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Research Article

Endpoints in T_0-Quasimetric Spaces: Part II

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We continue our work on endpoints and startpoints in T_0-quasimetric spaces. In particular we specialize some of our earlier results to the case of two-valued T_0-quasimetrics, that is, essentially, to partial orders. For instance, we observe that in a complete lattice the startpoints (resp., endpoints) in our sense are exactly the completely join-irreducible (resp., completely meet-irreducible) elements. We also discuss for a partially ordered set the connection between its Dedekind-MacNeille completion and the q-hyperconvex hull of its natural T_0-quasimetric space.

1. Introduction

During his investigations on the hyperconvex (or injective) hull of a metric space Isbell [1] introduced the concept of an endpoint of a metric space and proved among other things that the hyperconvex hull of a compact metric space is equal to the hyperconvex hull of the subspace consisting of its endpoints (cf. also [2, 3]). A theory for T_0-quasimetric spaces similar to the one for metric spaces due to Isbell can be developed (see, e.g., [4, 5]). In particular each T_0-quasimetric space has a q-hyperconvex (or injective) hull. For instance, it turns out that the hyperconvex hull of a metric space X is isometric to the largest metric subspace containing the canonical copy of X in the q-hyperconvex hull of X (see [5, Theorem 6]).

In [6] the authors defined the concept of an endpoint in an arbitrary T_0-quasimetric space. In the quasimetric context it turned out to be natural to consider also the dual concept of an endpoint, which we called a startpoint.

Improving on a result from [6] in this note we will show that for any join compact T_0-quasimetric space (X, d) the set of the endpoints (resp., startpoints) of X is equal to the set of the endpoints (resp., startpoints) of its q-hyperconvex hull (Q_X, D).

We also specialize some of our earlier results in [6] to two-valued T_0-quasimetric spaces. It is well known that they are, essentially, the partially ordered sets and that in the category of partially ordered sets the injective hull coincides with the Dedekind-MacNeille completion (see, e.g., [5, 7, 8]). We will observe that in the case of a complete lattice our startpoints (resp., endpoints) turn out to be exactly the completely join-irreducible (resp., the completely meet-irreducible) elements. We also discuss for a partially ordered set in some detail the connection between its Dedekind-MacNeille completion and the q-hyperconvex hull of its natural T_0-quasimetric space. Our results can, for instance, be used to analyze the similarity between the following result due to Isbell [1]:

> *A compact injective metric space Y has a smallest closed subset B such that the hyperconvex hull of B is equal to Y;*

and the following well-known result from order theory (see, e.g., [9, Theorem 7.42]):

> *A lattice L with no infinite chains is order isomorphic to the Dedekind-MacNeille completion of the partially ordered set $\mathcal{J}(L) \cup \mathcal{M}(L)$, where $\mathcal{J}(L)$ denotes the set of join-irreducible elements of L and $\mathcal{M}(L)$ denotes the set of meet-irreducible elements of L. Furthermore $\mathcal{J}(L) \cup \mathcal{M}(L)$ is the smallest subset of L which is both join- and meet-dense in L.*

2. Preliminaries

In this section we first recall some of the basic definitions from asymmetric topology needed to read this paper. Then we recall some fundamental facts of the theory of the q-hyperconvex hull of a T_0-quasimetric space.

Definition 1. Let X be a set and $d : X \times X \to [0,\infty)$ a function. (Here $[0,\infty)$ denotes the set of the nonnegative reals.) Then d is *quasipseudometric* on X if

(a) $d(x,x) = 0$ whenever $x \in X$, and

(b) $d(x,z) \leq d(x,y) + d(y,z)$ whenever x, y, and $z \in X$.

We will say that (X,d) is a T_0*-quasimetric space* provided that d also satisfies the following condition: for each $x, y \in X$, $d(x,y) = 0 = d(y,x)$ implies that $x = y$.

Let d be quasipseudometric on a set X. Then $d^{-1} : X \times X \to [0,\infty)$ defined by $d^{-1}(x,y) = d(y,x)$ whenever $x, y \in X$ is also quasipseudometric, called the *conjugate quasipseudometric* of d. Observe that if d is a T_0-quasimetric on X, then $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is a metric on X.

Given $x \in X$ and a nonnegative real number r we set $C_d(x,r) = \{y \in X : d(x,y) \leq r\}$. Note that this set is $\tau(d^{-1})$-closed, where $\tau(d)$ is the topology having the balls $B_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}$ with $x \in X$ and $\epsilon > 0$ as basic (open) sets.

A map $f : (X,d) \to (Y,e)$ between quasipseudometric spaces (X,d) and (Y,e) is called *isometric* provided that $d(x,y) = e(f(x), f(y))$ whenever $x, y \in X$. Each isometric map with a T_0-quasimetric domain is a one-to-one map.

Furthermore a map $f : (X,d) \to (Y,e)$ between quasipseudometric spaces (X,d) and (Y,e) is called *nonexpansive* provided that $e(f(x), f(y)) \leq d(x,y)$ whenever $x, y \in X$.

Given two real numbers a and b we will write $a \dot{-} b$ for $\max\{a-b, 0\}$, which we will also denote by $(a-b) \vee 0$. Note that $u(x,y) = x \dot{-} y$ with $x, y \in \mathbb{R}$ defines the standard T_0-quasimetric on the set $\mathbb{R}$ of the reals.

Given a T_0-quasimetric space (X,d), we recall that the *specialization (partial) order* $\leq_d$ *of* d is defined as follows: for each $x, y \in X$, set $x \leq_d y$ if and only if $d(x,y) = 0$.

For further basic concepts used from the theory of asymmetric topology we refer the reader to [10–12].

Many facts about hyperconvexity in metric spaces can be found in [13–15]. Connections between that theory and order theory are explored in [7, 16]. Throughout we will assume familiarity of the reader with the results of [6].

We next recall some facts mainly from [4] belonging to the theory of the q-hyperconvex hull of a T_0-quasimetric space (see also [5, 8, 17–19] for some related investigations).

Let (X,d) be a T_0-quasimetric space. We will say that a function pair $f = (f_1, f_2)$ on (X,d), where $f_i : X \to [0,\infty)$ $(i = 1,2)$ is *ample* provided that $d(x,y) \leq f_2(x) + f_1(y)$ whenever $x, y \in X$.

Let P_X denote the set of all ample function pairs on (X,d). (In such situations we may also write $P_{(X,d)}$ in cases where d is not obvious.) For each $f, g \in P_X$ we set

$$D(f,g) = \sup_{x\in X}\left(f_1(x) \dot{-} g_1(x)\right) \vee \sup_{x\in X}\left(g_2(x) \dot{-} f_2(x)\right). \quad (1)$$

Then D is an extended (if we replace in the definition of a quasipseudometric $[0,\infty)$ by $[0,\infty]$ we obtain the definition of an *extended* quasipseudometric. Of course, the triangle inequality for extended quasipseudometrics is interpreted in the self-explanatory way. (Indeed some authors prefer to work with extended quasipseudometrics throughout; see, e.g., [8]). For the purpose of this paper however real-valued T_0-quasimetrics seem to be more appropriate; cf. Section 7) T_0-quasimetric on P_X.

We will call a function pair f *minimal* on (X,d) (among the ample function pairs on (X,d)) if it is ample and whenever g is ample on (X,d), and for each $x \in X$ we have $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$ (for any function pairs f and g satisfying this relation we will write $g \leq f$); then $g = f$. It is well known that Zorn's Lemma implies that below each ample function pair there is a minimal ample pair (cf., e.g., [2, 20]). By Q_X we will denote the set of all minimal ample pairs on (X,d) equipped with the restriction of D to $Q_X \times Q_X$, which we will also denote by D. Recall that D is a (real-valued) T_0-quasimetric on $Q_X \times Q_X$ [4, Remark 6].

