2k}^{-}}{2k+1},$$
$$B_{i,2k}^{-} = \frac{-2^{k+1+1/2}}{b_0^i}\left(a_{i,2k}^{-} + \frac{2k+1}{i}b_{i-1,2k+1}^{-}\right) \times \int_0^{\pi/2} \sin^{2k}\theta\cos^{2i+1}\theta d\theta, \quad 1 \le i \le n. \tag{29}$$

Proof. Note that along $\widehat{AA_1}$, $x = h/a_0 - (1/2a_0)y^2$. Then, inserting it into (14) follows that

$$M_1^{+}(h,\delta) = \sum_{i+j=0}^{n-1} \overline{p}_{ij}^{+} \int_{-\sqrt{2h}}^{\sqrt{2h}} \left(\frac{h}{a_0} - \frac{1}{2a_0}y^2\right)^{i+1} y^j dy = \sum_{i+2k=0}^{n-1} \frac{\overline{p}_{i,2k}^{+}}{2^i a_0^{i+1}} \int_0^{\sqrt{2h}} \left(2h - y^2\right)^{i+1} y^{2k} dy. \tag{30}$$

Let $y = \sqrt{2h}\sin\theta$. Then we have $dy = \sqrt{2h}\cos\theta\, d\theta$ and the above integral can be carried into

$$\int_0^{\sqrt{2h}} \left(2h - y^2\right)^{i+1} y^{2k} dy = (2h)^{i+1+k+(1/2)} \times \int_0^{\pi/2} \sin^{2k}\theta\cos^{2(i+1)+1}\theta\, \mathrm{d}\theta. \tag{31}$$

Thus, using (30) and the above equation we can write (13) as

$$M^{+}(h,\delta) = \sum_{i+2k=0}^{n-1} \frac{2^{k+1+1/2}\overline{p}_{i,2k}^{+}}{a_0^{i+1}} \times \int_0^{\pi/2} \sin^{2k}\theta\cos^{2(i+1)+1}\theta d\theta \times h^{i+1+k+1/2} + \sum_{2k=0}^{n} \frac{2^{k+1+1/2}a_{0,2k}^{+}}{2k+1} h^{k+1/2} = h^{1/2} \sum_{i+2k=0}^{n} B_{i,2k}^{+} h^{i+k}, \tag{32}$$

where

$$B_{0,2k}^{+} = \frac{2^{k+1+1/2}a_{0,2k}^{+}}{2k+1},$$
$$B_{i,2k}^{+} = \frac{2^{k+1+1/2}\overline{p}_{i-1,2k}^{+}}{a_0^i} \int_0^{\pi/2} \sin^{2k}\theta\cos^{2i+1}\theta\, d\theta, \quad 1 \le i \le n, \tag{33}$$

which gives (i) by (15). Thus, (i) holds and we can prove (ii) in the same way by (16)–(18). This ends the proof. □

Lemma 4. *Let system* (5a) *satisfy* (3) *and* (4). *Then*

(i) *If* $a_1 > 0$, $a_0 = 0$, $M^{+}(h,\delta)$ *has the expression*

$$M^{+}(h,\delta) = \sqrt{h} \sum_{i+2k=0}^{n} A_{i,2k}^{+}\left(\sqrt{h}\right)^{i+2k}, \tag{34}$$

where

$$A_{0,2k}^{+} = \frac{2\left(\sqrt{2}\right)^{2k+1}a_{0,2k}^{+}}{2k+1},$$
$$A_{i,2k}^{+} = \frac{2\left(\sqrt{2}\right)^{2k+1+i}}{\left(\sqrt{a_1}\right)^i}\left(b_{i-1,2k+1}^{+} + \frac{i}{2k+1}a_{i,2k}^{+}\right) \times \int_0^{\pi/2} \sin^{i-1}\theta\cos^{2k+2}\theta\, d\theta, \quad i \ge 1. \tag{35}$$

(ii) *If* $a_1 > 0$, $a_0 \neq 0$, $M^{+}(h,\delta)$ *can be written as*

$$M^{+}(h,\delta) = \sqrt{h}\left[\sum_{i+2k=0}^{n-1}\left(b_{i,2k+1}^{+} + \frac{i+1}{2k+1}a_{i+1,2k}^{+}\right)\phi_{ik}^{+}(h) + \sum_{2k=0}^{n}\frac{2^{k+1+1/2}a_{0,2k}^{+}}{2k+1}h^k\right] + \sum_{i+2k=0}^{n-1}\left(b_{i,2k+1}^{+} + \frac{i+1}{2k+1}a_{i+1,2k}^{+}\right) \times \sum_{r=0,\ r\ even}^{i}\alpha_{irk}^{+}\left(2h + \frac{a_0^2}{a_1}\right)^{k+r/2}\overline{I}_{00}^{+}(h,\delta), \tag{36}$$

or

$$M^{+}(h,\delta) = \sqrt{h}\psi_{[n/2]}^{+}(h,\delta) + \left(2h + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+}\left(2h + \frac{a_0^2}{a_1},\delta\right) \times \left(\frac{\pi}{2} - \arcsin\frac{a_0}{\sqrt{2a_1h + a_0^2}}\right), \tag{37}$$

where

$$\overline{I}_{00}^{+}(h,\delta) = \int_{a_0/\sqrt{a_1}}^{\sqrt{2h + a_0^2/a_1}} \sqrt{2h + \frac{a_0^2}{a_1} - v^2}\, dv, \tag{38}$$

each α_{irk}^{+} *is a nonzero constant and* ϕ_{ik}^{+}, $\psi_{[n/2]}^{+}$, $\varphi_{[(n-1)/2]}^{+}$ *are polynomials of degree* $k + [(i+1)/2]$, $[n/2]$, $[(n-1)/2]$, *respectively.*

Proof. Since $y = \pm\sqrt{2h - a_1x^2 - 2a_0x}$ along the curve $\widehat{AA_1}$, $M_1^+(h,\delta)$ in (14) becomes

$$\begin{aligned} M_1^+(h,\delta) &= \sum_{i+j=0}^{n-1} \overline{q}_{ij}^+ \int_{\widehat{AA_1}} x^i y^{j+1} dx \\ &= \sum_{i+j=0}^{n-1} \overline{q}_{ij}^+ \left[\int_0^{(-a_0+\sqrt{2a_1h+a_0^2})/a_1} x^i \left(2h - a_1x^2 - 2a_0x\right)^{(j+1)/2} dx \right] \\ &\quad + \int_{(-a_0+\sqrt{2a_1h+a_0^2})/a_1}^{0} x^i (-1)^{j+1} \left(2h - a_1x^2 - 2a_0x\right)^{(j+1)/2} dx \\ &= \sum_{i+2k=0}^{n-1} 2\overline{q}_{i,2k}^+ I_{i,2k}^+(h,\delta), \end{aligned} \tag{39}$$

where

$$\begin{aligned} I_{i,2k}^+(h,\delta) &= \int_0^{(-a_0+\sqrt{2a_1h+a_0^2})/a_1} x^i \left[2h + \frac{a_0^2}{a_1} - a_1\left(x + \frac{a_0}{a_1}\right)^2 \right]^{k+1/2} dx. \end{aligned} \tag{40}$$

Let $v = \sqrt{a_1}(x + a_0/a_1)$. Then, we have $dv = \sqrt{a_1}dx$ and the above equation becomes

$$\begin{aligned} I_{i,2k}^+(h,\delta) &= \frac{1}{\left(\sqrt{a_1}\right)^{i+1}} \int_{a_0/\sqrt{a_1}}^{\sqrt{2h+a_0^2/a_1}} \left(v - \frac{a_0}{\sqrt{a_1}}\right)^i \\ &\quad \times \left(2h + \frac{a_0^2}{a_1} - v^2\right)^{k+(1/2)} dv. \end{aligned} \tag{41}$$

For $a_0 = 0$, make the transformation $v = \sqrt{2h}\sin\theta$. Then, we have by (41)

$$I_{i,2k}^+(h,\delta) = \frac{\left(\sqrt{2h}\right)^{2k+1+i+1}}{\left(\sqrt{a_1}\right)^{i+1}} \int_0^{\pi/2} \sin^i\theta\cos^{2k+2}\theta\, d\theta. \tag{42}$$

Substituting the above formula into (37), together with (13), gives that

$$\begin{aligned} M^+(h,\delta) &= \sum_{i+2k=0}^{n-1} \frac{2\overline{q}_{i,2k}^+}{\left(\sqrt{a_1}\right)^{i+1}} \\ &\quad \times \int_0^{\pi/2} \sin^i\theta\cos^{2k+2}\theta\, d\theta \times \left(\sqrt{2h}\right)^{2k+1+i+1} \\ &\quad + \sum_{2k=0}^{n} \frac{2a_{0,2k}^+}{2k+1}\left(\sqrt{2h}\right)^{2k+1} \\ &= \sum_{i+2k=0}^{n} A_{i,2k}^+ \left(\sqrt{h}\right)^{2k+1+i}, \end{aligned} \tag{43}$$

where

$$\begin{aligned} A_{0,2k}^+ &= \frac{2\left(\sqrt{2}\right)^{2k+1} a_{0,2k}^+}{2k+1}, \\ A_{i,2k}^+ &= \frac{2\left(\sqrt{2}\right)^{2k+1+i} \overline{q}_{i-1,2k}^+}{\left(\sqrt{a_1}\right)^i} \\ &\quad \times \int_0^{\pi/2} \sin^{i-1}\theta\cos^{2k+2}\theta\, d\theta, \quad 1 \le i \le n. \end{aligned} \tag{44}$$

Thus, by (15) and the above discussion we know that (i) holds.

