

Research Article

New Wavelets Collocation Method for Solving Second-Order Multipoint Boundary Value Problems Using Chebyshev Polynomials of Third and Fourth Kinds

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This paper is concerned with introducing two wavelets collocation algorithms for solving linear and nonlinear multipoint boundary value problems. The principal idea for obtaining spectral numerical solutions for such equations is employing third- and fourth-kind Chebyshev wavelets along with the spectral collocation method to transform the differential equation with its boundary conditions to a system of linear or nonlinear algebraic equations in the unknown expansion coefficients which can be efficiently solved. Convergence analysis and some specific numerical examples are discussed to demonstrate the validity and applicability of the proposed algorithms. The obtained numerical results are comparing favorably with the analytical known solutions.

1. Introduction

Spectral methods are one of the principal methods of discretization for the numerical solution of differential equations. The main advantage of these methods lies in their accuracy for a given number of unknowns (see, e.g., [1–4]). For smooth problems in simple geometries, they offer exponential rates of convergence/spectral accuracy. In contrast, finite difference and finite-element methods yield only algebraic convergence rates. The three most widely used spectral versions are the Galerkin, collocation, and tau methods. Collocation methods [5, 6] have become increasingly popular for solving differential equations, also they are very useful in providing highly accurate solutions to nonlinear differential equations.

Many practical problems arising in numerous branches of science and engineering require solving high even-order and high odd-order boundary value problems. Legendre polynomials have been previously used for obtaining numerical spectral solutions for handling some of these kinds of problems (see, e.g., [7, 8]). In [9], the author has constructed some algorithms by selecting suitable combinations of Legendre

polynomials for solving the differentiated forms of high-odd-order boundary value problems with the aid of Petrov-Galerkin method, while in the two papers [10, 11], the authors handled third- and fifth-order differential equations using Jacobi tau and Jacobi collocation methods.

Multipoint boundary value problems (BVPs) arise in a variety of applied mathematics and physics. For instance, the vibrations of a guy wire of uniform cross-section composed of N parts of different densities can be set up as a multipoint BVP, as in [12]; also, many problems in the theory of elastic stability can be handled by the method of multipoint problems [13]. The existence and multiplicity of solutions of multipoint boundary value problems have been studied by many authors; see [14–17] and the references therein. For two-point BVPs, there are many solution methods such as orthonormalization, invariant imbedding algorithms, finite difference, and collocation methods (see, [18–20]). However, there seems to be little discussion about numerical solutions of multipoint boundary value problems.

Second-order multipoint boundary value problems (BVP) arise in the mathematical modeling of deflection of cantilever beams under concentrated load [21, 22], deformation

of beams and plate deflection theory [23], obstacle problems [24], Troesch's problem relating to the confinement of a plasma column by radiation pressure [25, 26], temperature distribution of the radiation fin of trapezoidal profile [21, 27], and a number of other engineering applications. Many authors have used numerical and approximate methods to solve second-order BVPs. The details about the related numerical methods can be found in a large number of papers (see, for instance, [21, 23, 24, 28]). The Walsh wavelets and the semiorthogonal B-spline wavelets are used in [23, 29] to construct some numerical algorithms for the solution of second-order BVPs with Dirichlet and Neumann boundary conditions. Na [21] has found the numerical solution of second-, third-, and fourth-order BVPs by converting them into initial value problems and then applying a class of methods like nonlinear shooting, method of reduced physical parameters, method of invariant imbedding, and so forth. The presented approach in this paper can be applied to both BVPs and IVPs with a slight modification, but without the transformation of BVPs into IVPs or vice versa.

Wavelets theory is a relatively new and an emerging area in mathematical research. It has been applied to a wide range of engineering disciplines; particularly, wavelets are very successfully used in signal analysis for wave form representation and segmentations, time frequency analysis, and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms, (see [30, 31]).

The application of Legendre wavelets for solving differential and integral equations is thoroughly considered by many authors (see, for instance, [32, 33]). Also, Chebyshev wavelets are used for solving some fractional and integral equations (see, [34, 35]).