Furthermore $f = (f_1, f_2) \in Q_X$ if and only if the following equations $(*)$ are satisfied:

$$\begin{aligned} f_1(x) &= \sup\left\{d(y,x) \dot{-} f_2(y) : y \in X\right\}, \\ f_2(x) &= \sup\left\{d(x,y) \dot{-} f_1(y) : y \in X\right\} \end{aligned} \quad (*)$$

whenever $x \in X$ (cf. [21, Remark 2]). In particular note that such pairs are ample on (X,d).

Obviously the second component f_2 of a minimal ample pair (f_1, f_2) on (X,d) satisfies the following equation $(**)$:

$$f_2(x) = \sup_{y\in X}\left(d(x,y) \dot{-} \sup_{y'\in X}\left(d(y',y) \dot{-} f_2(y')\right)\right) \quad (**)$$

whenever $x \in X$.

Given any real-valued function $f : X \to [0,\infty)$, satisfying $(**)$, we can set $f_1(x) := \sup\{d(y,x) \dot{-} f(y) : y \in X\}$ whenever $x \in X$.

One readily checks that (f_1, f) is an ample function pair on (X,d).

Furthermore, of course,

$$\begin{aligned} f_1(x) &= \sup\left\{d(y,x) \dot{-} f(y) : y \in X\right\}, \\ f(x) &= \sup\left\{d(x,y) \dot{-} f_1(y) : y \in X\right\} \end{aligned} \quad (2)$$

whenever $x \in X$. Hence (f_1, f) is a minimal ample pair on (X,d), and thus $(f_1, f) \in Q_X$.

Hence $(**)$ characterizes exactly those functions $f : X \to [0,\infty)$ that are second component of minimal ample pairs on (X,d) (cf., e.g., [5, 8] concerning the underlying Isbell conjugation adjunction; see also [22]). Of course, an analogous result holds for the first component of minimal ample pairs on (X,d).

It is known (see [4, Lemma 3]) that $f \in Q_X$ implies that $f_1(x) - f_1(y) \leq d(y,x)$ and $f_2(x) - f_2(y) \leq d(x,y)$ whenever $x, y \in X$.

Moreover $\sup_{x\in X}(f_1(x) \dot{-} g_1(x)) = \sup_{x\in X}(g_2(x) \dot{-} f_2(x))$ whenever $f, g \in Q_X$ (cf., [4, Lemma 7]).

For each $x \in X$ we can define the minimal function pair

$$f_x(y) = (d(x,y), d(y,x)) \quad (3)$$

whenever $y \in X$ on (X,d). The map e defined by $x \mapsto f_x$ whenever $x \in X$ defines an isometric embedding of (X,d) into (Q_X, D) (see [4, Lemma 1]).

Then (Q_X, D) is called the *q-hyperconvex hull* of (X,d). A T_0-quasimetric space X is said to be *q-hyperconvex* if $f \in Q_X$ implies that there is an $x \in X$ such that $f = f_x$ (cf. [4, Corollary 4]). An intrinsic characterization of q-hyperconvexity of a T_0-quasimetric space (X,d) can, for instance, be found in [4, Definition 2]: a T_0-quasimetric space (X,d) is q-hyperconvex if and only if, given $A \subseteq X$ and families of nonnegative reals $(r_x)_{x\in A}$ and $(s_x)_{x\in A}$ such that $d(x,y) \le r_x + s_y$ whenever $x, y \in A$, we have that $\bigcap_{x\in A}(C_d(x,r_x) \cap C_{d^{-1}}(x,s_x)) \ne \emptyset$ (see [4, Remark 2]).

Note also that $D(f, f_x) = f_1(x)$ and $D(f_x, f) = f_2(x)$ whenever $x \in X$ and $f \in Q_X$ [4, Lemma 8].

The following important result (see [4, Remark 7]) is best understood as a kind of density of X in Q_X. For any $y_1, y_2 \in Q_X$, we have that

$$D(y_1, y_2) = \sup\Big\{\Big(D\big(f_{x_1}, f_{x_2}\big) - D\big(f_{x_1}, y_1\big) - D\big(y_2, f_{x_2}\big)\Big) \vee 0 : x_1, x_2 \in X\Big\}. \tag{4}$$

Our first example shows that in that formula the step of taking the supremum with 0 cannot be avoided in general.

Example 2. Let $a, b \in [0,\infty)$ be such that $a, b > 0$, and let $Y = [0,a] \times [0,b]$. Set

$$D((\alpha_1,\alpha_2),(\beta_1,\beta_2)) = (\alpha_1 \dot{-} \beta_1) \vee (\alpha_2 \dot{-} \beta_2) \tag{5}$$

whenever $(\alpha_1,\alpha_2), (\beta_1,\beta_2) \in Y$. It is known that Y can be identified with the q-hyperconvex hull of the subspace $X = \{(a,0),(0,b)\}$ of Y (see [6, Example 4]).

Furthermore

$$\begin{aligned}
&[D((a,0),(0,b)) - D((a,0),(0,0)) - D((a,b),(0,b))] \\
&\vee [D((0,b),(a,0)) - D((b,0),(0,0)) - D((a,b),(a,0))] \\
&= [a-a-a] \vee [b-b-b] \ne 0 = D((0,0),(a,b)).
\end{aligned} \tag{6}$$

3. The Concept of Collinearity in Quasipseudometric Spaces

The following definition was given in [6] (cf. [3]). Let (X,d) be a quasipseudometric space.

(1) A finite sequence $(x_1, x_2, \ldots, x_n)$ in X is called *collinear* in (X,d) provided that $i < j < k \le n$ implies that $d(x_i,x_k) = d(x_i,x_j) + d(x_j,x_k)$.

(2) An element $x \in X$ is called an *endpoint* of (X,d) provided that there exists an element y in (X,d) such that $d(y,x) > 0$ and such that for any $z \in X$ collinearity of (y,x,z) in (X,d) implies that $x = z$. We will say that y *witnesses* that x is an endpoint.

(3) An element $x \in X$ is called a *startpoint of* (X,d) if it is an endpoint of (X,d^{-1}).

Let us finally recall that a quasipseudometric space (X,d) is called *join compact* provided that $\tau(d^s)$ is compact.

The next result says intuitively that the points in the remainder of the q-hyperconvex hull Q_X of a join compact T_0-quasimetric space X lie between the points of X.

Proposition 3. *Let (X,d) be a join compact T_0-quasimetric space, and let $e : (X,d) \to (Q_X,D)$ be the canonical isometric embedding of (X,d) into its q-hyperconvex hull (Q_X,D). Consider any $f \in Q_X \setminus e(X)$. Then there are a startpoint s and an endpoint e in (X,d) such that (s,f,e) is collinear in (Q_X,D).*

Proof. As stated above, for $y \in X$ we will identify f_y with y.

Fix $x \in X$. Since (Q_X,D) is a T_0-space, we have that $D(f,x) > 0$ or $D(x,f) > 0$. We consider only the first case. The second one is analogous. By [6, Lemma 2] there is $y \in X$ such that (y,f,x) is collinear in (Q_X,D).

Then $D(y,x) \ge D(f,x) > 0$. Therefore by [6, Corollary 3] there are a startpoint s and an endpoint e in (X,d) such that (s,y,x,e) is collinear in (Q_X,D).