For $a_0 \neq 0$, (41) can be represented as

$$I_{i,2k}^+(h,\delta) = \frac{1}{\left(\sqrt{a_1}\right)^{i+1}} \sum_{r=0}^{i} C_i^r \left(-\frac{a_0}{\sqrt{a_1}}\right)^{i-r} \overline{I}_{rk}^+(h,\delta), \tag{45}$$

where

$$\overline{I}_{rk}^+(h,\delta) = \int_{a_0/\sqrt{a_1}}^{\sqrt{2h+a_0^2/a_1}} v^r \left(2h + \frac{a_0^2}{a_1} - v^2\right)^{k+1/2} dv. \tag{46}$$

Recall that

$$\begin{aligned} &\int v^r \left(2h + \frac{a_0^2}{a_1} - v^2\right)^{k+1/2} dv \\ &\quad = \frac{v^{r+1}\left(2h + a_0^2/a_1 - v^2\right)^{k+1/2}}{2k+2+r} \\ &\qquad + \frac{(2k+1)\left(2h + a_0^2/a_1\right)}{2k+2+r} \\ &\qquad \times \int v^r \left(2h + \frac{a_0^2}{a_1} - v^2\right)^{k-1/2} dv. \end{aligned} \tag{47}$$

Then, by (46) and the above equation we obtain that

$$\begin{aligned} \overline{I}_{rk}^+(h,\delta) &= -\frac{a_0^{r+1}(2h)^{k+1/2}}{a_1^{(r+1)/2}(2k+2+r)} \\ &\quad + \frac{(2k+1)\left(2h+a_0^2/a_1\right)}{2k+2+r} \overline{I}_{r,k-1}^+(h,\delta), \quad k \ge 1,\ r \ge 0. \end{aligned} \tag{48}$$

It follows that

$$\begin{aligned} \overline{I}_{rk}^+(h,\delta) &= -\frac{a_0^{r+1}}{a_1^{(r+1)/2}} \sqrt{2h}\,\widetilde{\varphi}_{rk}^+(h) \\ &\quad + \widetilde{\alpha}_{rk}^+ \left(2h + \frac{a_0^2}{a_1}\right)^k \overline{I}_{r0}^+(h,\delta), \quad k \ge 1,\ r \ge 0, \end{aligned} \tag{49}$$

where

$$\widetilde{\alpha}_{rk}^{+} = \frac{(2k+1)(2k-1)(2k-3)\times\cdots\times 3}{(2k+2+r)(2k+r)(2k-2+r)\times\cdots\times(4+r)},$$

$$\begin{aligned}\widetilde{\varphi}_{rk}^{+}(h) = {} & \frac{(2h)^k}{2k+2+r} + \frac{2k+1}{(2k+2+r)(2k+r)} \\ & \times\left(2h+\frac{a_0^2}{a_1}\right)(2h)^{k-1} \\ & + \frac{(2k+1)(2k-1)}{(2k+2+r)(2k+r)(2k-2+r)} \\ & \times\left(2h+\frac{a_0^2}{a_1}\right)^2(2h)^{k-2}+\cdots \\ & + \frac{(2k+1)(2k-1)\times\cdots\times 5}{(2k+2+r)(2k+r)(2k-2+r)\times\cdots\times(4+r)} \\ & \times\left(2h+\frac{a_0^2}{a_1}\right)^{k-1}2h,\end{aligned} \tag{50}$$

which is a polynomial of degree k in h. For convenience, introduce

$$\overline{\varphi}_{rk}^{+}(h) = \begin{cases}\widetilde{\varphi}_{rk}^{+}(h), & k\geq 1,\\ 0, & k=0,\end{cases} \qquad \overline{\alpha}_{rk}^{+} = \begin{cases}\widetilde{\alpha}_{rk}^{+}, & k\geq 1,\\ 1, & k=0.\end{cases} \tag{51}$$

Then, combining (49) and (51) gives that

$$\begin{aligned}\overline{I}_{rk}^{+}(h,\delta) = {} & -\frac{a_0^{r+1}}{a_1^{(r+1)/2}}\sqrt{2h}\overline{\varphi}_{rk}^{+}(h) \\ & + \overline{\alpha}_{rk}^{+}\left(2h+\frac{a_0^2}{a_1}\right)^k \overline{I}_{r0}^{+}(h,\delta), \quad k\geq 0,\ r\geq 0.\end{aligned} \tag{52}$$

Further, by using the formula

$$\begin{aligned}\int v^r\left(2h+\frac{a_0^2}{a_1}-v^2\right)^{1/2}dv = {} & \frac{-v^{r-1}\left(2h+a_0^2/a_1-v^2\right)^{3/2}}{r+2} \\ & + \frac{(r-1)\left(2h+a_0^2/a_1\right)}{r+2} \\ & \times\int v^{r-2}\left(2h+\frac{a_0^2}{a_1}-v^2\right)^{1/2}dv,\end{aligned} \tag{53}$$

we have that

$$\begin{aligned}\overline{I}_{r0}^{+}(h,\delta) = {} & \frac{a_0^{r-1}(2h)^{3/2}}{a_1^{(r-1)/2}(r+2)} \\ & + \frac{r-1}{r+2}\left(2h+\frac{a_0^2}{a_1}\right)\overline{I}_{r-2,0}^{+}(h,\delta), \quad r\geq 1.\end{aligned} \tag{54}$$

It follows that

$$\begin{aligned}\overline{I}_{r0}^{+}(h,\delta) = {} & (2h)^{3/2}\widetilde{\varphi}_r^{+}(h) \\ & + \widetilde{\alpha}_r^{+}\left(2h+\frac{a_0^2}{a_1}\right)^{r/2}\overline{I}_{00}(h,\delta), \quad r\geq 1,\end{aligned} \tag{55}$$

where

$$\widetilde{\alpha}_r^{+} = \begin{cases}0, & r \text{ odd},\\ \dfrac{(r-1)(r-3)\times\cdots\times 3\times 1}{(r+2)r(r-2)\times\cdots\times 6\times 4}, & r \text{ even},\end{cases}$$

$$\begin{aligned}\widetilde{\varphi}_r^{+}(h) = {} & \frac{a_0^{r-1}}{(r+2)a_1^{(r-1)/2}} + \frac{(r-1)a_0^{r-3}}{(r+2)ra_1^{(r-3)/2}}\left(2h+\frac{a_0^2}{a_1}\right) \\ & + \frac{(r-1)(r-3)a_0^{r-5}}{(r+2)r(r-2)a_1^{(r-5)/2}}\left(2h+\frac{a_0^2}{a_1}\right)^2+\cdots \\ & + \Bigg((r-1)(r-3)\times\cdots\times\left(r+1-2\left[\frac{(r-1)}{2}\right]\right) \\ & \times a_0^{r-1-2[(r-1)/2]}\Bigg) \\ & \times\Bigg((r+2)r(r-2)\times\cdots\times\left(r+2-2\left[\frac{(r-2)}{2}\right]\right) \\ & \times a_1^{(r-1)/2-[(r-1)/2]}\left[\frac{(r-1)}{2}\right]\Bigg)^{-1} \\ & \times\left(2h+\frac{a_0^2}{a_1}\right)^{[(r-1)/2]},\end{aligned} \tag{56}$$

which is a polynomial of degree $[(r-1)/2]$ in h. Let

$$\overline{\varphi}_r^{+}(h) = \begin{cases}\widetilde{\varphi}_r^{+}(h), & r\geq 1,\\ 0, & r=0,\end{cases} \qquad \overline{\alpha}_r^{+} = \begin{cases}\widetilde{\alpha}_r^{+}, & r\geq 1,\\ 1, & r=0.\end{cases} \tag{57}$$

Then, we have that by (55) and the above

$$\begin{aligned}\overline{I}_{r0}^{+}(h,\delta) = {} & (2h)^{3/2}\overline{\varphi}_r^{+}(h) \\ & + \overline{\alpha}_r^{+}\left(2h+\frac{a_0^2}{a_1}\right)^{r/2}\overline{I}_{00}^{+}(h,\delta), \quad r\geq 0.\end{aligned} \tag{58}$$

Substituting the above equation into (52), one can find that

$$\begin{aligned}\overline{I}_{rk}^{+}(h,\delta) = {} & \sqrt{h}\overline{\psi}_{rk}^{+}(h) + \overline{\alpha}_{rk}\overline{\alpha}_r\left(2h+\frac{a_0^2}{a_1}\right)^{k+r/2} \\ & \times\overline{I}_{00}^{+}(h,\delta), \quad r\geq 0,\ k\geq 0,\end{aligned} \tag{59}$$

where $\overline{\alpha}_r = 0$ for r odd, $\overline{\alpha}_r > 0$ for r even, and

$$\begin{aligned}\overline{\psi}_{rk}^{+}(h,\delta) = {} & -\frac{\sqrt{2}a_0^{r+1}}{a_1^{(r+1)/2}}\overline{\varphi}_{rk}^{+}(h) \\ & + 2\sqrt{2}h\overline{\alpha}_{rk}\left(2h+\frac{a_0^2}{a_1}\right)^k\overline{\varphi}_r^{+}(h),\end{aligned} \tag{60}$$

which is a polynomial of degree $k + [(r+1)/2]$ in h. Combining (37), (45), and (59) gives that

$$\begin{aligned} M_1^+(h,\delta) = {} & \sqrt{h}\sum_{i+2k=0}^{n-1}\frac{2\overline{q}_{i,2k}^+}{\left(\sqrt{a_1}\right)^{i+1}}\sum_{r=0}^{i}C_i^r\left(-\frac{a_0}{\sqrt{a_1}}\right)^{i-r}\overline{\psi}_{rk}^+(h) \\ & + \sum_{i+2k=0}^{n-1}\frac{2\overline{q}_{i,2k}^+}{\left(\sqrt{a_1}\right)^{i+1}} \\ & \times \sum_{r=0,\ r\,\text{even}}^{i}C_i^r\overline{\alpha}_{rk}\overline{\alpha}_r\left(-\frac{a_0}{\sqrt{a_1}}\right)^{i-r} \\ & \times \left(2h+\frac{a_0^2}{a_1}\right)^{k+r/2}\overline{I}_{00}^+(h,\delta), \end{aligned} \tag{61}$$

which implies (35), together with (13) and (15).