Chebyshev polynomials have become increasingly crucial in numerical analysis, from both theoretical and practical points of view. It is well known that there are four kinds of Chebyshev polynomials, and all of them are special cases of the more widest class of Jacobi polynomials. The first and second kinds are special cases of the symmetric Jacobi polynomials (i.e., ultraspherical polynomials), while the third and fourth kinds are special cases of the nonsymmetric Jacobi polynomials. In the literature, there is a great concentration on the first and second kinds of Chebyshev polynomials $T_n(x)$ and $U_n(x)$ and their various uses in numerous applications, (see, for instance, [36]). However, there are few articles that concentrate on the other two types of Chebyshev polynomials, namely, third and fourth kinds $V_n(x)$ and $W_n(x)$, either from theoretical or practical point of view and their uses in various applications (see, e.g., [37]). This motivates our interest in such polynomials. We therefore intend in this work to use them in a marvelous application of multipoint BVPs arising in physics.

There are several advantages of using Chebyshev wavelets approximations based on collocation spectral method. First, unlike most numerical techniques, it is now well established that they are characterized by exponentially decaying errors. Second, approximation by wavelets handles singularities in the problem. The effect of any such singularities will appear

in some form in any scheme of the numerical solution, and it is well known that other numerical methods do not perform well near singularities. Finally, due to their rapid convergence, Chebyshev wavelets collocation method does not suffer from the common instability problems associated with other numerical methods.

The main aim of this paper is to develop two new spectral algorithms for solving second-order multipoint BVPs based on shifted third- and fourth-kind Chebyshev wavelets. The method reduces the differential equation with its boundary conditions to a system of algebraic equations in the unknown expansion coefficients. Large systems of algebraic equations may lead to greater computational complexity and large storage requirements. However the third- and fourth-kind Chebyshev wavelets collocation method reduces drastically the computational complexity of solving the resulting algebraic system.

The structure of the paper is as follows. In Section 2, we give some relevant properties of Chebyshev polynomials of third and fourth kinds and their shifted ones. In Section 3, the third- and fourth-kind Chebyshev wavelets are constructed. Also, in this section, we ascertain the convergence of the Chebyshev wavelets series expansion. Two new shifted Chebyshev wavelets collocation methods for solving second-order linear and nonlinear multipoint boundary value problems are implemented and presented in Section 4. In Section 5, some numerical examples are presented to show the efficiency and the applicability of the presented algorithms. Some concluding remarks are given in Section 6.

2. Some Properties of $V_k(x)$ and $W_k(x)$

The Chebyshev polynomials $V_k(x)$ and $W_k(x)$ of third and fourth kinds are polynomials of degree k in x defined, respectively, by (see [38])

$$V_k(x) = \frac{\cos(k + (1/2))\theta}{\cos(\theta/2)}, \quad W_k(x) = \frac{\sin(k + (1/2))\theta}{\sin(\theta/2)}, \quad (1)$$

where $x = \cos\theta$; also they can be obtained explicitly as two particular cases of Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ for the two nonsymmetric cases correspond to $\beta = -\alpha = \pm 1/2$. Explicitly, we have

$$V_k(x) = \frac{(2^k k!)^2}{(2k)!} P_k^{(-1/2, 1/2)}(x), \quad (2)$$

$$W_k(x) = \frac{(2^k k!)^2}{(2k)!} P_k^{(1/2, -1/2)}(x).$$

It is readily seen that

$$W_k(x) = (-1)^k V_k(-x). \quad (3)$$

Hence, it is sufficient to establish properties and relations for $V_n(x)$ and then deduce their corresponding properties and relations for $W_n(x)$ (by replacing x by $-x$).

The polynomials $V_n(x)$ and $W_n(x)$ are orthogonal on $(-1, 1)$; that is,

$$\begin{aligned} & \int_{-1}^1 \omega_1(x) V_k(x) V_j(x) dx \\ &= \int_{-1}^1 \omega_2(x) W_k(x) W_j(x) dx \quad (4) \\ &= \begin{cases} \pi, & k = j, \\ 0, & k \neq j, \end{cases} \end{aligned}$$

where

$$\omega_1(x) = \sqrt{\frac{1+x}{1-x}}, \quad \omega_2(x) = \sqrt{\frac{1-x}{1+x}}, \quad (5)$$

and they may be generated by using the two recurrence relations

$$V_k(x) = 2xV_{k-1}(x) - V_{k-2}(x), \quad k = 2, 3, \dots, \quad (6)$$

with the initial values

$$V_0(x) = 1, \quad V_1(x) = 2x - 1, \quad (7)$$

$$W_k(x) = 2x W_{k-1}(x) - W_{k-2}(x), \quad k = 2, 3, \dots, \quad (8)$$

with the initial values

$$W_0(x) = 1, \quad W_1(x) = 2x + 1. \quad (9)$$

The shifted Chebyshev polynomials of third and fourth kinds are defined on $[0, 1]$, respectively, as

$$V_n^*(x) = V_n(2x - 1), \quad W_n^*(x) = W_n(2x - 1). \quad (10)$$

All results of Chebyshev polynomials of third and fourth kinds can be easily transformed to give the corresponding results for their shifted ones.