It follows that $D(s,f) + D(f,e) \le (D(s,y) + D(y,f)) + (D(f,x) + D(x,e)) = D(s,y) + D(y,x) + D(x,e) = (D(s,y) + D(y,x)) + D(x,e) = D(s,x) + D(x,e) = D(s,e)$, and hence (s,f,e) is collinear in (Q_X,D). □

Lemma 4. *Let (X,d) be a T_0-quasimetric space. If x is an endpoint with witness y and (z,y,x) is collinear in (X,d), then z is also a witness that x is an endpoint in (X,d). Similarly, if x is a startpoint with witness y and (x,y,z) is collinear in (X,d), then z is also a witness that x is a startpoint in (X,d).*

Proof. Assume that x is an endpoint with witness y and that (z,y,x) is collinear in (X,d). Note that $d(z,x) \ge d(y,x) > 0$. Suppose that for some $a \in X$, (z,x,a) is collinear in (X,d). Then inequalities $d(z,a) \le d(z,y) + d(y,a) \le d(z,y) + d(y,x) + d(x,a) = d(z,x) + d(x,a) = d(z,a)$ imply by subtracting $d(z,y)$ that $d(y,a) = d(y,x) + d(x,a)$ (cf., [6, Lemma 1]); hence (y,x,a) is collinear in (X,d), and thus $x = a$ by our assumption. Therefore z witnesses that x is an endpoint of (X,d). The dual result is proved analogously. □

Example 5. Let (X,d) be a join compact T_0-quasimetric space with $y_1, y_2 \in X$ such that $d(y_1,y_2) > 0$.

According to the proofs of [6, Proposition 3, Corollary 3] there exist a startpoint s in (X,d) with witness y_2 and an endpoint e in (X,d) with witness s such that (s,y_1,y_2,e) is collinear in (X,d). By Lemma 4 it follows that both s witnesses that e is an endpoint and e witnesses that s is a startpoint in (X,d).

We next prove the result mentioned in Section 1.

Proposition 6. *Let (X,d) be a join compact T_0-quasimetric space. Then (Q_X,D) has exactly the same endpoints and startpoints as (X,d).*

Proof. In [6, Proposition 4] it was shown that each endpoint of (X,d) is an endpoint of (Q_X,D). In fact, it was proved that if x is an endpoint in (X,d) with witness $y \in X$, then f_y witnesses that f_x is an endpoint in (Q_X,D).

Here we will show that each endpoint of (Q_X, D) is an endpoint of (X, d).

Suppose that $f \in Q_X$ witnesses that $g \in Q_X$ is an endpoint in (Q_X, D). Then $D(f, g) > 0$. Thus by [20, Proposition 5] there are sequences $x_n, y_n \in X$ such that the increasing sequence $(D(x_n, y_n) - D(x_n, f) - D(g, y_n))_{n\in\mathbb{N}}$ converges to $D(f, g)$ with respect to the usual topology on $\mathbb{R}$. (Note that here and in the following, for any $x \in X$, we will identify x with f_x.)

By join compactness of (X, d) indeed we have that there is a subsequence $(n_k)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ and $x, y \in X$ such that $d^s(x_{n_k}, x) \to 0$ and $d^s(y_{n_k}, y) \to 0$. Note that, for instance, $|D(x_{n_k}, f) - D(x, f)| \le d^s(x_{n_k}, x)$ for any $k \in \mathbb{N}$.

Taking limits, therefore $D(f, g) = D(x, y) - D(x, f) - D(g, y)$, and thus $D(x, y) = D(x, f) + D(f, g) + D(g, y)$. Consequently (f, g, y) is collinear in (Q_X, D), since $D(x, f) + D(f, g) + D(g, y) = D(x, y) \le D(x, f) + D(f, y) \le D(x, f) + D(f, g) + D(g, y)$. Hence $D(x, f) + D(f, g) + D(g, y) = D(x, f) + D(f, y)$, and the statement follows.

We conclude that $g = y$, since f witnesses that g is an endpoint in (Q_X, D). Indeed then $D(x, y) = D(x, f) + D(f, y)$, and (x, f, g) is collinear in (Q_X, D). By Lemma 4 therefore x is a witness belonging to X that $g = y \in X$ is an endpoint in (Q_X, D). It follows that x witnesses that y is an endpoint in (X, d), because (X, d) is a subspace of (Q_X, D).

The assertion about startpoints is proved analogously. □

4. T_0-Quasimetrics Induced by Partial Orders

Let $(X, \le)$ be a partially ordered set and $y \in X$. In the following, we set $\uparrow y := \{x \in X : y \le x\}$ and $\downarrow y := \{x \in X : y \ge x\}$.

Given a partially ordered set $(X, \le)$, we equip it with the T_0-quasimetric d given by setting, for all $x, y \in X, d(x, y) = 0$ if $x \le y$ and $d(x, y) = 1$ otherwise. Indeed d is a T_0-ultraquasimetric, but we will not use this fact in this paper (see, e.g., [23]).

We will call d the *natural T_0-quasimetric* of $\le$. Note that a map $f : (X, d) \to (\{0, 1\}, u)$ is nonexpansive if and only if f is monotonically increasing. (Of course, here u also denotes the restriction of u to $\{0, 1\}^2$.)

Observe that, if d is the natural T_0-quasimetric induced by a partial order $\le$, then d^{-1} is the natural T_0-quasimetric induced by the partial order $\ge$.

As we noted in the Preliminaries section, with the help of the specialization order we can equip each T_0-quasimetric space (X, d) with a partial order. In [6, Proposition 1] we considered the following more sophisticated method.

Fix $y \in X$. For $a_1, a_2 \in X$, set $a_1 \le_y a_2$ if (y, a_1, a_2) is collinear in (X, d). Then $\le_y$ is a partial order on X according to [6, Proposition 1]. Our next result shows that, if the T_0-quasimetric space (X, d) originates from a partial order $\le$, then $\le_y$ can be readily described.

Example 7. Let $(X, \le)$ be a partially ordered set, and let d be its natural T_0-quasimetric. Furthermore let $y \in X$. Then $\le_y$ agrees with $\le$ if the two relations are restricted to $(\uparrow y)^2$ or to $(X\setminus \uparrow y)^2$. Furthermore $\le_y$ restricted to $(X\setminus \uparrow y) \times \uparrow y$ is empty, and $\le_y$ restricted to $\uparrow y \times (X\setminus \uparrow y)$ agrees with the complement of $\le$ restricted to $\uparrow y \times (X\setminus \uparrow y)$.

> Hence $\le_y$ can be described as follows. For $a, b \in X$, we have $a \le_y b$ if Case 1 is $a \le b$ and [(i) $a, b \in \uparrow y$ or (ii) $a, b \in (X\setminus \uparrow y)$] or if Case 2 is $a \nleq b, a \in \uparrow y$ and $b \in (X\setminus \uparrow y)$.
>
> Note that obviously $\le_y$ is a linear order if $\le$ is a linear order.

Let us consider two specific examples of this construction.

Example 8. (a) Let $(X, \le)$ be a partially ordered set with the smallest element 0. Then $\le_0 = \le$.

(b) Let $(X, \le)$ be a partially ordered set with the largest element 1. Then $\le_1$ is obtained on X by removing the top element 1 from X and adding it as a new smallest element to $X \setminus \{1\}$.

Let (X, d) be a T_0-quasimetric space. Given $a, c \in X$ we set that $L_{a,c} := \{b \in X : (a, b, c) \text{ is collinear in } (X, d)\}$ (cf. [2, (4.2)]). We have that $a, c \in L_{a,c}$ and $L_{a,c}$ is obviously $\tau(d^s)$-closed.

In the special case of T_0-quasimetrics originating from partial orders the set $L_{a,c}$ can be described as follows.

Example 9. Let $(X, \le)$ be a partially ordered set with natural T_0-quasimetric d, and let $a, c \in X$. Then $L_{a,c}$ is equal to the interval $[a, c] = \{x \in X : a \le x \le c\}$ if $a \le c$ and equal to $\uparrow a \cup \downarrow c$ if $a \nleq c$.