Note that

$$\int\sqrt{a^2-x^2}dx = \frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{|a|}\right]. \tag{62}$$

Then, we have

$$\begin{aligned} \overline{I}_{00}^+(h,\delta) = {} & -\frac{a_0}{2\sqrt{a_1}}\sqrt{2h}+\frac{1}{2}\left(2h+\frac{a_0^2}{a_1}\right) \\ & \times\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right). \end{aligned} \tag{63}$$

Inserting the above formula into (35), we can obtain (36). Hence, the proof is finished. □

Similar to Lemma 4, we can obtain the following lemma about $M^-(h,\delta)$.

Lemma 5. *Let system* (5b) *satisfy* (3) *and* (4). *Then*

(i) *If* $b_1 > 0, b_0 = 0$, $M^-(h,\delta)$ *in* (11) *has the expression*

$$M^-(h,\delta) = \sqrt{h}\sum_{i+2k=0}^{n}A_{i,2k}^-\left(\sqrt{h}\right)^{i+2k}, \tag{64}$$

where

$$\begin{aligned} A_{0,2k}^- &= -\frac{2\left(\sqrt{2}\right)^{2k+1}a_{0,2k}^-}{2k+1}, \\ A_{i,2k}^- &= \frac{2\left(\sqrt{2}\right)^{2k+1+i}}{\left(\sqrt{b_1}\right)^i}\left(b_{i-1,2k+1}^- + \frac{i}{2k+1}a_{i,2k}^-\right) \\ &\quad\times\int_{-\pi/2}^{0}\sin^{i-1}\theta\cos^{2k+2}\theta\,d\theta, \quad 1\le i\le n. \end{aligned} \tag{65}$$

(ii) *If* $b_1 > 0,\ b_0 \neq 0$, $M^-(h,\delta)$ *in* (11) *has the form*

$$\begin{aligned} M^-(h,\delta) = {} & \sqrt{h}\left[\sum_{i+2k=0}^{n-1}\left(b_{i,2k+1}^- + \frac{i+1}{2k+1}a_{i+1,2k}^-\right)\phi_{ik}^-(h)\right. \\ & \left. - \sum_{2k=0}^{n}\frac{2^{k+1+1/2}a_{0,2k}^-}{2k+1}h^k\right] \\ & + \sum_{i+2k=0}^{n-1}\left(b_{i,2k+1}^- + \frac{i+1}{2k+1}a_{i+1,2k}^-\right) \\ & \times\sum_{r=0,\ r\,even}^{i}\alpha_{irk}^-\left(2h+\frac{b_0^2}{b_1}\right)^{k+r/2}\overline{I}_{00}^-(h,\delta), \end{aligned} \tag{66}$$

or

$$\begin{aligned} M^-(h,\delta) = {} & \sqrt{h}\psi_{[n/2]}^-(h,\delta)+\left(2h+\frac{b_0^2}{b_1}\right)\varphi_{[(n-1)/2]}^- \\ & \times\left(2h+\frac{b_0^2}{b_1},\delta\right)\left(\frac{\pi}{2}+\arcsin\frac{b_0}{\sqrt{2b_1h+b_0^2}}\right), \end{aligned} \tag{67}$$

where

$$\overline{I}_{00}^-(h,\delta) = \int_{-\sqrt{2h+b_0^2/b_1}}^{b_0/\sqrt{b_1}}\sqrt{2h+\frac{b_0^2}{b_1}-v^2}dv, \tag{68}$$

each α_{irk}^- *is nonzero constant and* $\phi_{ik}^-, \psi_{[n/2]}^-, \varphi_{[(n-1)/2]}^-$ *are polynomials of degree* $k + [(i+1)/2], [n/2], [(n-1)/2]$, *respectively.*

3. Proof of Theorem 1

In this section, we will prove the main results. Obviously, under (9) there are the following 9 subcases:

(1) $a_1 = b_1 = 0,\ a_0 > 0,\ b_0 < 0$,

(2) $a_1 > 0, b_1 > 0,\ a_0 = b_0 = 0$,

(3) $a_1 > 0, b_0 < 0,\ a_0 = b_1 = 0$,

(4) $a_1 > 0, a_0 > 0,\ b_1 = 0,\ b_0 < 0$,

(5) $a_1 > 0, a_0 > 0,\ b_1 > 0,\ b_0 = 0$,

(6) $a_1 > 0, a_0 > 0,\ b_1 > 0,\ b_0 < 0$,

(7) $a_0 > 0, b_1 > 0,\ a_1 = b_0 = 0$,

(8) $a_1 = 0, a_0 > 0,\ b_1 > 0,\ b_0 < 0$,

(9) $a_1 > 0, a_0 = 0,\ b_1 > 0,\ b_0 < 0$.

We only give the proof of Subcases 1, 2, 3, 4, 5, and 6. And the Subcases 7, 8, and 9 can be verified, similar to Subcases 3, 4, and 5, respectively.

Subcase 1. $a_1 = b_1 = 0,\quad a_0 > 0,\quad b_0 < 0$. From (12) and Lemma 3, one can obtain that

$$\begin{aligned} M(h,\delta) &= M^+(h,\delta) + M^-(h,\delta) \\ &= h^{1/2}\sum_{i+2k=0}^{n} B_{i,2k}^{+} h^{i+k} + h^{1/2}\sum_{i+2k=0}^{n} B_{i,2k}^{-} h^{i+k} \\ &= h^{1/2}\sum_{i+2k=0}^{n}\left(B_{i,2k}^{+} + B_{i,2k}^{-}\right) h^{i+k}, \end{aligned} \tag{69}$$

which implies that $M(h,\delta)$ has at most n isolated positive zeros for $h > 0$. To show that this bound can be reached, take $a_{ij}^{-} = b_{ij}^{\pm} = 0, a_{ij}^{+} = 0,\ j \geq 1$. Then, by (27) and (29), (69) has the form

$$M(h,\delta) = h^{1/2}\sum_{i=0}^{n} B_{i0}^{+} h^{i}, \tag{70}$$

where

$$B_{00}^{+} = 2^{1+1/2} a_{00}^{+}, \qquad B_{i0}^{+} = \frac{2^{1+1/2}}{a_0^i}\frac{(2i)!!}{(2i+1)!!} a_{i0}^{+}, \quad i \geq 1. \tag{71}$$

Hence, using (70) we can take $a_{i0}^{+}, i = 0, 1, \ldots, n$ as free parameters to produce n simple positive zeros of $M(h,\delta)$ near $h = 0$, which gives n limit cycles correspondingly near the origin. Thus, $N(n) \geq n$ in this case. This ends the proof.

Subcase 2. $a_1 > 0,\ b_1 > 0,\ a_0 = b_0 = 0$ Similar to the above and using (32) and (64), $M(h,\delta)$ in (12) has the expression of the form

$$\begin{aligned} M(h,\delta) &= \sum_{i+2k=0}^{n} A_{i,2k}^{+}\left(\sqrt{h}\right)^{2k+1+i} + \sum_{i+2k=0}^{n} A_{i,2k}^{-}\left(\sqrt{h}\right)^{2k+1+i} \\ &= \sqrt{h}\sum_{l=0}^{n} A_l\left(\sqrt{h}\right)^{l}, \end{aligned} \tag{72}$$

where $A_l = \sum_{i+2k=l}(A_{i,2k}^{+} + A_{i,2k}^{-})$. Further, taking $a_{ij}^{-} = b_{ij}^{\pm} = 0,\ a_{ij}^{+} = 0,\ j \geq 1$, then, by (34) and (65), $M(h,\delta)$ in (72) becomes

$$M(h,\delta) = h^{1/2}\sum_{i=0}^{n} A_{i0}^{+}\left(\sqrt{h}\right)^{i}, \tag{73}$$

where

$$\begin{aligned} A_{00}^{+} &= \frac{2a_{00}^{+}}{2k+1}, \\ A_{i0}^{+} &= \frac{2ia_{i0}^{+}}{\left(\sqrt{a_1}\right)^{i}}\int_0^{\pi/2} \sin^{i-1}\theta\cos^2\theta\, d\theta, \quad i \geq 1. \end{aligned} \tag{74}$$

Thus, from (72) and (73), we can discuss similar to Subcase 1. This finishes the proof.