The orthogonality relations of $V_n^*(t)$ and $W_n^*(t)$ on $[0, 1]$ are given by

$$\begin{aligned} & \int_0^1 \omega_1^* V_m^*(t) V_n^*(t) dt \\ &= \int_0^1 \omega_2^* W_m^*(t) W_n^*(t) dt \quad (11) \\ &= \begin{cases} \frac{\pi}{2}, & m = n, \\ 0, & m \neq n, \end{cases} \end{aligned}$$

where

$$\omega_1^* = \sqrt{\frac{t}{1-t}}, \quad \omega_2^* = \sqrt{\frac{1-t}{t}}. \quad (12)$$

3. Shifted Third- and Fourth-Kind Chebyshev Wavelets

Wavelets constitute of a family of functions constructed from dilation and translation of single function called the mother

wavelet. When the dilation parameter a and the translation parameter b vary continuously, then we have the following family of continuous wavelets:

$$\psi_{a,b}(t) = |a|^{-1/2} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0. \quad (13)$$

Each of the third- and fourth-kind Chebyshev wavelets $\psi_{nm}(t) = \psi(k, n, m, t)$ has four arguments: $k, n \in \mathbb{N}$, m is the order of the polynomial $V_m^*(t)$ or $W_m^*(t)$, and t is the normalized time. They are defined explicitly on the interval $[0, 1]$ as

$$\begin{aligned} & \psi_{nm}(t) \\ &= \begin{cases} \frac{2^{(k+1)/2}}{\sqrt{\pi}} V_m^*(2^k t - n), \\ \text{resp.,} \\ \frac{2^{(k+1)/2}}{\sqrt{\pi}} W_m^*(2^k t - n), & t \in \left[\frac{n}{2^k}, \frac{n+1}{2^k}\right], \quad 0 \leq m \leq M, \\ 0, & 0 \leq n \leq 2^k - 1, \\ \text{otherwise.} \end{cases} \quad (14) \end{aligned}$$

3.1. Function Approximation. A function $f(t)$ defined over $[0, 1]$ may be expanded in terms of Chebyshev wavelets as

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t), \quad (15)$$

where

$$c_{nm} = (f(t), \psi_{nm}(t))_{\omega_i^*} = \int_0^1 \omega_i^* f(t) \psi_{nm}(t) dt, \quad (16)$$

and the weights ω_i^* , $i = 1, 2$, are given in (12).

Assume that $f(t)$ can be approximated in terms of Chebyshev wavelets as

$$f(t) \approx \sum_{n=0}^{2^k-1} \sum_{m=0}^M c_{nm} \psi_{nm}(t). \quad (17)$$

3.2. Convergence Analysis. In this section, we state and prove a theorem to ascertain that the third- and fourth-kind Chebyshev wavelets expansion of a function $f(t)$, with bounded second derivative, converges uniformly to $f(t)$.

Theorem 1. Assume that a function $f(t) \in L^2_{\omega_1^*}[0, 1]$, $\omega_1^* = \sqrt{t/(1-t)}$ with $|f''(t)| \leq L$, can be expanded as an infinite series of third-kind Chebyshev wavelets; then this series converges uniformly to $f(t)$. Explicitly, the expansion coefficients in (16) satisfy the following inequality:

$$|c_{nm}| < \frac{2\sqrt{2\pi} L m^2}{(n+1)^{5/2} (m^4 - 1)}, \quad \forall n \geq 0, \quad m > 1. \quad (18)$$

Proof. From (16), it follows that

$$c_{nm} = \frac{2^{(k+1)/2}}{\sqrt{\pi}} \int_{n/2^k}^{(n+1)/2^k} f(t) V_m^*(2^k t - n) \omega_1^*(2^k t - n) dt. \tag{19}$$