Example 10. Let $(X, \le)$ be a partially ordered set and d its natural T_0-quasimetric. Suppose that (x_1, x_2, x_3, x_4) is collinear in (X, d) and $d(x_2, x_3) > 0$. Then the sequence $(d(x_i, x_{i+1}))_{i=1,2,3}$ is equal to 010.

Proof. Of course $d(x_2, x_3) = 1$. Furthermore $i \in \{1, 2\}$ with $d(x_i, x_{i+1}) = 1 = d(x_{i+1}, x_{i+2})$ is impossible, since (x_i, x_{i+1}, x_{i+2}) is collinear in (X, d). Thus we have that the sequence $(d(x_i, x_{i+1}))_{i=1,2,3}$ is alternating in $\{0, 1\}$. □

We are next going to illustrate the concepts of a startpoint (resp., endpoint) in the case of T_0-quasimetrics induced by partial orders. Let us first note that [6, Proposition 3] is useless for an infinite partially ordered set X equipped with its natural T_0-quasimetric d, since d^s is the discrete metric, and therefore $\tau(d^s)$ is compact if and only if X is finite.

Lemma 11. *Let $(X, \le)$ be a partially ordered set, d its natural T_0-quasimetric, and $x, y \in X$. Then x is a startpoint of (X, d) witnessed by y if and only if x is a minimal element in $X\setminus \downarrow y$.*

Proof. Suppose that x is a startpoint of (X, d) witnessed by $y \in X$. Then $d(x, y) = 1$, and thus $x \nleq y$. Furthermore for all $a \in X, d(a, x) + d(x, y) = d(a, y)$ implies that $x = a$. Hence $a \le x$ and $a \in X\setminus \downarrow y$ imply that $x = a$. Thus x is minimal in $X\setminus \downarrow y$.

Assume that x is minimal in $X\setminus \downarrow y$. Then $d(x, y) > 0$. Suppose that, for some $a \in X, (a, x, y)$ is collinear in (X, d). Then $d(a, y) = d(a, x) + d(x, y) = d(a, x) + 1 = 1$; thus $a \nleq y$

and $a \leq x$. Therefore $a = x$, since x is minimal in $X \setminus \downarrow y$. Hence x is a startpoint in (X, d) witnessed by y. □

Corollary 12. *Let $(X, \leq)$ be a partially ordered set, d its natural T_0-quasimetric, and $x, y \in X$. Then x is an endpoint in (X, d) witnessed by y if and only if x is a maximal element in $X \setminus \uparrow y$.*

We next illustrate Lemma 11 and its corollary by two simple examples.

Example 13. Let X be a set having at least two points and equipped with the discrete order $=$. Then the T_0-quasimetric d induced by $=$ on X is the discrete metric. Note that each point of X is an endpoint and a startpoint in (X, d), witnessed by any other point.

Example 14. Let X be a complete lattice, and let $k \in X$ be compact (see [24, Definition I-1.1.] for the definition), but such that $k \neq 0$. Since $k \neq 0$, there is $y \in X$ such that $k \nleq y$. By Zorn's Lemma, let C be a maximal chain in $X \setminus \uparrow k$ containing y. Set $m = \vee C$. Then $m \in X \setminus \uparrow k$, since k is compact. Furthermore $m \in C$, since C is a maximal chain. Thus m is maximal in $X \setminus \uparrow k$. Consequently m is an endpoint witnessed by k in (X, d), where d denotes the natural T_0-quasimetric of X.

The following definition can essentially be found in [24, Definition I-4.21]. Let X be a partially ordered set. An element $x \in X$ is called *completely m-irreducible* if either x is maximal in X but different from the largest element or the set $\uparrow x \setminus \{x\}$ has a least element, which will be denoted by x^+. Dually one defines *completely j-irreducible* elements in X.

Proposition 15. *(a) Each completely m-irreducible element x in a partially ordered set $(X, \leq)$ is an endpoint in (X, d), where d denotes the natural T_0-quasimetric of X.*

(b) Let X be a complete lattice equipped with its natural T_0-quasimetric d and $x \in X$ (cf. [24, Remark I-4.23]). If x is an endpoint of (X, d), then x is completely m-irreducible.

Proof. (a) Suppose first that x is maximal but not the largest element of X. Then there is $y \in X$ such that $y \nleq x$. Therefore x is maximal in $X \setminus \uparrow y$, and x is an endpoint by Corollary 12.

Suppose now that $\uparrow x \setminus \{x\}$ has a least element x^+. It will suffice by Corollary 1 to show that x is maximal in $X \setminus \uparrow x^+$. So let $a \in X$ with $x < a$. Then $a \in \uparrow x \setminus \{x\}$ and consequently $x^+ \leq a$. Hence $a \notin X \setminus \uparrow x^+$ and x is maximal in $X \setminus \uparrow x^+$. Therefore x is an endpoint in (X, d).

(b) For the convenience of the reader we include a proof, which follows [24, page 126]. Let $k \in X$ witness that x is an endpoint in (X, d). If x is maximal, then x is completely m-irreducible, as it cannot be the largest element in X, because $k \nleq x$.

If x is not maximal in X, then $\emptyset \neq \uparrow x \setminus \{x\} \subseteq \uparrow k$, since x is maximal in $X \setminus \uparrow k$ by Corollary 12. Hence $x^+ := \wedge(\uparrow x \setminus \{x\})$ exists and $x^+ > x$, as $x^+ \geq k$. Thus $\uparrow x \setminus \{x\}$ has a least element. We have shown that x is completely m-irreducible in either case. □

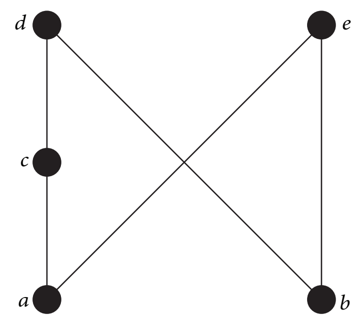

Figure 1: Hasse diagram of P_4.

Corollary 16. *(a) Each completely j-irreducible element x in a partially ordered set $(X, \leq)$ is a startpoint in (X, d), where d denotes the natural T_0-quasimetric of X.*

(b) Let X be a complete lattice equipped with its natural T_0-quasimetric d and $x \in X$ (cf., [24, Remark I-4.23]). If x is a startpoint of (X, d), then x is completely j-irreducible.

Recall that an element x in a complete lattice X is called *completely join-irreducible* if, for each subset S of X, $x = \vee S$ implies that $x \in S$. Completely meet-irreducible elements are defined dually (see [9, Definition 10.26]).

Of course, in a complete lattice the completely j-irreducible elements are exactly the completely join-irreducible elements and the completely m-irreducible elements are exactly the completely meet-irreducible elements.

Corollary 17. *Let X be a complete lattice and d its natural T_0-quasimetric. Then $x \in X$ is a startpoint in (X, d) if and only if x is completely join-irreducible. Similarly, $x \in X$ is an endpoint in (X, d) if and only if x is completely meet-irreducible in (X, d).*

Example 18. In a partially ordered set X that is not complete and is equipped with its natural T_0-quasimetric d, an endpoint of (X, d) need not be completely m-irreducible. As an example consider the partially ordered set P_4 from [9, page 169] (see Figure 1).

Our characterizations (Lemma 11 and Corollary 12) immediately yield the set of startpoints $\{a, b, c, e\}$ and the set of endpoints $\{b, c, d, e\}$. In particular b is not completely m-irreducible, although it is an endpoint.