Subcase 3. $a_1 > 0,\ b_0 < 0,\ a_0 = b_1 = 0$ By Lemmas 3, and 4 and (12), we can have that

$$\begin{aligned} M(h,\delta) &= \sum_{i+2k=0}^{n} A_{i,2k}^{+}\left(\sqrt{h}\right)^{2k+1+i} + h^{1/2}\sum_{i+2k=0}^{n} B_{i,2k}^{-} h^{i+k} \\ &= h^{1/2}\left(\sum_{i+2k=0}^{n} A_{i,2k}^{+}\left(\sqrt{h}\right)^{i+2k} + \sum_{i+2k=0}^{n} B_{i,2k}^{-} h^{i+k}\right) \\ &= h^{1/2} M^*(h,\delta). \end{aligned} \tag{75}$$

Let us prove that $M^*(h,\delta)$ has at most $n + [(n+1)/2]$ zeros on the open interval $(0, +\infty)$. For the purpose, let $\sqrt{h} = \lambda$. Then, for $n = 2l,\ l \geq 1$, $M^*(h,\delta)$ in (75) has the expression

$$\begin{aligned} M^*(h,\delta) &= \sum_{i+2k=0}^{n} A_{i,2k}^{+}\lambda^{i+2k} + \sum_{i+2k=0}^{n} B_{i,2k}^{-}\lambda^{2i+2k} \\ &= \sum_{j=0}^{2l} C_j\lambda^j + \sum_{j=1}^{l} C_{2l+j}\lambda^{2j+2l} \triangleq \overline{M}(\lambda,\delta), \end{aligned} \tag{76}$$

where

$$\begin{aligned} C_j &= \sum_{i+2k=j} A_{i,2k}^{+} + \sum_{2i+2k=j} B_{i,2k}^{-}, \quad j = 0, 1, 2, \ldots, 2l, \\ C_{2l+j} &= \sum_{i+k=j+l} B_{i,2k}^{-}, \quad j = 1, 2, \ldots, l. \end{aligned} \tag{77}$$

To prove $M^*(h,\delta)$ has at most $n + [(n+1)/2]$ zeros, it suffices to prove $\overline{M}(\lambda,\delta)$ has at most $n + [(n+1)/2] = 3l$ zeros for $\lambda > 0$. By Rolles theorem we need only to prove that $d^{2l}\overline{M}(\lambda,\delta)/d\lambda^{2l}$ has at most l zeros for $\lambda \in (0, +\infty)$. From (76), we can have that

$$\frac{d^{2l}\overline{M}(\lambda,\delta)}{d\lambda^{2l}} = C_{2l}(2l)! + \sum_{j=1}^{l} C_{2l+j} A_{2l+2j}^{2l}\lambda^{2j}, \tag{78}$$

which shows that $d^{2l}\overline{M}(\lambda,\delta)/d\lambda^{2l}$ has at most l zeros for $\lambda > 0$. Thus, $M(h,\delta)$ has at most $3l$ zeros for $h > 0$. To prove $3l$ zeros can appear, we only need to prove that $M^*(h,\delta)$ in (75) can appear $3l$ zeros for $h > 0$ small. Let $b_{ij}^{\pm} = 0,\ a_{ij}^{\pm} = 0,\ j \geq 1, a_{i0}^{-} = 0,\ 0 \leq i \leq l$, and $a_{2l}^{-} \neq 0$. Then $M^*(h,\delta)$ in (75) can be expressed as by (29) and (34)

$$\begin{aligned} M^*(h,\delta) &= \sum_{i=0}^{2l} A_{i0}^{+}\left(\sqrt{h}\right)^{i} + \sum_{i=l+1}^{2l} B_{i0}^{-}\left(\sqrt{h}\right)^{2i} \\ &= A_{00}^{+} + A_{10}^{+}\sqrt{h} + A_{20}^{+}\left(\sqrt{h}\right)^{2} + \cdots \\ &\quad + A_{2l-1,0}^{+}\left(\sqrt{h}\right)^{2l-1} + A_{2l,0}^{+}\left(\sqrt{h}\right)^{2l} \\ &\quad + B_{l+1,0}^{-}\left(\sqrt{h}\right)^{2l+2} + B_{l+2,0}^{-}\left(\sqrt{h}\right)^{2l+4} \\ &\quad + \cdots + B_{2l,0}^{-}\left(\sqrt{h}\right)^{4l}, \end{aligned} \tag{79}$$

where

$$\begin{aligned} A_{00}^{+} &= 2\sqrt{2}a_{00}^{+}, \\ A_{i0}^{+} &= \frac{2\left(\sqrt{2}\right)^{i+1}}{\left(\sqrt{a_1}\right)^{i}} i a_{i0}^{+}, \quad i = 1, 2, \ldots, 2l, \\ B_{i0}^{-} &= \frac{-2\sqrt{2}}{b_0^{i}} a_{i0}^{-} \int_0^{\pi/2} \cos^{2i+1}\theta d\theta \\ &= \frac{-2\sqrt{2}}{b_0^{i}} \frac{(2i)!!}{(2i+1)!!} a_{i0}^{-}, \quad i = l+1, \ldots, 2l. \end{aligned} \tag{80}$$

Thus, by changing the sign of $a_{2l,0}^{-}, a_{2l-1,0}^{-}, \ldots, a_{l+1,0}^{-}$, $a_{2l,0}^{+}$, $a_{2l-1,0}^{+}, \ldots, a_{00}^{+}$ in turn such that

$$\begin{aligned} a_{i-1,0}^{-} a_{i0}^{-} < 0, &\qquad i = 2l,\ 2l-1, \ldots, l+2, \\ a_{l+1}^{-} a_{2l,0}^{+} > 0, &\qquad a_{i-1,0}^{+} a_{i0}^{+} < 0, \quad i = 2l, 2l-1, \ldots, 1, \\ 0 < \left|a_{00}^{+}\right| &\ll \left|a_{10}^{+}\right| \ll \cdots \ll \left|a_{2l,0}^{+}\right| \\ &\ll \left|a_{l+1}^{-}\right| \ll \left|a_{l+2}^{-}\right| \ll \left|a_{2l-1}^{-}\right| \ll 1, \end{aligned} \tag{81}$$

we can find $3l$ simply positive zeros $h_1, h_2, \ldots, h_{3l}$ with $0 < h_{3l} < h_{3l-1} < \cdots < h_1 \ll 1$. For $n = 2l+1, l = 0, 1, \ldots$, we can discuss in a similar way. Thus, this bound can be reached and $N(n) \geq n + [(n+1)/2]$. The proof is finished.

Subcase 4. $a_1 > 0$, $a_0 > 0$, $b_1 = 0$, $b_0 < 0$ From (12) and Lemmas 3 and 4, we get that

$$\begin{aligned} M(h,\delta) &= \sqrt{h}\psi_{[n/2]}^{+}(h,\delta) + \left(2h + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+} \\ &\quad \times \left(2h + \frac{a_0^2}{a_1}, \delta\right)\left(\pi - 2\arcsin\frac{a_0}{\sqrt{2a_1 h + a_0^2}}\right) \\ &\quad + \sqrt{h}\sum_{i+2k=0}^{n} B_{i,2k}^{-} h^{i+k} \\ &= \sqrt{h} f_n(h,\delta) + \left(2h + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+} \\ &\quad \times \left(2h + \frac{a_0^2}{a_1}, \delta\right)\left(\pi - 2\arcsin\frac{a_0}{\sqrt{2a_1 h + a_0^2}}\right), \end{aligned} \tag{82}$$

where

$$f_n(h,\delta) = \psi_{[n/2]}^{+}(h,\delta) + \sum_{i+2k=0}^{n} B_{i,2k}^{-} h^{i+k}, \tag{83}$$

which is a polynomial of degree n in h. Let $\lambda = \sqrt{h}$. Then $h = \lambda^2$, $\lambda \in (0, +\infty)$, and (82) becomes

$$\begin{aligned} M(h,\delta) &= \lambda f_n\left(\lambda^2\right) \\ &\quad + \nu(\lambda,\delta)\left(\frac{\pi}{2} - \arcsin\frac{a_0}{\sqrt{2a_1\lambda^2 + a_0^2}}\right) \\ &\triangleq \widetilde{M}(\lambda,\delta), \end{aligned} \tag{84}$$

where

$$\nu(\lambda,\delta) = \left(2\lambda^2 + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+}\left(2\lambda^2 + \frac{a_0^2}{a_1}, \delta\right). \tag{85}$$

One can see that $(d/d\lambda)(\widetilde{M}(\lambda,\delta)/\nu(\lambda,\delta)) = u(\lambda,\delta)/\nu^2(\lambda,\delta)$, where

$$\begin{aligned} u(\lambda,\delta) &= \left(2\lambda^2 + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+}\left(2\lambda^2 + \frac{a_0^2}{a_1}, \delta\right) \\ &\quad \times \frac{d}{d\lambda}\left(\lambda f_n\left(\lambda^2,\delta\right)\right) - \lambda f_n\left(\lambda^2,\delta\right) \\ &\quad \times \frac{d}{d\lambda}\left[\left(2\lambda^2 + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+}\left(2\lambda^2 + \frac{a_0^2}{a_1}, \delta\right)\right] \\ &\quad + \frac{\sqrt{2}a_0}{\sqrt{a_1}}\left(2\lambda^2 + \frac{a_0^2}{a_1}\right) \\ &\quad \times \left(\varphi_{[(n-1)/2]}^{+}\left(2\lambda^2 + \frac{a_0^2}{a_1}, \delta\right)\right)^2. \end{aligned} \tag{86}$$

Denote by $\#\{\lambda \in (0, +\infty) \mid f(\lambda) = 0\}$ the number of zeros of the function in the interval $(0, +\infty)$ taking into account their multiplicities. Note that

$$\deg \nu = 2\left[\frac{n+1}{2}\right], \qquad \deg u = 2\left(n + \left[\frac{n+1}{2}\right]\right), \tag{87}$$

and they are even functions in λ. Therefore,

$$\begin{aligned} \#\{\lambda \in (0, +\infty) \mid \nu(\lambda,\delta)\} &\leq \left[\frac{n-1}{2}\right], \\ \#\{\lambda \in (0, +\infty) \mid u(\lambda,\delta)\} &\leq n + \left[\frac{n+1}{2}\right]. \end{aligned} \tag{88}$$

Then, from [8], we can obtain that

$$\begin{aligned} &\#\left\{\lambda \in (0, +\infty) \mid \widetilde{M}(\lambda,\delta) = 0\right\} \\ &\quad \leq \#\{\lambda \in (0, +\infty) \mid \nu(\lambda,\delta)\} \\ &\qquad + \#\{\lambda \in (0, +\infty) \mid u(\lambda,\delta)\} + 1 \\ &\quad \leq \left[\frac{n-1}{2}\right] + n + \left[\frac{n+1}{2}\right] + 1 \\ &\quad = n + 2\left[\frac{n+1}{2}\right], \end{aligned} \tag{89}$$

which implies that $Z(n) \le n + 2[(n+1)/2]$. Now, we verify $Z(n) \ge n + [(n+1)/2]$.