If we make use of the substitution $2^k t - n = \cos \theta$ in (19), then we get

$$\begin{aligned} c_{nm} &= \frac{2^{(-k+1)/2}}{\sqrt{\pi}} \int_0^\pi f\left(\frac{\cos \theta + n}{2^k}\right) \\ &\quad \times \frac{\cos(m + (1/2))\theta}{\cos(\theta/2)} \sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}} \sin \theta d\theta \\ &= \frac{2^{(-k+3)/2}}{\sqrt{\pi}} \int_0^\pi f\left(\frac{\cos \theta + n}{2^k}\right) \cos\left(m + \frac{1}{2}\right) \theta \cos\left(\frac{\theta}{2}\right) d\theta \\ &= \frac{2^{(-k+1)/2}}{\sqrt{\pi}} \int_0^\pi f\left(\frac{\cos \theta + n}{2^k}\right) \\ &\quad \times [\cos(m + 1)\theta + \cos m\theta] d\theta, \end{aligned} \tag{20}$$

which in turn, and after performing integration by parts two times, yields

$$c_{nm} = \frac{1}{2^{5k/2} \sqrt{2\pi}} \int_0^\pi f''\left(\frac{\cos \theta + n}{2^k}\right) \gamma_m(\theta) d\theta, \tag{21}$$

where

$$\begin{aligned} \gamma_m(\theta) &= \frac{1}{m+1} \left(\frac{\sin m\theta}{m} - \frac{\sin(m+2)\theta}{m+2} \right) \\ &\quad + \frac{1}{m} \left(\frac{\sin(m-1)\theta}{m-1} - \frac{\sin(m+1)\theta}{m+1} \right). \end{aligned} \tag{22}$$

Now, we have

$$\begin{aligned} |c_{nm}| &= \left| \frac{1}{2^{5k/2} \sqrt{2\pi}} \int_0^\pi f''\left(\frac{\cos \theta + n}{2^k}\right) \gamma_m(\theta) d\theta \right| \\ &= \frac{1}{2^{5k/2} \sqrt{2\pi}} \left| \int_0^\pi f''\left(\frac{\cos \theta + n}{2^k}\right) \gamma_m(\theta) d\theta \right| \\ &\leq \frac{L}{2^{5k/2} \sqrt{2\pi}} \int_0^\pi |\gamma_m(\theta)| d\theta \\ &\leq \frac{L\sqrt{\pi}}{2^{(5k+1)/2}} \left[\frac{1}{m+1} \left(\frac{1}{m} + \frac{1}{m+2} \right) \right. \\ &\quad \left. + \frac{1}{m} \left(\frac{1}{m-1} + \frac{1}{m+1} \right) \right] \\ &= \frac{L\sqrt{2\pi}}{2^{5k/2}} \left[\frac{1}{m^2 + 2m} + \frac{1}{m^2 - 1} \right] \\ &< \frac{2L\sqrt{2\pi}}{2^{5k/2}} \left(\frac{m^2}{m^4 - 1} \right). \end{aligned} \tag{23}$$

Finally, since $n \leq 2^k - 1$, we have

$$|c_{nm}| < \frac{2\sqrt{2\pi} L m^2}{(n+1)^{5/2} (m^4 - 1)}. \tag{24}$$

□

Remark 2. The estimation in (18) is also valid for the coefficients of fourth-kind Chebyshev wavelets expansion. The proof is similar to the proof of Theorem 1.

4. Solution of Multipoint BVPs

In this section, we present two Chebyshev wavelets collocation methods, namely, third-kind Chebyshev wavelets collocation method (3CWCM) and fourth-kind Chebyshev wavelets collocation method (4CWCM), to numerically solve the following multipoint boundary value problem (BVP):

$$\begin{aligned} a(x) y''(x) + b(x) y'(x) + c(x) y(x) \\ + f(x, y') + g(x, y) = 0, \quad 0 \leq x \leq 1, \end{aligned} \tag{25}$$

$$\alpha_0 y(0) + \alpha_1 y'(0) = \sum_{i=1}^{m_0} \lambda_i y(\xi_i) + \delta_0, \tag{26}$$

$$\beta_0 y(1) + \beta_1 y'(1) = \sum_{i=1}^{m_1} \mu_i y(\eta_i) + \delta_1,$$

where $a(x)$, $b(x)$, and $c(x)$ are piecewise continuous on $[0, 1]$; $a(0)a(1)$ may equal zero; $0 < \xi_i, \eta_i < 1$; $\alpha_i, \beta_i, \lambda_i, \mu_i$, and δ_i are constants such that $(\alpha_0^2 + \alpha_1^2)(\beta_0^2 + \beta_1^2) \neq 0$; f is a nonlinear function of y' , and g is a nonlinear function in y .