As usual, an element x in a complete lattice X will be called *completely join-prime* if $x \leq \vee S$ for some subset S of X means that there is $s \in S$ such that $x \leq s$. The concept of a *completely meet-prime element* is defined dually. It is easy to see that each completely join-prime element is completely join-irreducible and each completely meet-prime element is completely meet-irreducible (see [9, Definition 10.26]).

Example 19. Let X be a complete lattice and d its natural T_0-quasimetric. Furthermore let $x \in X$ be completely join-prime. It follows that $y = \vee(X \setminus \uparrow x)$ is completely meet-prime. Furthermore one proves that $\downarrow y = X \setminus \uparrow x$. Indeed (x, y) is a completely prime pair in the sense of [9, page 246].

We conclude that y is an endpoint with witness x in (X, d), since y is obviously maximal in $X \setminus \uparrow x$, and x is a startpoint with witness y in (X, d), since x is minimal in $X \setminus \downarrow y$.

We recall that a subset E of a partially ordered set X is called *join-dense* in X provided that for each $x \in X$ there exists $E' \subseteq E$ such that $x = \vee_X E'$ (see [9, page 53]). Dually one defines the concept of a *meet-dense* subset of a partially ordered set X.

It is known (see [24, Remark I-4.22]) that each meet-dense subset of X contains all completely m-irreducible elements. Similarly each join-dense subset of X contains all completely j-irreducible elements.

Proposition 20. *Let X be a partially ordered set and d its natural T_0-quasimetric.*

(a) *If E is a join-dense subset of X, then all startpoints of (X, d) belong to E. Dually, if E is a meet-dense subset in X, then all endpoints of (X, d) belong to E.*

(b) *If E is join- and meet-dense in X, then all startpoints (resp., endpoints) of X are startpoints (resp., endpoints) of E.*

Proof. (a) Suppose that x is a startpoint of (X, d). Then our characterization (see Lemma 11) of a startpoint gives $y \in X$ such that x is a minimal element of $X \setminus \downarrow y$.

Since $x \nleq y$, by join density of E in X, there must be $e \in E$ such that $e \leq x$ and $e \nleq y$. Hence by the minimality property of x we obtain $x = e$. Therefore x belongs to E. The dual result is proved analogously.

(b) We continue the proof of part (a) dealing with startpoints. By the additional assumption of meet density of E in X, in the proof of part (a) we can find $y' \in E$ such that $y \leq y'$ and $x = e \nleq y'$. Then x is minimal in $X \setminus \downarrow y'$. We conclude that $y' \in E$ witnesses that $x \in E$ is a startpoint in E, because $E \subseteq X$. The result on endpoints is proved similarly. □

5. Examples

In this section we will discuss various examples in the light of the results in Section 4. Some of the details of the arguments are left to the reader.

Example 21. Let $(X, \leq)$ be a partially ordered set with the smallest element 0 and the largest element 1, where $0 \neq 1$, equipped with its natural T_0-quasimetric d. Given $x \in X$, collinearity of $(0, 1, x)$ in (X, d) obviously implies that $x = 1$. However we cannot conclude that 1 is an endpoint of (X, d) with witness 0, since $d(0, 1) = 0$.

Indeed we have the following result.

Proposition 22. *Let $(X, \leq)$ be a partially ordered set equipped with its natural T_0-quasimetric d. A smallest element 0 of X cannot be a startpoint of (X, d). Similarly a largest element 1 of X is never an endpoint of (X, d).*

Proof. Since $0 \leq x$, thus $d(0, x) = 0$ whenever $x \in X$; there cannot be an element in X witnessing that 0 is a startpoint in (X, d). The dual statement is proved similarly. □

Let $(X, \leq)$ be a linearly ordered set, and let $a, b \in X$ be such that $a < b$, but there does not exist an element $z \in X$ such that $a < z < b$. As usual, the pair (a, b) is called a *jump* in X. Note that for partially ordered sets one also says that b *covers* a in this case; see, for instance, [9, page 11].

Proposition 23. *Let $(X, \leq)$ be a linearly ordered set equipped with its natural T_0-quasimetric d. The first elements of jumps in X are exactly the endpoints of (X, d). The second elements of jumps in X are exactly the startpoints of (X, d).*

Proof. Suppose that (a, b) is a jump in X. Then b is minimal in $X \setminus \downarrow a$. Hence b is a startpoint of (X, d) by Lemma 11.

In order to prove the converse suppose that b is a startpoint of (X, d). Then there is $y \in X$ such that b is minimal in $X \setminus \downarrow y$. Since $\leq$ is a linear order, we have that $y < b$ and (y, b) is a jump. Similarly one proves the stated result on endpoints. □

Corollary 24. *In $\{0, 1\}$, equipped with its usual linear order and natural T_0-quasimetric, 0 is an endpoint but not a startpoint, while 1 is a startpoint but not an endpoint.*

In the set $\mathbb{Z}$ of integers equipped with the usual linear order and its natural T_0-quasimetric, each point is an endpoint and a startpoint.

The closed unit interval of the set of rational numbers equipped with its usual linear order and the natural T_0-quasimetric induced by that order does not have any endpoints nor any startpoints.

Corollary 25. *Let $(X, \leq)$ be a partially ordered set equipped with its natural T_0-quasimetric d. If, for some $c \in X, \downarrow c$ consists of two elements, then c is a startpoint in (X, d). Similarly, if, for some $c \in X, \uparrow c$ consists of two elements, then c is an endpoint in (X, d).*

Proof. Suppose that $\downarrow c = \{c, m\}$ with $m \neq c$. Obviously c is a minimal element of $X \setminus \downarrow m$. The result follows from Lemma 11. The dual result is proved analogously. □

Corollary 26. *Each atom in a partially ordered set X (see [9, page 113] for the definition) with the smallest element and equipped with its natural T_0-quasimetric d is a startpoint of (X, d). Similarly each coatom in a partially ordered set X with a largest element and equipped with its natural T_0-quasimetric d is an endpoint of (X, d).*

Example 27. For a set X with at least one element consider the complete lattice $(\mathscr{P}(X), \subseteq)$ equipped with its natural T_0-quasimetric d, where $\mathscr{P}(X)$ is the power set of X. Then the startpoints of $(\mathscr{P}(X), d)$ are exactly the singletons. The endpoints of $(\mathscr{P}(X), d)$ are exactly the complements of the singletons.

In fact, for each $x \in X$, the startpoint $\{x\}$ witnesses that $X \setminus \{x\}$ is an endpoint, and the endpoint $X \setminus \{x\}$ witnesses that

$\{x\}$ is a startpoint. Observe that for each $x \in X$, $(\{x\}, X \setminus \{x\})$ is a completely prime pair.

Example 28. Let $\mathscr{R}$ be the usual topology on the set $\mathbb{R}$ of the reals equipped with set-theoretic inclusion as a partial order, and let d be its natural T_0-quasimetric. Then there are no startpoints, and exactly the complements of singletons are the endpoints in $(\mathscr{R}, d)$.

Let us give a proof of this statement just using the basic definitions. Suppose that $H \in \mathscr{R}$ is a startpoint with witness $H' \in \mathscr{R}$. In particular $H \nsubseteq H'$. Let $h \in H \setminus H'$. Then we find $H'' \in \mathscr{R}$ such that $h \in H'' \subset H$. It follows that (H'', H, H') is collinear in $(\mathscr{R}, d)$—a contradiction. We conclude that there are no startpoints in $(\mathscr{R}, d)$.

On the other hand fix $x \in \mathbb{R}$. Then collinearity of $(X, X \setminus \{x\}, G)$ with $G \in \mathscr{R}$ implies that $G = X \setminus \{x\}$. Thus X witnesses that $X \setminus \{x\}$ is an endpoint in $(\mathscr{R}, d)$.