Make the transformation $u = \sqrt{v^2/2h - a_0^2/2a_1 h}$. Then $\overline{I}_{00}^{+}(h,\delta)$ in (35) becomes

$$\overline{I}_{00}^{+}(h,\delta) = \frac{(2h)^{3/2}}{\sqrt{a_0^2/a_1}} \int_0^1 \frac{u\left(1-u^2\right)^{1/2}}{\sqrt{1+\left(2ha_1u^2/a_0^2\right)}}du, \tag{90}$$

which follows that as $h > 0$ small

$$\begin{aligned}\overline{I}_{00}^{+}(h,\delta) &= \frac{(2h)^{3/2}}{\sqrt{a_0^2/a_1}} \int_0^1 u\left(1-u^2\right)^{1/2} \\ &\quad\times\left[1+\sum_{m=1}^{+\infty}\frac{(-1)^m(2m-1)!!}{(2m)!!}\left(\frac{2a_1}{a_0^2}\right)^m h^m u^{2m}\right]du.\end{aligned} \tag{91}$$

Note that

$$\begin{aligned}&\int_0^1 u^{2m+1}\left(1-u^2\right)^{1/2}du \\ &\quad= \int_0^{\pi/2}\sin^{2m+1}\theta\cos^2\theta d\theta \quad (\text{Let } u=\sin\theta) \\ &\quad= \frac{(2m)!!}{(2m+3)!!}, \quad m = 0,1,2,\ldots.\end{aligned} \tag{92}$$

Inserting the above formula into (91) gives that

$$\overline{I}_{00}(h,\delta) = 2\sqrt{h}\sqrt{\frac{2a_1}{a_0^2}}\sum_{m=0}^{+\infty} C_m\left(\frac{2a_1}{a_0^2}\right)^m h^{m+1}, \tag{93}$$

where

$$C_m = \frac{(-1)^m}{(2m+1)(2m+3)}, \quad m \ge 0. \tag{94}$$

Take $b_{ij}^{-} = a_{ij}^{+} = 0, a_{ij}^{-} = 0,\ j \ge 1,\ b_{ij}^{+} = 0,\ i \ge 1$. Then, by (28), (35), and (93), we can obtain that for $h > 0$ small

$$\begin{aligned}M(h,\delta) &= \sqrt{h}\left[\sum_{i=0}^{n}\frac{-2\sqrt{2}\,(2i)!!}{(2i+1)!!b_0^i}a_{i0}^{-}h^i + \sum_{2k=0}^{n-1} b_{0,2k+1}^{+}\psi_{0k}^{+}(h)\right. \\ &\quad+\sum_{2k=0}^{n-1}2^{k+1}\alpha_{00k}^{+}\sqrt{\frac{2a_1}{a_0^2}}b_{0,2k+1}^{+}\left(h+\frac{a_0^2}{2a_1}\right)^k \\ &\quad\left.\times\sum_{m=0}^{+\infty}C_m\left(\frac{2a_1}{a_0^2}\right)^m h^{m+1}\right] \\ &= \sqrt{h}\sum_{i\ge 0}v_i h^i,\end{aligned} \tag{95}$$

where

$$\begin{aligned}v_i &= \frac{-2\sqrt{2}\,(2i)!!}{(2i+1)!!b_0^i}a_{i0}^{-} \\ &\quad+L_i\left(b_{01}^{+},b_{03}^{+},\ldots,b_{0,2[(n-1)/2]+1}^{+}\right), \quad i=0,1,2,\ldots,n \\ v_{n+1+i} &= \sum_{2k=0}^{n-1}\alpha_{00k}^{+}2^{k+1}\left(\frac{2a_1}{a_0^2}\right)^{n+i-k+1/2} \\ &\quad\times\sum_{r=0}^{k}C_k^r C_{n+i-r}b_{0,2k+1}^{+}, \quad i\ge 0,\end{aligned} \tag{96}$$

with $L_i, i = 0,1,\ldots,n$ being linear combination, $L_i(0,0,\ldots,0) = 0$. One can find that

$$\begin{aligned}&\frac{\partial\left(v_0,v_1,\ldots,v_n,v_{n+1},v_{n+2},\ldots,v_{n+[(n+1)/2]}\right)}{\partial\left(a_{00}^{-},a_{10}^{-},\ldots,a_{n0}^{-},b_{01}^{+},b_{03}^{+},\ldots,b_{0,2[(n-1)/2]+1}^{+}\right)} \\ &\quad= \begin{pmatrix} A_1 & A_2 \\ 0 & 2\sqrt{\dfrac{2a_1}{a_0^2}}A_3\end{pmatrix} \equiv A,\end{aligned} \tag{97}$$

where A_2 is a $(n+1)\times[(n+1)/2]$ matrix,

$$A_1 = \begin{pmatrix} -2\sqrt{2} & 0 & 0 & \cdots & 0 \\ 0 & \dfrac{-4\sqrt{2}}{b_0} & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{-16\sqrt{2}}{15b_0^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{-2\sqrt{2}\,(2n)!!}{(2n+1)!!b_0^n}\end{pmatrix}, \tag{98}$$

$$\begin{aligned}A_3 = \big(&\alpha_{000}^{+}\beta_0, 2\alpha_{001}^{+}\beta_1, 2^2\alpha_{002}^{+}\beta_2,\ldots, \\ &2^{[(n-1)/2]}\alpha_{00,[(n-1)/2]}^{+}\beta_{[(n-1)/2]}\big),\end{aligned}$$

with β_i are $[(n+1)/2]\times 1$ matrix satisfying

$$\beta_i = \begin{pmatrix} \left(\dfrac{2a_1}{a_0^2}\right)^{n-i}\displaystyle\sum_{r=0}^{i}C_i^r C_{n-r} \\ \left(\dfrac{2a_1}{a_0^2}\right)^{n+1-i}\displaystyle\sum_{r=0}^{i}C_i^r C_{n+1-r} \\ \vdots \\ \left(\dfrac{2a_1}{a_0^2}\right)^{n+[(n-1)/2]-i}\displaystyle\sum_{r=0}^{i}C_i^r C_{n+[(n-1)/2]-r}\end{pmatrix}, \tag{99}$$

$$i = 0,1,2,\ldots,\left[\frac{n-1}{2}\right].$$

Hence, we can obtain that from (97)

$$|A| = \left(2\sqrt{\frac{2a_1}{a_0^2}}\right)^{[(n+1)/2]} |A_1||A_3|$$
$$= \left(2\alpha_0^+\sqrt{\frac{2a_1}{a_0^2}}\right)^{[(n+1)/2][(n-1)/2]} \prod_{i=0}^{} 2^i\alpha_{00i}^+ \prod_{i=0}^{n} \frac{-2\sqrt{2}\,(2i)!!}{(2i+1)!!b_0^i}\,|B|, \tag{100}$$

where

$$B = (\beta_0, \beta_1, \ldots, \beta_{[(n-1)/2]}), \tag{101}$$

and β_i are given in (99). We claim that $|A| \neq 0$. We only need to prove $|B| \neq 0$ by the above formula. Using elementary transformations to $|B|$ by multiplying ith column by $(2a_1/a_0^2)^{i-1}$, $i = 2, 3, \ldots, [(n+1)/2]$, we can obtain that by (99) and (101)

$$|B| = \left(\frac{2a_1}{a_0^2}\right)^{n[(n+1)/2]} |B_1|, \tag{102}$$

where

$$|B_1| = \begin{vmatrix} C_n & \sum_{r=0}^{1} C_1^r C_{n-r} & \cdots & \sum_{r=0}^{[(n-1)/2]} C_{[(n-1)/2]}^r C_{n-r} \\ C_{n+1} & \sum_{r=0}^{1} C_1^r C_{n+1-r} & \cdots & \sum_{r=0}^{[(n-1)/2]} C_{[(n-1)/2]}^r C_{n+1-r} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n+[(n-1)/2]} & \sum_{r=0}^{1} C_1^r C_{n+[(n-1)/2]-r} & \cdots & \sum_{r=0}^{[(n-1)/2]} C_{[(n-1)/2]}^r C_{n+[(n-1)/2]-r} \end{vmatrix}. \tag{103}$$

Now we will use elementary transformations to B_1 as follows.

(1) Add the first column multiplying by -1 to ith column, $i = 2, 3, \ldots, [(n+1)/2]$.

(2) Add the second column multiplying by $-C_{i-1}^1$ to ith column, $i = 3, 4, \ldots, [(n+1)/2]$.

(3) Add the third column multiplying by $-C_{i-1}^2$ to ith column, $i = 4, 5, \ldots, [(n+1)/2]$

$\vdots$

$[(n-1)/2]$. Add the $[(n-1)/2]$th column multiplying by $-C_{[(n-1)/2]}^{[(n-3)/2]}$ to $[(n+1)/2]$th column,

$[(n+1)/2]$. multiply ith column by $(-1)^{i-1}$, $i = 2, 3, \ldots, [(n+1)/2]$.