Consider an approximate solution to (25) and (26) which is given in terms of Chebyshev wavelets as

$$y_{k,M}(x) = \sum_{n=0}^{2^k-1} \sum_{m=0}^M c_{nm} \psi_{nm}(x); \tag{27}$$

then the substitution of (27) into (25) enables one to write the residual of (25) in the form

$$\begin{aligned} R(x) &= \sum_{n=0}^{2^k-1} \sum_{m=2}^M c_{nm} a(x) \psi''_{nm}(x) \\ &\quad + \sum_{n=0}^{2^k-1} \sum_{m=1}^M c_{nm} b(x) \psi'_{nm}(x) \\ &\quad + \sum_{n=0}^{2^k-1} \sum_{m=0}^M c_{nm} c(x) \psi_{nm}(x) \\ &\quad + f\left(x, \sum_{n=0}^{2^k-1} \sum_{m=1}^M c_{nm} \psi'_{nm}(x)\right) \\ &\quad + g\left(x, \sum_{n=0}^{2^k-1} \sum_{m=0}^M c_{nm} \psi_{nm}(x)\right). \end{aligned} \tag{28}$$

Now, the application of the typical collocation method (see, e.g., [5]) gives

$$R(x_i) = 0, \quad i = 1, 2, \dots, 2^k(M + 1) - 2, \quad (29)$$

where x_i are the first $(2^k(M + 1) - 2)$ roots of $V_{2^k(M+1)}^*(x)$ or $W_{2^k(M+1)}^*(x)$. Moreover, the use of the boundary conditions (26) gives

$$\begin{aligned} & \alpha_0 \sum_{n=0}^{2^k-1} \sum_{m=0}^M c_{nm} \psi_{nm}(0) \\ & + \alpha_1 \sum_{n=0}^{2^k-1} \sum_{m=1}^M c_{nm} \psi'_{nm}(0) \\ & = \sum_{i=1}^{m_0} \sum_{n=0}^{2^k-1} \sum_{m=0}^M \lambda_i c_{nm} \psi_{nm}(\xi_i) + \delta_0, \\ & \beta_0 \sum_{n=0}^{2^k-1} \sum_{m=0}^M c_{nm} \psi_{nm}(1) + \beta_1 \sum_{n=0}^{2^k-1} \sum_{m=1}^M c_{nm} \psi'_{nm}(1) \\ & = \sum_{i=1}^{m_1} \sum_{n=0}^{2^k-1} \sum_{m=0}^M \mu_i c_{nm} \psi_{nm}(\eta_i) + \delta_1. \end{aligned} \quad (30)$$

Equations (29) and (30) generate $2^k(M + 1)$ equations in the unknown expansion coefficients, c_{nm} , which can be solved with the aid of the well-known Newton's iterative method. Consequently, we get the desired approximate solution $y_{k,M}(x)$ given by (27).

5. Numerical Examples

In this section, the presented algorithms in Section 4 are applied to solve both of linear and nonlinear multipoint BVPs. Some examples are considered to illustrate the efficiency and applicability of the two proposed algorithms.

Example 1. Consider the second-order nonlinear BVP (see [6, 28]):

$$\begin{aligned} y'' + \frac{3}{8}y + \frac{2}{1089}(y')^2 + 1 &= 0, \quad 0 < x < 1, \\ y(0) = 0, \quad y\left(\frac{1}{3}\right) &= y(1). \end{aligned} \quad (31)$$

The two proposed methods are applied to the problem for the case corresponding to $k = 0$ and $M = 8$. The numerical solutions are shown in Table 1. Due to nonavailability of the exact solution, we compare our results with Haar wavelets method [6], ADM solution [28] and ODEs Solver from Mathematica which is carried out by using Runge-Kutta method. This comparison is also shown in Table 1.

Example 2. Consider the second-order linear BVP (see, [39, 40]):

$$\begin{aligned} y'' &= \sinh x - 2, \quad 0 < x < 1, \\ y'(0) = 0, \quad y(1) &= 3y\left(\frac{3}{5}\right). \end{aligned} \quad (32)$$

The exact solution of problem (32) is given by

$$y(x) = \frac{1}{2} \left(\sinh 1 - 3 \sinh \frac{3}{5} \right) + \sinh x - x^2 - x + \frac{11}{25}. \quad (33)$$

In Table 2, the maximum absolute error E is listed for $k = 1$ and various values of M , while in Table 3, we give a comparison between the best errors resulted from the application of various methods for Example 2, while in Figure 1, we give a comparison between the exact solution of (32) with three approximate solutions.