Assume that H is an endpoint in $(\mathscr{R}, d)$. Then there is a witness $G \in \mathscr{R}$ such that $G \nsubseteq H$. Let $g \in G \setminus H$. Then $(G, H, X \setminus \{g\})$ is collinear in $(\mathscr{R}, d)$. By our assumption $H = X \setminus \{g\}$. Hence exactly the complements of singletons are the endpoints in $(\mathscr{R}, d)$.

Given a nonempty subset A of a set X, by $\mathrm{fil}\{A\}$ we denote the filter generated by the filter base $\{A\}$ on X.

Corollary 29. *Let $\mathscr{P}$ be the set of filters (partially ordered under set-theoretic inclusion) on an infinite set X and equipped with its natural T_0-quasimetric d. Then the set of endpoints of $(\mathscr{P}, d)$ consists of all the ultrafilters on X, and the set of startpoints of $(\mathscr{P}, d)$ consists of all the filters $\mathrm{fil}\{X \setminus \{x\}\}$ with $x \in X$.*

Proof. It is well known that each filter on X is the intersection of ultrafilters on X and that the maximal elements in $\mathscr{P}$ are the ultrafilters. Hence the endpoints in $\mathscr{P}$ are exactly the ultrafilters on X (see Propositions 15 and 20).

Furthermore $\mathscr{P}$ has a smallest element, namely, $\{X\}$. Since the set of $\mathrm{fil}\{A\}$ with nonempty $A \subseteq X$ is join-dense in $\mathscr{P}$, the startpoints can only be of the form $\mathrm{fil}\{A\}$ with proper $A \subseteq X$ by Propositions 20 and 22. We know by Corollary 26 that, for all $x \in X$, $\mathrm{fil}\{X \setminus \{x\}\}$ is a startpoint in $\mathscr{P}$. We finally show that there are no other startpoints in $(\mathscr{P}, d)$.

Let $A \subseteq X$ be such that $X \setminus A$ contains at least two points. Take any $\mathscr{G} \in \mathscr{P}$ such that $\mathrm{fil}\{A\} \nsubseteq \mathscr{G}$. We consider two cases.

Case 1. $(\cap\{G : G \in \mathscr{G}\}) \setminus A = \emptyset$. Then choose $a' \in X \setminus A$. It follows that $\mathrm{fil}\{A\}$ is not minimal in $\mathscr{P} \setminus \downarrow \mathscr{G}$, since $\mathrm{fil}\{A \cup \{a'\}\} \subset \mathrm{fil}\{A\}$ and also $\mathrm{fil}\{A \cup \{a'\}\} \nsubseteq \mathscr{G}$. Indeed if $A \cup \{a'\} \in \mathscr{G}$, then there is $Y' \in \mathscr{G}$ such that $a' \notin Y'$; hence $(A \cup \{a'\}) \cap Y' \subseteq A$ and then $\mathrm{fil}\{A\} \subseteq \mathscr{G}$—a contradiction.

Case 2. There is $a' \in (\cap\{G : G \in \mathscr{G}\}) \setminus A$. By our assumption about A we can choose $a'' \in X \setminus (A \cup \{a'\})$. Then $\mathrm{fil}\{A \cup \{a''\}\} \subset \mathrm{fil}\{A\}$ and also $\mathrm{fil}\{A \cup \{a''\}\} \nsubseteq \mathscr{G}$, because $A \cup \{a''\} \notin \mathscr{G}$, since $a' \notin A \cup \{a''\}$. It follows that $\mathrm{fil}\{A\}$ is not minimal in $\mathscr{P} \setminus \downarrow \mathscr{G}$.

Thus we are done, since no $\mathscr{G} \in \mathscr{P}$ exists that could witness that $\mathrm{fil}\{A\}$ is a startpoint of $(\mathscr{P}, d)$. □

6. The Dedekind-MacNeille Completion versus the q-Hyperconvex Hull

Some of the results mentioned in the previous sections may have reminded the reader of the theory of the Dedekind-MacNeille completion (cf. also [16]). Of course this is not accidental but can be explained categorically (see, e.g., [5, 8]).

For the following discussion we need some basic facts from the theory of the Dedekind-MacNeille completion of a partially ordered set (see, e.g., [9, page 166]).

Let $(X, \leq)$ be a partially ordered set, and let $A \subseteq X$. Then we define the *set of upper bounds* of A, that is, $A^u = \{x \in X : a \leq x$ whenever $a \in A\}$ and the *set of lower bounds* of A, that is, $A^\ell = \{x \in X : a \geq x$ whenever $a \in A\}$. Let $\mathrm{DM}(X) = \{A \subseteq X : A^{u\ell} = A\}$. The partially ordered set $(\mathrm{DM}(X), \subseteq)$ is a complete lattice. It is known as the *Dedekind-MacNeille completion of* X. Furthermore $\phi : X \to \mathrm{DM}(X)$ defined by $\phi(x) = \downarrow x$ is an order embedding such that $\phi(X)$ is both join-dense and meet-dense in $\mathrm{DM}(X)$. This is indeed the characteristic property of the Dedekind-MacNeille completion (cf. [9, Theorem 7.41]).

Proposition 30. *Let $(X, \leq)$ be a partially ordered set and d its natural T_0-quasimetric. Furthermore let D be the natural T_0-quasimetric of $(DM(X), \subseteq)$. Then (X, d) and $(DM(X), D)$ have the same startpoints (resp., endpoints).*

Proof. For the proof we consider X a subset of $\mathrm{DM}(X)$. By Proposition 20(b) each startpoint (resp., endpoint) of $(\mathrm{DM}(X), D)$ is a startpoint (resp., endpoint) of (X, d), since X is both join-dense and meet-dense in $\mathrm{DM}(X)$. Suppose now that x is a startpoint of (X, d) with witness $y \in X$. Let $f \in \mathrm{DM}(X)$ be such that (f, x, y) is collinear in $(\mathrm{DM}(X), D)$. Thus $x \nleq y$, and, therefore, $\downarrow x \nsubseteq \downarrow y$, $f \subseteq \downarrow x$, and $f \nsubseteq \downarrow y$. Since X, is join-dense in $\mathrm{DM}(X)$, there is $f' \in X$ such that $\downarrow f' \subseteq f$ and $\downarrow f' \nsubseteq \downarrow y$. Thus $\downarrow f' \subseteq f \subseteq \downarrow x$ and (f', x, y) is collinear in (X, d). Hence $f' = x$ and therefore $f = \downarrow x$, too. Consequently x is a startpoint with witness y in $(\mathrm{DM}(X), D)$. The dual result is proved analogously. □

Example 31. We continue the discussion of Example 18 (see [9, page 169] and Figure 2).

Considering P_4 as a subset of $\mathrm{DM}(P_4)$, in the light of Proposition 30 and according to Corollary 17 in the complete lattice $\mathrm{DM}(P_4)$, the set of the startpoints of P_4 becomes the set of the (completely) join-irreducible elements of $\mathrm{DM}(P_4)$, and the set of the endpoints of P_4 becomes the set of the (completely) meet-irreducible elements of $\mathrm{DM}(P_4)$.

We will say that a partially ordered set $(X, \leq)$ with its natural T_0-quasimetric d is *o-hyperconvex* if, for any $A \subseteq X$ and any families $(r_x)_{x\in A}, (s_x)_{x\in A}$ in $\{0, 1\}$ satisfying that $d(x, y) \leq r_x + s_y$ whenever $x, y \in A$, it follows that $\bigcap_{x\in A}(C_d(x, r_x) \cap C_{d^{-1}}(x, s_x)) \neq \emptyset$. Observe that (X, d) is o-hyperconvex provided that (X, d) is q-hyperconvex.