Then, $|B_1|$ becomes, together with (94)

$$|B_1| = (-1)^{[(n-1)/2][(n+1)/2]/2} \det\left(\widetilde{\beta}_0, \widetilde{\beta}_1, \ldots, \widetilde{\beta}_{[(n-1)/2]}\right)$$
$$= 2^{[(n+1)/2]}(-1)^{n[(n+1)/2]} \det\left(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{[(n-1)/2]}\right)$$
$$\triangleq 2^{[(n+1)/2]}(-1)^{n[(n+1)/2]} |B_2|, \tag{104}$$

where

$$\widetilde{\beta}_i = \begin{pmatrix} \frac{(-1)^n}{(2(n-i)+1)(2(n-i)+3)} \\ \frac{(-1)^{n+1}}{(2(n+1-i)+1)(2(n+1-i)+3)} \\ \vdots \\ \frac{(-1)^{n+[(n-1)/2]}}{(2(n+[(n-1)/2]-i)+1)(2(n+[(n-1)/2]-i)+3)} \end{pmatrix},$$
$$\overline{\beta}_i = \begin{pmatrix} \frac{1}{2(n-i)+1} - \frac{1}{2(n-i)+3} \\ \frac{1}{2(n+1-i)+1} - \frac{1}{2(n+1-i)+3} \\ \vdots \\ \frac{1}{2(n+[(n-1)/2]-i)+1} - \frac{1}{2(n+[(n-1)/2]-i)+3} \end{pmatrix}, \tag{105}$$

with $i = 0, 1, \ldots, [(n-1)/2]$. For B_2 in (104) by adding a column on the left and a row on the above, we can obtain that, together with adding ith column to $(i+1)$th column with $i = 1, 2, \ldots, [(n+1)/2]$

$$|B_2| = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{2n+3} & \frac{1}{2n+1} & \frac{1}{2n-1} & \cdots & \frac{1}{2(n-[(n-1)/2])+1} \\ \frac{1}{2n+5} & \frac{1}{2n+3} & \frac{1}{2n+1} & \cdots & \frac{1}{2(n-[(n-1)/2])+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2(n+[(n-1)/2])+3} & \frac{1}{2(n+[(n-1)/2])+1} & \frac{1}{2(n+[(n-1)/2])-1} & \cdots & \frac{1}{2n+1} \end{vmatrix}, \tag{106}$$

which implies that $|A_3| \neq 0$ by (102) and (104) if $|B_2| \neq 0$. We claim that $|B_2| \neq 0$ and

$$|B_3| = \begin{vmatrix} \frac{1}{2n+3} & \frac{1}{2n+1} & \frac{1}{2n-1} & \cdots & \frac{1}{2(n-[(n-1)/2])+1} \\ \frac{1}{2n+5} & \frac{1}{2n+3} & \frac{1}{2n+1} & \cdots & \frac{1}{2(n-[(n-1)/2])+3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2(n+[(n-1)/2])+3} & \frac{1}{2(n+[(n-1)/2])+1} & \frac{1}{2(n+[(n-1)/2])-1} & \cdots & \frac{1}{2n+1} \\ \frac{1}{2(n+[(n+1)/2])+3} & \frac{1}{2(n+[(n+1)/2])+1} & \frac{1}{2(n+[(n+1)/2])-1} & \cdots & \frac{1}{2n+3} \end{vmatrix} \neq 0. \tag{107}$$

Now, we prove them by induction on n. For $n = 1, 2$ we have

$$\begin{vmatrix} 1 & 1 \\ \frac{1}{5} & \frac{1}{3} \end{vmatrix} = \frac{2}{15}, \qquad \begin{vmatrix} 1 & 1 \\ \frac{1}{7} & \frac{1}{5} \end{vmatrix} = \frac{2}{35}, \qquad \begin{vmatrix} \frac{1}{5} & \frac{1}{3} \\ \frac{1}{7} & \frac{1}{5} \end{vmatrix} = \frac{-4}{525}, \qquad \begin{vmatrix} \frac{1}{7} & \frac{1}{5} \\ \frac{1}{9} & \frac{1}{7} \end{vmatrix} = \frac{-4}{2205}, \tag{108}$$

which means that (106) and (107) hold for $n = 1, 2$. Suppose (106) and (107) hold for $n = 2l - 1$, $2l, l \geq 1$. That is, we have for $n = 2l$

$$|B_2| = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{4l+3} & \frac{1}{4l+1} & \frac{1}{4l-1} & \cdots & \frac{1}{2l+3} \\ \frac{1}{4l+5} & \frac{1}{4l+3} & \frac{1}{4l+1} & \cdots & \frac{1}{2l+5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{6l+1} & \frac{1}{6l-1} & \frac{1}{6l-3} & \cdots & \frac{1}{4l+1} \end{vmatrix} \triangleq |C| \neq 0,$$
$$|B_3| = \begin{vmatrix} \frac{1}{4l+3} & \frac{1}{4l+1} & \frac{1}{4l-1} & \cdots & \frac{1}{2l+3} \\ \frac{1}{4l+5} & \frac{1}{4l+3} & \frac{1}{4l+1} & \cdots & \frac{1}{2l+5} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{6l+1} & \frac{1}{6l-1} & \frac{1}{6l-3} & \cdots & \frac{1}{4l+1} \\ \frac{1}{6l+3} & \frac{1}{6l+1} & \frac{1}{6l-1} & \cdots & \frac{1}{4l+3} \end{vmatrix} \neq 0. \tag{109}$$

Then for $n = 2l + 1$, we have

$$|B_2| = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{4l+5} & \frac{1}{4l+3} & \frac{1}{4l+1} & \cdots & \frac{1}{2l+3} \\ \frac{1}{4l+7} & \frac{1}{4l+5} & \frac{1}{4l+3} & \cdots & \frac{1}{2l+5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{6l+5} & \frac{1}{6l+3} & \frac{1}{6l+1} & \cdots & \frac{1}{4l+3} \end{vmatrix},$$
$$|B_3| = \begin{vmatrix} \frac{1}{4l+5} & \frac{1}{4l+3} & \frac{1}{4l+1} & \cdots & \frac{1}{2l+3} \\ \frac{1}{4l+7} & \frac{1}{4l+5} & \frac{1}{4l+3} & \cdots & \frac{1}{2l+5} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{6l+5} & \frac{1}{6l+3} & \frac{1}{6l+1} & \cdots & \frac{1}{4l+3} \\ \frac{1}{6l+7} & \frac{1}{6l+5} & \frac{1}{6l+3} & \cdots & \frac{1}{4l+5} \end{vmatrix}. \tag{110}$$

Note that by the first equation of (109) there only exist $\alpha_1, \alpha_2, \ldots, \alpha_{l+1}$ such that

$$\begin{aligned} |B_2| &= \left[\frac{1}{6l+5} - \left(\frac{\alpha_1}{6l+3} + \frac{\alpha_2}{6l+1} + \cdots + \frac{\alpha_{l+1}}{4l+3}\right)\right]|C| \\ &= \frac{-2}{6l+5}\left[\frac{\alpha_1}{6l+3} + \frac{2\alpha_2}{6l+1} + \cdots + \frac{(l+1)\alpha_{l+1}}{4l+3}\right]|C| \end{aligned} \tag{111}$$

since $\alpha_1 + \alpha_2 + \cdots + \alpha_{l+1} = 1$, where C is given in (109). If $|B_2| = 0$, then we can obtain that from (109) and (111)

$$\frac{\alpha_1}{6l+3} + \frac{2\alpha_2}{6l+1} + \cdots + \frac{(l+1)\alpha_{l+1}}{4l+3} = 0. \tag{112}$$

Note that

$$\begin{aligned} \frac{\alpha_1}{4l+3} + \frac{\alpha_2}{4l+1} + \cdots + \frac{\alpha_{l+1}}{2l+3} &= \frac{1}{4l+5}, \\ \frac{\alpha_1}{4l+5} + \frac{\alpha_2}{4l+3} + \cdots + \frac{\alpha_{l+1}}{2l+5} &= \frac{1}{4l+7}, \\ &\vdots \\ \frac{\alpha_1}{6l+1} + \frac{\alpha_2}{6l-1} + \cdots + \frac{\alpha_{l+1}}{4l+1} &= \frac{1}{6l+3}, \end{aligned} \tag{113}$$

which follows that, together with (112)

$$\begin{aligned}
&\frac{\alpha_1}{4l+3}+\frac{2\alpha_2}{4l+1}+\cdots+\frac{(l+1)\,\alpha_{l+1}}{2l+3}=0,\\
&\qquad\vdots\\
&\frac{\alpha_1}{6l+1}+\frac{2\alpha_2}{6l-1}+\cdots+\frac{(l+1)\,\alpha_{l+1}}{4l+1}=0,
\end{aligned}\qquad
\begin{aligned}
&\frac{\alpha_1}{4l+5}+\frac{2\alpha_2}{4l+3}+\cdots+\frac{(l+1)\,\alpha_{l+1}}{2l+5}=0,\\
&\qquad\vdots\\
&\frac{\alpha_1}{6l+3}+\frac{2\alpha_2}{6l+1}+\cdots+\frac{(l+1)\,\alpha_{l+1}}{4l+3}=0.
\end{aligned}\tag{114}$$