Example 3. Consider the second-order singular nonlinear BVP (see [40, 41]):

$$\begin{aligned} x(1-x)y'' + 6y' + 2y + y^2 \\ = 6 \cosh x + (2+x-x^2+\sinh x) \sinh x, \\ 0 < x < 1, \end{aligned} \quad (34)$$

$$\begin{aligned} y(0) + y\left(\frac{2}{3}\right) &= \sinh\left(\frac{2}{3}\right), \\ y(1) + \frac{1}{2}y\left(\frac{4}{5}\right) &= \frac{1}{2} \sinh\left(\frac{4}{5}\right) + \sinh 1, \end{aligned}$$

with the exact solution $y(x) = \sinh x$. In Table 4, the maximum absolute error E is listed for $k = 0$ and various values of M , while in Table 5 we give a comparison between the best errors resulted from the application of various methods for Example 3. This table shows that our two algorithms are more accurate if compared with the two methods developed in [40, 41].

Example 4. Consider the second-order nonlinear BVP (see [42]):

$$\begin{aligned} y'' + (1+x+x^3)y^2 &= f(x), \quad 0 < x < 1, \\ y(0) &= \frac{1}{6}y\left(\frac{2}{9}\right) + \frac{1}{3}y\left(\frac{7}{9}\right) - 0.0286634, \\ y(1) &= \frac{1}{5}y\left(\frac{2}{9}\right) + \frac{1}{2}y\left(\frac{7}{9}\right) - 0.0401287, \end{aligned} \quad (35)$$

where

$$\begin{aligned} f(x) &= \frac{1}{9} \left[-6 \cos(x-x^2) + \sin(x-x^2) \right. \\ & \quad \left. \times (-3(1-2x)^2 + (1+x+x^3) \sin(x-x^2)) \right]. \end{aligned} \quad (36)$$

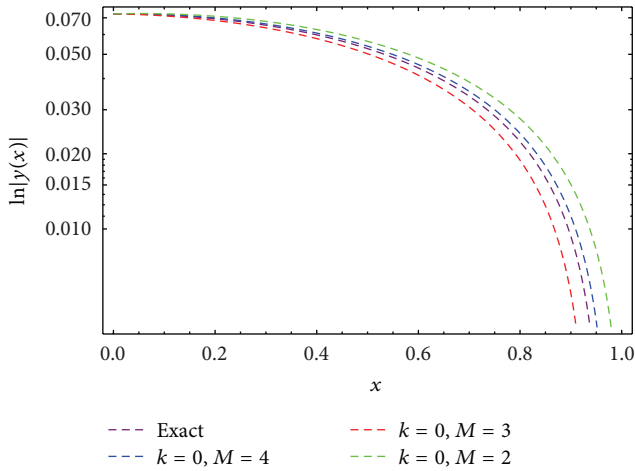


FIGURE 1: Different solutions of Example 2.

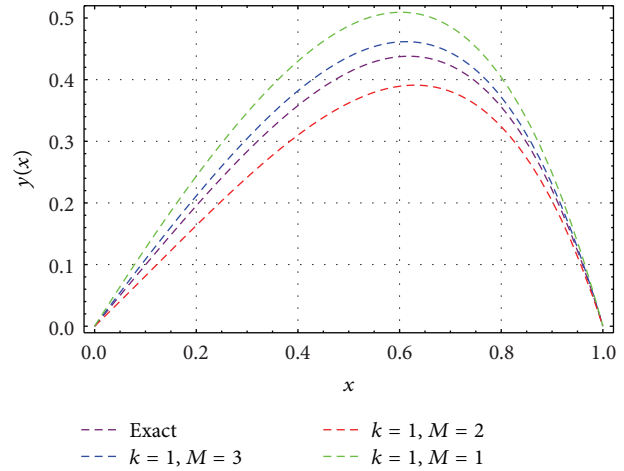


FIGURE 2: Different solutions of Example 5.

The exact solution of (35) is given by $y(x) = (1/3) \sin(x - x^2)$. In Table 6, the maximum absolute error E is listed for $k = 2$ and various values of M , and in Table 7 we give a comparison between best errors resulted from the application of various methods for Example 4. This table shows that our two algorithms are more accurate if compared with the method developed in [42].