Proposition 32. *Let $(X, \leq)$ be a complete lattice and d its natural T_0-quasimetric on X. Then (X, d) is o-hyperconvex.*

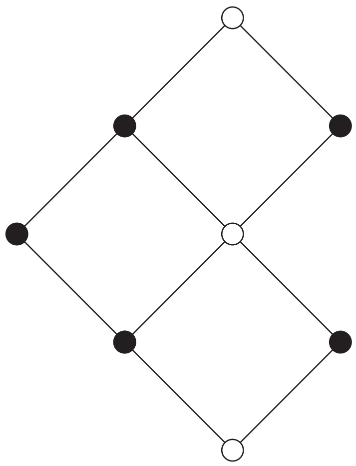

Figure 2: Hasse diagram of $\mathrm{DM}(P_4)$.

Proof. Let $A \subseteq X$, and let $(r_x)_{x\in A}$ and $(s_x)_{x\in A}$ be families in $\{0,1\}$ such that $d(x,y) \leq r_x + s_y$ whenever $x, y \in A$. Set $R = \{x \in A : r_x = 0\}$ and $S = \{x \in A : s_x = 0\}$. Note that $R \cap S$ contains at most one element, since (X,d) is a T_0-space. We observe that $x, y \in R \cap S$ implies that $d(x,y) \leq r_x + s_y$ and $d(y,x) \leq r_y + s_x$; thus $d(x,y) = 0 = d(y,x)$, and hence $x = y$.

Furthermore by our assumption, obviously we have that $R \subseteq S^{\ell}$, since $d(r,s) \leq 0 + 0$ whenever $r \in R$ and $s \in S$. Since X is a complete lattice, $\vee R$ exists.

Then

$$\begin{aligned}\vee R &\in \left(\bigcap_{x\in R} \uparrow x\right) \cap \left(\bigcap_{x\in S} \downarrow x\right) \\ &= \left(\bigcap_{x\in A} C_d(x, r_x)\right) \cap \left(\bigcap_{x\in A} C_{d^{-1}}(x, s_x)\right),\end{aligned} \tag{7}$$

since $C_d(x,r_x) = X$ if $x \in A \backslash R$ and $C_{d^{-1}}(x,s_x) = X$ whenever $x \in A \setminus S$. Thus (X,d) is o-hyperconvex. □

As an illustration it may be useful to include here the following simple example (see [4, Example 8]).

Example 33. Let $X = \{0,1\}$ be equipped with its usual order $\leq$ and with its natural T_0-quasimetric d. Then (Q_X, D) can be identified with $([0,1], u)$ under the obvious inclusion $X \to [0,1]$. Hence (X,d) is not q-hyperconvex, although $(X,\leq)$ is a complete lattice.

For the following result compare the discussion preceding [8, Lemma 2.5].

Proposition 34. *Let (X,d) be a bounded q-hyperconvex T_0-quasimetric space and $\leq$ its specialization order. Then $(X,\leq)$ is a complete lattice.*

Proof. Suppose that $g \in [0,\infty)$ is an upper bound of d.

Let $A \subseteq X$ and $x \in X$. Set $r_x = 0$ if $x \in A$, and $r_x = g$ otherwise. Furthermore let $s_x = 0$ if $x \in A^u$, and $s_x = g$ otherwise.

Consider now arbitrary $x, y \in X$. Assume first that $x \in A$ and $y \in A^u$. Then $x \leq y$ and $d(x,y) = 0 = r_x + s_y$.

Suppose now that $x \notin A$ or $y \notin A^u$. Consequently $r_x + s_y \geq g \geq d(x,y)$. We have shown that for $((C_d(x,r_x))_{x\in X}$; $(C_{d^{-1}}(x,s_x))_{x\in X})$ the hypothesis of the condition of q-hyperconvexity is satisfied.

We conclude that there is

$$\begin{aligned} y &\in \left(\bigcap_{x\in X} C_d(x, r_x)\right) \cap \left(\bigcap_{x\in X} C_{d^{-1}}(x, s_x)\right) \\ &\subseteq \left(\bigcap_{a\in A} \uparrow a\right) \cap \left(\bigcap_{b\in A^u} \downarrow b\right).\end{aligned} \tag{8}$$

Consequently $y = \vee A$. (Of course, similarly one could show that $\wedge A$ exists.) We deduce that $(X,\leq)$ is a complete lattice. □

Example 35. Recall that $(\mathbb{R}, u)$ is a q-hyperconvex T_0-quasimetric space (see [4, Example 1]). The specialization order $\leq$ of that space is the standard order on $\mathbb{R}$; hence $(\mathbb{R},\leq)$ is not a complete lattice. So boundedness cannot be omitted in Proposition 34.

Remark 36. Let $(X,\leq)$ be a partially ordered set and d its natural T_0-quasimetric. If (X,d) is o-hyperconvex, then $(X,\leq)$ is a complete lattice. This is a consequence of the proof of Proposition 34 by setting $g = 1$, since we have $\leq_d = \leq$.

Indeed, more generally, our next result shows explicitly how, given a partially ordered set $(X,\leq)$ equipped with its natural T_0-quasimetric d, the q-hyperconvex hull $(Q_{(X,d)}, D)$ of (X,d) contains the Dedekind-MacNeille completion of X.

Lemma 37. *Let $(X,\leq)$ be a partially ordered set and d its natural T_0-quasimetric. Furthermore let F_X be the set of all those minimal ample function pairs (f_1,f_2) on (X,d) that only attain the values 0 and 1.*

Consider an arbitrary pair (f_1,f_2) of functions $X \to \{0,1\}$. Then the following conditions are equivalent.

(a) $(f_1,f_2) \in F_X$.

(b) $f_1^{-1}\{0\} = (f_2^{-1}\{0\})^u$ *and* $f_2^{-1}\{0\} = (f_1^{-1}\{0\})^{\ell}$.

(c) $(f_2^{-1}\{0\})^{u\ell} = f_2^{-1}\{0\}$ *and* $f_1(x) = \sup_{y\in X}(d(y,x) \dot{-} f_2(y))$ *whenever* $x \in X$.

Proof. (a) $\to$ (b) : given $x \in X$, consider

$$f_1(x) = \sup_{y\in X}\left(d(y,x) \dot{-} f_2(y)\right) \tag{9}$$

(see $(*)$ from Section 2).

Since the functions f_1, f_2 and d attain only the values 0 and 1, we see that these equations are equivalent to $f_1^{-1}\{0\} = (f_2^{-1}\{0\})^u$. Indeed, given $x \in X$, we have that $f_1(x) = 0$ if and only if, for any $y \in X$, $d(y,x) = 1$ implies that $f_2(y) = 1$, and if and only if, for any $y \in X$, $f_2(y) = 0$ implies that $d(y,x) = 0$ if and only if $x \in (f_2^{-1}\{0\})^u$.

Similarly one verifies that $f_2^{-1}\{0\} = (f_1^{-1}\{0\})^\ell$ is equivalent to $f_2(x) = \sup_{y\in X}(d(x, y)\dot{-}f_1(y))$ whenever $x \in X$.

Since $(f_1, f_2) \in F_X \subseteq Q_X$, condition (b) is satisfied.

(b) $\rightarrow$ (c) : by (b) we conclude that $(f_2^{-1}\{0\})^{u\ell} = f_2^{-1}\{0\}$. Furthermore the second part of (c) is equivalent to $f_1^{-1}\{0\} = (f_2^{-1}\{0\})^u$, as we have just noted above. Hence condition (c) is satisfied.