By the second equation in (109) and the above formula, we have

$$\alpha_1=\alpha_2=\cdots=\alpha_{l+1}=0.\tag{115}$$

This is a contradiction with $\alpha_1+\alpha_2+\cdots+\alpha_{l+1}=1$. Hence $\alpha_1/(6l+3)+2\alpha_2/(6l+1)+\cdots+(l+1)\alpha_{l+1}/(4l+3)\neq 0$, which means that (106) holds for $n=2l+1$. Since $|B_2|\neq 0$ in (110), there only exist $\beta_1,\beta_2,\ldots,\beta_{l+1}$ such that

$$\begin{aligned}
&\beta_1+\frac{\beta_2}{4l+5}+\frac{\beta_3}{4l+7}+\cdots+\frac{\beta_{l+1}}{6l+5}=\frac{1}{6l+7},\\
&\beta_1+\frac{\beta_2}{4l+3}+\frac{\beta_3}{4l+5}+\cdots+\frac{\beta_{l+1}}{6l+3}=\frac{1}{6l+5},\\
&\qquad\vdots\\
&\beta_1+\frac{\beta_2}{2l+3}+\frac{\beta_3}{2l+5}+\cdots+\frac{\beta_{l+1}}{4l+3}=\frac{1}{4l+5},
\end{aligned}\tag{116}$$

with $\beta_1\neq 0$ since the last row in the second formula of (110) is linearly independent with all rows in the first formula of (110), which means that (107) holds for $n=2l+1$. In a similar way, we can prove (106) and (107) hold for $n=2l+2$. Hence, the claim holds. So, from (99) we can know that $a_{00}^-,a_{10}^-,\ldots,a_{n0}^-,b_{01}^+,b_{03}^+,\ldots,b_{0,2[(n-1)/2]+1}^+$ can be taken as free parameters. So we can choose these values such that

$$\begin{aligned}
&v_iv_{i+1}<0,\quad i=0,1,\ldots,n+\left[\frac{(n-1)}{2}\right],\\
&0<\left|v_0\right|\ll\left|v_1\right|\ll\cdots\ll\left|v_{n+[(n+1)/2]}\right|\ll 1,
\end{aligned}\tag{117}$$

which yields that by (97) and (96) $M(h,\delta)$ can appear $n+[(n+1)/2]$ positive zeros for $h>0$ small. We also can know that $N(n)\geq n+[(n+1)/2]$. Hence, the conclusion is proved.

Subcase 5. $a_1>0$, $a_0>0$, $b_1>0$, $b_0=0$ By (36) and (67), one can see that

$$\begin{aligned}
M(h,\delta)=&\sqrt{h}\psi_{[n/2]}^+(h,\delta)+\left(2h+\frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^+\\
&\times\left(2h+\frac{a_0^2}{a_1},\delta\right)\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right)\\
&+\sqrt{h}\sum_{i+2k=0}^{n}A_{i,2k}^-\left(\sqrt{h}\right)^{i+2k}\\
=&\sqrt{h}g_{[n/2]}(h,\delta)+\overline{g}_{[(n+1)/2]}(h,\delta)\\
&+\left(2h+\frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^+\left(2h+\frac{a_0^2}{a_1},\delta\right)\\
&\times\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right),
\end{aligned}\tag{118}$$

where

$$\begin{aligned}
g_{[n/2]}(h,\delta)&=\psi_{[n/2]}^+(h,\delta)+\sum_{l=0}^{[n/2]}\sum_{i+2k=2l}A_{i,2k}^-h^l,\\
\overline{g}_{[(n+1)/2]}(h,\delta)&=\sum_{l=0}^{[(n-1)/2]}\sum_{i+2k=2l+1}A_{i,2k}^-h^{l+1},\quad n\geq 1.
\end{aligned}\tag{119}$$

For convenience, we denote by g_n any polynomial of degree n although its coefficients may be different when it appears in different place. Then, we claim that for any $2\leq k\leq[(n-1)/2]$

$$\begin{aligned}
\frac{d^kM(h,\delta)}{dh^k}=&\sum_{j=0}^{k}h^{1/2-k+j}g_{[n/2]-j}+\frac{d^k\overline{g}_{[(n+1)/2]}}{dh^k}\\
&+\sum_{j=0}^{k-2}g_jh^{-j-1/2}\left(2h+\frac{a_0^2}{a_1}\right)^{-j-1}\\
&+\frac{d^k}{dh^k}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^+\left(2h+\frac{a_0^2}{a_1},\delta\right)\right]\\
&\times\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right).
\end{aligned}\tag{120}$$

Now, we verify this claim by induction on k. For $k = 2$, by (118) we can obtain that

$$\begin{aligned}\frac{dM(h,\delta)}{dh} &= \frac{1}{2}h^{-1/2}g_{[n/2]}(h,\delta) \\ &\quad + h^{1/2}\frac{dg_{[n/2]}(h,\delta)}{dh} + \frac{d\overline{g}_{[(n+1)/2]}}{dh} \\ &\quad + \frac{d}{dh}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^+_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right] \\ &\quad \times\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right) \\ &\quad + \left(2h+\frac{a_0^2}{a_1}\right)\varphi^+_{[(n-1)/2]} \\ &\quad \times\left(2h+\frac{a_0^2}{a_1},\delta\right)\frac{a_0}{\sqrt{2a_1}\sqrt{h}\left(2h+a_0^2/a_1\right)} \\ &= h^{-1/2}g_{[n/2]} + h^{1/2}g_{[n/2]-1} + \frac{d\overline{g}_{[(n+1)/2]}}{dh} \\ &\quad + \frac{d}{dh}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^+_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right] \\ &\quad \times\left(\frac{\pi}{2}-\arcsin\frac{\mathrm{a}_0}{\sqrt{2\mathrm{a}_1h+\mathrm{a}_0^2}}\right),\end{aligned} \tag{121}$$

which follows that

$$\begin{aligned}\frac{d^2M(h,\delta)}{dh^2} &= h^{-3/2}g_{[n/2]} + h^{-1/2}g_{[n/2]-1} \\ &\quad + h^{1/2}g_{[n/2]-2} + \frac{d^2\overline{g}_{[(n+1)/2]}}{dh^2} + \frac{g_0}{\sqrt{h}\left(2h+a_0^2/a_1\right)} \\ &\quad + \frac{d^2}{dh^2}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^+_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right] \\ &\quad \times\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right).\end{aligned} \tag{122}$$

Hence, (120) holds for $k = 2$. Suppose (120) holds for $k, 2 \le k \le [(n-1)/2] - 1$. Then for $k+1$, we have

$$\begin{aligned}&\frac{d^{k+1}M(h,\delta)}{dh^{k+1}} \\ &= \sum_{j=0}^{k}\left(\frac{1}{2}-k+j\right)h^{1/2-(k+1)+j}g_{[n/2]-j} \\ &\quad + \sum_{j=0}^{k}h^{1/2-k+j}g_{[n/2]-1-j} + \frac{d^{k+1}\overline{g}_{[(n+1)/2]}}{dh^{k+1}} \\ &\quad + \frac{d^{k+1}}{dh^{k+1}}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^+_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right] \\ &\quad \times\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right) \\ &\quad + \frac{d^k}{dh^k}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^+_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right] \\ &\quad \times\frac{a_0}{\sqrt{2a_1}\sqrt{h}\left(2h+a_0^2/a_1\right)} \\ &\quad + \sum_{j=0}^{k-2}\left(h\left(2h+\frac{a_0^2}{a_1}\right)\frac{d}{dh}g_j\right. \\ &\qquad \left.-g_j\left[2(j+1)h+\left(j+\frac{1}{2}\right)\left(2h+\frac{a_0^2}{a_1}\right)\right]\right) \\ &\quad \times\left(h^{j+3/2}\left(2h+\frac{a_0^2}{a_1}\right)^{j+2}\right)^{-1},\end{aligned} \tag{123}$$

which implies that (120) holds for $k+1$. Thus, the claim is proved. Then, taking $k = [(n-1)/2]$, we can obtain that by differentiating it

$$\begin{aligned}\frac{d^{[(n+1)/2]}M(h,\delta)}{dh^{[(n+1)/2]}} &= \sum_{j=0}^{[n/2]}h^{1/2-[(n+1)/2]+j}g_{[n/2]-j} \\ &\quad + \sum_{j=0}^{[(n+1)/2]-2}g_jh^{-j-1/2}\left(2h+\frac{a_0^2}{a_1}\right)^{-j-1} \\ &\quad + \frac{d^{[(n+1)/2]}\overline{g}_{[(n+1)/2]}}{dh^{[(n+1)/2]}} + \frac{d^{[(n+1)/2]}}{dh^{[(n+1)/2]}} \\ &\quad \times\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^+_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right] \\ &\quad \times\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right).\end{aligned} \tag{124}$$