Example 5. Consider the second-order singular linear BVP:

$$y'' + f(x)y = g(x), \quad 0 < x < 1,$$

$$y(0) + 16y\left(\frac{1}{4}\right) = 3\sqrt[4]{e}, \quad (37)$$

$$y(1) + 16y\left(\frac{3}{4}\right) = 3\sqrt[4]{e^3},$$

where

$$f(x) = \begin{cases} 3x, & 0 \leq x \leq \frac{1}{2}, \\ 2x, & \frac{1}{2} < x \leq 1, \end{cases} \quad (38)$$

and $g(x)$ is chosen such that the exact solution of (37) is $y(x) = x(1-x)e^x$. In Table 8, the maximum absolute error E is listed for $k = 1$ and various values of M , while in Figure 2, we give a comparison between the exact solution of (37) with three approximate solutions.

Example 6. Consider the following nonlinear second-order BVP:

$$y'' + (y')^2 - 64y = 32, \quad 0 < x < 1, \quad (39)$$

$$y(0) + y\left(\frac{1}{4}\right) = 1, \quad (40)$$

$$4y\left(\frac{1}{2}\right) - y(1) = 0, \quad (41)$$

with the exact solution $y(x) = 16x^2$. We solve (39) using 3CWCM for the case corresponding to $k = 0$ and $M = 3$,

to obtain an approximate solution of $y(x)$. If we make use of (27), then the approximate solution $y_{0,3}(x)$ can be expanded in terms of third-kind Chebyshev wavelets as

$$y_{0,3}(x) = c_{0,0} \sqrt{\frac{2}{\pi}} + c_{0,1} \sqrt{\frac{2}{\pi}} (4x - 3) + c_{0,2} \sqrt{\frac{2}{\pi}} (16x^2 - 20x + 5). \quad (42)$$

If we set

$$v_i = \sqrt{\frac{2}{\pi}} c_{0,i}, \quad i = 0, 1, 2, \quad (43)$$

then (42) reduces to the form

$$y_{0,3}(x) = v_0 + v_1(4x - 3) + v_2(16x^2 - 20x + 5). \quad (44)$$

If we substitute (44) into (39), then the residual of (39) is given by

$$R(x) = 2v_2 + [v_1 + v_2(8x - 5)]^2 - 4[v_0 + v_1(4x - 3) + v_2(16x^2 - 20x + 5)] - 2. \quad (45)$$

We enforce the residual to vanish at the first root of $V_3^*(x) = 64x^3 - 112x^2 + 56x - 7$, namely, at $x_1 = 0.18825509907063323$, to get

$$6.10388v_2^2 + 0.5v_1^2 - 3.49396v_2v_1 - 2.60388v_2 + 4.49396v_1 - 2v_0 = 1. \quad (46)$$

Furthermore, the use of the boundary conditions (40) and (41) yields

$$\begin{aligned} 2v_0 - 5v_1 + 6v_2 &= 1, \\ 3v_0 - 5v_1 - 5v_2 &= 0. \end{aligned} \quad (47)$$

TABLE 1: Comparison between different solutions for Example 1.

x	3CWCM	4CWCM	Haar method [6]	ADM method [28]	ODEs solver from Mathematica
0.1	0.06560	0.06560	0.06561	0.0656	0.06560
0.3	0.16587	0.16587	0.16588	0.1658	0.16587
0.5	0.22369	0.22369	0.22369	0.2236	0.22369
0.7	0.23820	0.23820	0.23821	0.2382	0.23820
0.9	0.20920	0.20920	0.20910	0.2092	0.20920

TABLE 2: The maximum absolute error E for Example 2.

k	M	10	11	12	13	14	15
0	3CWCM	$5.173 \cdot 10^{-9}$	$7.184 \cdot 10^{-11}$	$4.418 \cdot 10^{-12}$	$5.018 \cdot 10^{-14}$	$2.442 \cdot 10^{-15}$	$2.220 \cdot 10^{-16}$
	4CWCM	$3.324 \cdot 10^{-10}$	$2.533 \cdot 10^{-11}$	$2.811 \cdot 10^{-12}$	$1.665 \cdot 10^{-14}$	$1.110 \cdot 10^{-15}$	$2.220 \cdot 10^{-16}$
	M	4	5	6	7	8	9
1	3CWCM	$4.644 \cdot 10^{-6}$	$1.001 \cdot 10^{-8}$	$1.840 \cdot 10^{-9}$	$2.547 \cdot 10^{-10}$	$6.247 \cdot 10^{-11}$	$5.681 \cdot 10^{-12}$
	4CWCM	$2.15 \cdot 10^{-6}$	$5.247 \cdot 10^{-9}$	$7.548 \cdot 10^{-10}$	$1.004 \cdot 10^{-10}$	$2.154 \cdot 10^{-11}$	$4.257 \cdot 10^{-12}$

TABLE 3: The best errors for Example 2.