(c) $\rightarrow$ (a): observe that $(f_2^{-1}\{0\})^{u\ell} = f_2^{-1}\{0\}$ and $f_1^{-1}\{0\} = (f_2^{-1}\{0\})^u$ together imply that $f_2^{-1}\{0\} = (f_1^{-1}\{0\})^\ell$. But the latter equality is equivalent to $f_2(x) = \sup_{y\in X}(d(x, y)\dot{-}f_1(y))$ whenever $x \in X$, as we have observed above. Thus $(f_1, f_2) \in F_X$, and condition (a) holds. □

Proposition 38. *Let $(X, \leq)$ be a partially ordered set with its natural T_0-quasimetric d, and let F_X be defined as in Lemma 37.*

Then the map $\psi : (F_X, \leq_D) \rightarrow (DM(X), \subseteq)$ defined by $(f_1, f_2) \mapsto f_2^{-1}\{0\}$ is an order isomorphism between F_X (equipped with the specialization order $\leq_D$ induced on F_X by the T_0-quasimetric D of the q-hyperconvex hull of (X, d)) and the Dedekind-MacNeille completion $(DM(X), \subseteq)$ of X. Furthermore for each $x \in X, \psi(f_x) = \downarrow x$.

Proof. By Lemma 37 each set $f_2^{-1}\{0\}$ belongs to the Dedekind-MacNeille completion, since $f \in F_X$. Moreover for each $x \in X$, $f_x \in F_X$. Also for each $x \in X$, obviously $(f_x)_2^{-1}\{0\} = \downarrow x$. The specialization order $\leq_D$ induced on F_X by the T_0-quasimetric D is defined by $f \preceq g$ if and only if $D(f, g) = 0$ whenever $f, g \in F_X$. Hence we have for any $f, g \in F_X$,

$$D(f, g) = \sup_{x\in X}(g_2(x)\dot{-}f_2(x)) = 0 \tag{10}$$

if and only if

$$\psi(f) = f_2^{-1}\{0\} \subseteq g_2^{-1}\{0\} = \psi(g). \tag{11}$$

(Of course this means exactly that $g_2 \leq f_2$ with respect to the usual pointwise order on real-valued functions.) In particular ψ is injective. Furthermore for $A \subseteq X$ with $A^{u\ell} = A$ we define $f_2 : X \rightarrow \{0, 1\}$ so that $A = f_2^{-1}\{0\}$ and $f_1(x) = \sup_{y\in X}(d(y, x)\dot{-}f_2(y))$ whenever $x \in X$. Then $f = (f_1, f_2) \in F_X$ according to Lemma 37 and $\psi(f) = A$, and hence ψ is surjective.

We conclude that the set F_X can be identified with the ground set of the Dedekind-MacNeille completion of X, and $\psi : (F_X, \leq_D) \rightarrow (DM(X), \subseteq)$ is an order isomorphism satisfying $\psi(f_x) = \downarrow x$. □

Remark 39. Given a partially ordered set $(X, \leq)$ equipped with its natural T_0-quasimetric d and its q-hyperconvex hull $Q_{(X,d)}$, the subspace S identified above with $DM(X)$ in $Q_{(X,d)}$ is obviously characterized by the property that it is the largest subspace of $Q_{(X,d)}$ containing $e(X)$ and such that the T_0-quasimetric D restricted to $S \times S$ attains only values in $\{0, 1\}$.

Example 40. Let $X = \{0, 1\}$ be equipped with the discrete order $=$. As we have observed above, the natural T_0-quasimetric on X is the discrete metric. Furthermore (Q_X, D) can be identified with the set $Y = [0, 1] \times [0, 1]$ equipped with the T_0-quasimetric

$$D((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = (\alpha_1\dot{-}\beta_1) \vee (\alpha_2\dot{-}\beta_2) \tag{12}$$

whenever $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in Y$, where 0 is identified with $(0, 1)$ and 1 is identified with $(1, 0)$ (see [6, Example 4]). Of course the Dedekind-MacNeille completion of (X, d) consists only of the four corner points of Y endowed with the induced specialization order on Y (see Example P_2 in [9, page 169]).

Problem 1. Given a T_0-quasimetric space (X, d), compare the set of the endpoints of (X, d) with the set of the endpoints of (Q_X, D).

In this paper we established that these two sets are equal if $\tau(d^s)$ is compact.

7. Extended T_0-Quasimetrics

Given a partially ordered set $(X, \leq)$, in the last three sections we worked with the natural T_0-quasimetric d of X. The choice of the value 1 in the definition of the natural T_0-quasimetric looked rather arbitrary. A more canonical approach can be given if one uses extended T_0-quasimetrics.

Remark 41. Let $(X, \leq)$ be a partially ordered set. For each $x, y \in X$, we set $\rho(x, y) = 0$ if $x \leq y$ and $\rho(x, y) = \infty$ otherwise (see, e.g., [8, Section 2.1]). Obviously, ρ is an extended T_0-quasimetric on X.

We note however that, for any $a \in [0, \infty]$, we have $\infty + a = \infty$, but $1 + 1 = 2$ (so that $\{0, 1\}$ is not closed under the operation of addition $+$). Therefore we obtain distinct theories of collinearity by considering (X, d) or (X, ρ), respectively. For our purposes the T_0-quasimetric d seems to lead to a more interesting theory. We finally show how our characterizations of endpoints and startpoints (see Lemma 11 and Corollary 12) have to be modified to be applied to our new setting.

Proposition 42. *Let $(X, \leq)$ be a partially ordered set, ρ its natural extended T_0-quasimetric, and $x, y \in X$. Then x is an endpoint with witness y in (X, ρ) if and only if one has $\{x\} = X\setminus \uparrow y$.*

Furthermore x is a startpoint with witness y in (X, ρ) if and only if one has $\{x\} = X\setminus \downarrow y$.

Proof. Suppose that x is an endpoint with witness y in (X, ρ). Therefore $\rho(y, x) = \infty$, and, for all $a \in X, \rho(y, x) + \rho(x, a) = \rho(y, a)$ implies that $x = a$. Hence $x \in X\setminus \uparrow y$. Consider any $a \in X\setminus \uparrow y$. Thus $\infty = \rho(y, x) + \rho(x, a) = \rho(y, a)$, which by our assumption implies that $x = a$. Therefore $\{x\} = X\setminus \uparrow y$.

For the converse suppose that $\{x\} = X\setminus \uparrow y$. Then $\rho(y, x) = \infty$. Let $a \in X$ be such that $\rho(y, x) + \rho(x, a) = \rho(y, a)$. Then $\rho(y, a) = \infty$ and $a \in X\setminus \uparrow y = \{x\}$. Consequently $a = x$. Thus x is an endpoint with witness y in (X, ρ). The dual result is proved analogously. □

Example 43. Let us consider the set $\mathbb{Z}$ of the integers equipped with its usual linear order, and let d be its natural T_0-quasimetric and, ρ its natural extended T_0-quasimetric respectively. Then $(\mathbb{Z}, \rho)$ does not have any endpoints nor any startpoints by Proposition 42, while in $(\mathbb{Z}, d)$ each point is an endpoint and a startpoint (see Corollary 24).

8. Conclusion

We showed that for any join compact T_0-quasimetric space (X, d) the set of endpoints (resp., startpoints) of (X, d) is equal to the set of endpoints (resp., startpoints) of its q-hyperconvex hull (Q_X, D). We also specialized some of our earlier results on endpoints contained in [6] to two-valued T_0-quasimetric spaces. In particular we observed that in the case of a complete lattice X and its natural T_0-quasimetric d the startpoints (resp., endpoints) of (X, d) are exactly the completely join-irreducible (resp., the completely meet-irreducible) elements. For a partially ordered set we finally explored the connection between its Dedekind-MacNeille completion and the q-hyperconvex hull of its natural T_0-quasimetric space.

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