One can find that

$$\begin{aligned}\frac{d^{[(n+1)/2]+1}M(h,\delta)}{dh^{[(n+1)/2]+1}} &= \sum_{j=0}^{[n/2]}h^{1/2-[(n+1)/2]-1+j}g_{[n/2]-j} \\ &\quad + \sum_{j=0}^{[(n+1)/2]-1}g_jh^{-j-1/2}\left(2h+\frac{a_0^2}{a_1}\right)^{-j-1}\end{aligned}$$

$$= h^{-[(n+1)/2]-1/2} \times \left(2h + \frac{a_0^2}{a_1}\right)^{-[(n+1)/2]} F(h,\delta), \tag{125}$$

where

$$F(h,\delta) = \left(2h + \frac{a_0^2}{a_1}\right)^{[(n+1)/2]} \sum_{j=0}^{[n/2]} h^j g_{[n/2]-j} + \sum_{j=0}^{[(n+1)/2]-1} g_j h^{[(n+1)/2]-j} \times \left(2h + \frac{a_0^2}{a_1}\right)^{[(n+1)/2]-j-1}, \tag{126}$$

where F is a polynomial of degree $[n/2] + [(n+1)/2]$. Since $M(0,\delta) = 0$ from (118), it is easy to see that $M(h,\delta)$ has at most $[n/2] + 2[(n+1)/2] = n + [(n+1)/2]$ zeros for $h > 0$ by Rolle theorem. As the above discussion, we only prove $Z(n) \geq n + [(n+1)/2]$ as $h > 0$ small, which implies $N(n) \geq n + [(n+1)/2]$. For the purpose, take $b_{ij}^- = a_{ij}^+ = 0, a_{ij}^- = 0,\ j \geq 1,\ b_{ij}^+ = 0,\ i \geq 1$. Then using (49), (51), we can write $M(h,\delta)$ in (12) as

$$M(h,\delta) = \sqrt{h}\left[-2\sqrt{2}a_{00}^- + \sum_{i=1}^{n} \frac{2\left(\sqrt{2}\right)^{i+1} i}{\left(\sqrt{b_1}\right)^i} a_{i0}^- \times \int_{-\pi/2}^{0} \sin^{i-1}\theta\cos^2\theta\, d\theta \left(\sqrt{h}\right)^i + \sum_{2k=0}^{n-1} b_{0,2k+1}^+ \psi_{0k}^+(h) + \sum_{2k=0}^{n-1} 2^{k+1}\alpha_{00k}^+ \sqrt{\frac{2a_1}{a_0^2}} b_{0,2k+1}^+ \times \left(h + \frac{a_0^2}{2a_1}\right)^k \sum_{m=0}^{+\infty} C_m \left(\frac{2a_1}{a_0^2}\right)^m h^{m+1}\right]. \tag{127}$$

For $n = 2l, l \geq 1$, (127) can be written as

$$M(h,\delta) = \sqrt{h}\left[\sum_{i=0}^{2l} \overline{v}_i \left(\sqrt{h}\right)^i + \sum_{i\geq 0} \overline{v}_{2l+1+i} h^{l+1+i}\right], \tag{128}$$

where

$$\overline{v}_0 = -2\sqrt{2}a_{00}^-.$$

$$\overline{v}_{2i+1} = \frac{2\left(\sqrt{2}\right)^{2i+2}(2i+1)}{\left(\sqrt{b_1}\right)^{2i+1}} a_{2i+1,0}^- \times \int_{-\pi/2}^{0} \sin^{2i}\theta\cos^2\theta\, d\theta, \quad i = 0,1,\ldots,l-1,$$

$$\overline{v}_{2i} = \frac{2\left(\sqrt{2}\right)^{2i+1} 2i}{\left(\sqrt{b_1}\right)^{2i}} a_{2i,0}^- \int_{-\pi/2}^{0} \sin^{2i-1}\theta\cos^2\theta\, d\theta + L_{2i}\left(b_{01}^+, b_{03}^+, \ldots, b_{0,2[(n-1)/2]+1}^+\right), \quad i = 1,2,\ldots,l,$$

$$\overline{v}_{2l+1+i} = \sum_{k=0}^{l-1} 2^{k+1}\alpha_{00k}^+ \left(\frac{2a_1}{a_0^2}\right)^{l-k+i+1/2} \times b_{0,2k+1}^+ \sum_{r=0}^{k} C_k^r C_{l+i-r}, \quad i = 0,1,\ldots,\left[\frac{n-1}{2}\right], \tag{129}$$

which implies that $M(h,\delta)$ can appear $n + [(n+1)/2]$ zeros in $h > 0$ small by using the same method with the Subcase 4. For $n = 2l+1,\ l = 0,1,\ldots$, we can discuss by (127) in a similar way. Hence, the conclusion holds.

Subcase 6. $a_1 > 0,\ a_0 > 0,\ b_1 > 0,\ b_0 < 0$. We have as the above

$$M(h,\delta) = \sqrt{h}\left(\psi_{[n/2]}^+(h,\delta) + \psi_{[n/2]}^-(h,\delta)\right) + \left(2h + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^+(h,\delta) \times \left(\frac{\pi}{2} - \arcsin\frac{a_0}{\sqrt{2a_1h + a_0^2}}\right) + \left(2h + \frac{b_0^2}{b_1}\right)\varphi_{[(n-1)/2]}^-(h,\delta) \times \left(\frac{\pi}{2} + \arcsin\frac{b_0}{\sqrt{2b_1h + b_0^2}}\right). \tag{130}$$

Similar to the Subcase 5, we can prove that for any $2 \leq k \leq [(n-1)/2]$

$$\frac{d^k M(h,\delta)}{dh^k} = \sum_{j=0}^{k} h^{1/2-k+j} g_{[n/2]-j}$$

$$
\begin{aligned}
&+\sum_{j=0}^{k-2} g_j h^{-j-1/2}\left(2h+\frac{a_0^2}{a_1}\right)^{-j-1}\\
&+\sum_{j=0}^{k-2} g_j h^{-j-1/2}\left(2h+\frac{b_0^2}{b_1}\right)^{-j-1}\\
&+\frac{d^k}{dh^k}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^+_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right]\\
&\times\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right)\\
&+\frac{d^k}{dh^k}\left[\left(2h+\frac{b_0^2}{b_1}\right)\varphi^+_{\left[\frac{(n-1)}{2}\right]}\left(2h+\frac{b_0^2}{a_1},\delta\right)\right]\\
&\times\left(\frac{\pi}{2}-\arcsin\frac{b_0}{\sqrt{2b_1h+b_0^2}}\right).
\end{aligned}
\tag{131}
$$

Taking $k=[(n-1)/2]$ and differentiating the above twice follow that

$$
\begin{aligned}
\frac{d^{[(n+1)/2]+1}M(h,\delta)}{dh^{[(n+1)/2]+1}} &= \sum_{j=0}^{[n/2]} h^{1/2-[(n+1)/2]-1+j} g_{[n/2]-j}\\
&+\sum_{j=0}^{[(n+1)/2]-1} g_j h^{-j-1/2}\left(2h+\frac{a_0^2}{a_1}\right)^{-j-1}\\
&+\sum_{j=0}^{[(n+1)/2]-1} g_j h^{-j-1/2}\left(2h+\frac{b_0^2}{b_1}\right)^{-j-1}.
\end{aligned}
\tag{132}
$$

If $a_0^2/a_1=b_0^2/b_1$, then it is easy to see that (132) has the same form with (125). Hence, we can know that $M(h,\delta)$ has at most $[n/2]+2[(n+1)/2]$ zeros for $h\in(0,+\infty)$. If $a_0^2/a_1\neq b_0^2/b_1$, then (132) can be written as

$$
\begin{aligned}
\frac{d^{[(n+1)/2]+1}M(h,\delta)}{dh^{[(n+1)/2]+1}} &= h^{-[(n+1)/2]-1/2}\left(2h+\frac{a_0^2}{a_1}\right)^{-[(n+1)/2]}\\
&\times\left(2h+\frac{b_0^2}{b_1}\right)^{-[(n+1)/2]}\overline{F}(h,\delta),
\end{aligned}
\tag{133}
$$

where

$$
\begin{aligned}
\overline{F}(h,\delta) &= \left(2h+\frac{a_0^2}{a_1}\right)^{[(n+1)/2]}\left(2h+\frac{b_0^2}{b_1}\right)^{[(n+1)/2]}\\
&\times\sum_{j=0}^{[n/2]} h^j g_{[n/2]-j}\\
&+\left(2h+\frac{b_0^2}{b_1}\right)^{[(n+1)/2]}\sum_{j=0}^{[(n+1)/2]-1} g_j h^{[(n+1)/2]-j}\\
&\times\left(2h+\frac{a_0^2}{a_1}\right)^{[(n+1)/2]-j-1}\\
&+\left(2h+\frac{a_0^2}{a_1}\right)^{[(n+1)/2]}\sum_{j=0}^{[(n+1)/2]-1} g_j h^{[(n+1)/2]-j}\\
&\times\left(2h+\frac{b_0^2}{b_1}\right)^{[(n+1)/2]-j-1},
\end{aligned}
\tag{134}
$$

where $\overline{F}$ is a polynomial of degree $[n/2]+2[(n+1)/2]$ in h. By Rolle theorem, we can obtain that $M(h,\delta)$ has at most $[n/2]+3[(n+1)/2]$ zeros for $h>0$ since $M(0,\delta)=0$. Now, we only need to prove $Z(n)\geq n$. Take $a_{ij}^-=b_{ij}^-=0$, $a_{ij}^+=b_{ij}^+=0$, $i\geq 1$. Then, by Lemmas 4, 5, one can see that for $h>0$ small

$$
\begin{aligned}
M(h,\delta) = \sqrt{h}\Bigg[&\sum_{2k=0}^{n}\frac{2^{k+1+1/2}a_{0,2k}^+}{2k+1}h^k+\sum_{2k=0}^{n-1}b_{0,2k+1}^+\psi_{0k}^+(h)\\
&+\sum_{2k=0}^{n-1}2^{k+1}\alpha_{00k}^+\sqrt{\frac{2a_1}{a_0^2}}b_{0,2k+1}^+\left(h+\frac{a_0^2}{2a_1}\right)^k\\
&\times\sum_{m=0}^{+\infty}C_m\left(\frac{2a_1}{a_0^2}\right)^m h^{m+1}\Bigg].
\end{aligned}
\tag{135}
$$

Similarly, we can discuss the above formula such that such that $M(h,\delta)$ can appear n zeros for $h>0$ small. Hence, the conclusion is proved.

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