Method in [39]	Method in [40]	3CWCM	4CWCM
$9.00 \cdot 10^{-6}$	$4.00 \cdot 10^{-5}$	$2.22 \cdot 10^{-16}$	$2.22 \cdot 10^{-16}$

TABLE 4: The maximum absolute error E for Example 3.

M	8	9	10	11	12	13
3CWCM	$4.444 \cdot 10^{-10}$	$2.759 \cdot 10^{-11}$	$4.389 \cdot 10^{-13}$	$1.643 \cdot 10^{-14}$	$4.441 \cdot 10^{-16}$	$2.220 \cdot 10^{-16}$
4CWCM	$4.478 \cdot 10^{-9}$	$2.179 \cdot 10^{-10}$	$1.527 \cdot 10^{-12}$	$2.975 \cdot 10^{-14}$	$4.441 \cdot 10^{-16}$	$2.220 \cdot 10^{-16}$

TABLE 5: The best errors for Example 3.

Method in [40]	Method in [41]	3CWCM	4CWCM
$3.00 \cdot 10^{-8}$	$6.60 \cdot 10^{-7}$	$2.22 \cdot 10^{-16}$	$2.22 \cdot 10^{-16}$

TABLE 6: The maximum absolute error E for Example 4.

M	4	5	6	7	8	9
3CWCM	$2.881 \cdot 10^{-3}$	$3.441 \cdot 10^{-3}$	$1.212 \cdot 10^{-5}$	$1.933 \cdot 10^{-5}$	$1.990 \cdot 10^{-6}$	$3.010 \cdot 10^{-8}$
4CWCM	$1.241 \cdot 10^{-3}$	$8.542 \cdot 10^{-4}$	$7.526 \cdot 10^{-6}$	$1.002 \cdot 10^{-6}$	$6.321 \cdot 10^{-7}$	$2.354 \cdot 10^{-9}$

TABLE 7: The best errors for Example 4.

Method in [42]	3CWCM	4CWCM
$8.00 \cdot 10^{-6}$	$3.010 \cdot 10^{-8}$	$2.35 \cdot 10^{-9}$

TABLE 8: The maximum absolute error E for Example 5.

M	7	8	9	10	11	12	13	14
3CWCM	$7.1 \cdot 10^{-6}$	$3.2 \cdot 10^{-7}$	$1.3 \cdot 10^{-8}$	$7.4 \cdot 10^{-10}$	$1.3 \cdot 10^{-11}$	$3.5 \cdot 10^{-13}$	$9.3 \cdot 10^{-15}$	$8.1 \cdot 10^{-16}$
4CWCM	$2.8 \cdot 10^{-5}$	$1.5 \cdot 10^{-6}$	$6.3 \cdot 10^{-8}$	$2.4 \cdot 10^{-9}$	$7.8 \cdot 10^{-11}$	$2.3 \cdot 10^{-12}$	$6.3 \cdot 10^{-14}$	$1.7 \cdot 10^{-15}$

The solution of the nonlinear system of (46) and (47) gives

$$v_0 = 10, \quad v_1 = 5, \quad v_2 = 1, \quad (48)$$

and consequently

$$y_{0,3}(x) = 10 + 5(4x - 3) + (16x^2 - 20x + 5) = 16x^2, \quad (49)$$

which is the exact solution.

Remark 3. It is worth noting here that the obtained numerical results in the previous solved six examples are very accurate, although the number of retained modes in the spectral expansion is very few, and again the numerical results are comparing favorably with the known analytical solutions.

6. Concluding Remarks

In this paper, two algorithms for obtaining numerical spectral wavelets solutions for second-order multipoint linear and nonlinear boundary value problems are analyzed and discussed. Chebyshev polynomials of third and fourth kinds are used. One of the advantages of the developed algorithms is their availability for application on singular boundary value problems. Another advantage is that high accurate approximate solutions are achieved using a few number of terms of the approximate expansion. The obtained numerical results are comparing favorably with the analytical ones.

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