

Research Article

Multiple Solutions for a Second-Order Impulsive Sturm-Liouville Equation

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We study the existence of solutions to a boundary value problem of a second-order impulsive Sturm-Liouville equation with a control parameter λ . By employing some existing critical point theorems, we find the range of the control parameter in which the boundary value problem admits at least three solutions. It is also shown that, under certain conditions, there exists an interval of the control parameter in which the boundary value problem possesses infinitely many solutions. Some examples are given to demonstrate the main results in this paper.

1. Introduction

The aim of this paper is to investigate the existence of multiple solutions of the following Neumann boundary value problem with impulsive Sturm-Liouville type equation:

$$\begin{aligned} -(p(t)u'(t))' + q(t)u(t) &= \lambda f(t, u(t)), \quad t \neq t_j, \quad t \in [0, 1], \\ \Delta(p(t_j)u'(t_j)) &= I_j(u(t_j)), \quad j = 1, 2, \dots, m, \\ u'(0) &= u'(1) = 0, \end{aligned} \quad (1)$$

where $p \in C^1([0, 1])$, $q \in C([0, 1])$, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, $I_j \in C(\mathbb{R}, \mathbb{R})$, $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = 1$, and λ is a positive real parameter. $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$, where $u'(t_j^+)$ and $u'(t_j^-)$ denote the right and left limits, respectively, of $u'(t)$ at $t = t_j$, $j = 1, 2, \dots, m$.

In the last few years, the existence of multiple solutions to Neumann problems has been widely investigated [1–5]. But little research has focused on the existence of multiple solutions for impulsive Sturm-Liouville equations whose right-hand side nonlinear term is depending on a parameter λ . Processes subject to sudden changes in their states are modeled by the impulsive differential equations and have

been investigated in various fields of science and technology. In the motion of spacecraft, one has to consider instantaneous impulses at a position with jump discontinuities in velocity, but no change in the position [6, 7]. This motivates us to consider problem (1).

In the literature, tools employed to establish the existence of solutions of impulsive differential equations include fixed point theorems, the upper and lower solutions method, the degree theory, critical point theory, and variational methods. See, for example, [8–20]. In this paper, our focus is on the existence of solutions of problem (1) with λ being the parameter. The problem is first transformed into the existence of critical points of some variational structure. Then with the help of critical point theory, results on the existence of at least three solutions and infinitely many solutions are established.

The rest of this paper is organized as follows. In Section 2 we present some preliminary results. Our main results and their proofs are given in Section 3.

2. Preliminaries

Throughout we assume that $p(t)$ and $q(t)$ satisfy

$$p_0 = \min_{t \in [0, 1]} p(t) > 0, \quad q_0 = \min_{t \in [0, 1]} q(t) > 0. \quad (2)$$

Take $X = W^{1,2}([0, 1])$ and define

$$\|u\|_X = \left(\int_0^1 (p(t) |u'(t)|^2 + q(t) |u(t)|^2) dt \right)^{1/2}, \quad u \in X. \tag{3}$$

For the norm in $C([0, 1])$, we put

$$\|u\|_\infty = \max_{t \in [0,1]} |u(t)|. \tag{4}$$

We have the following relation.

Lemma 1. *Let $M_1 = \min\{p_0, q_0\}$. Then*

$$\|u\|_\infty \leq \sqrt{\frac{2}{M_1}} \|u\|_X, \quad \forall u \in X. \tag{5}$$

Proof. For any $u \in X$, it follows from the mean value theorem that

$$\int_0^1 u(s) ds = u(\tau) \tag{6}$$

for some $\tau \in [0, 1]$. Hence, for $t \in [0, 1]$, using Hölder inequality and (2), we have

$$\begin{aligned} |u(t)| &= \left| u(\tau) + \int_\tau^t u'(s) ds \right| \leq |u(\tau)| + \int_0^1 |u'(s)| ds \\ &\leq \int_0^1 |u(s)| ds + \int_0^1 |u'(s)| ds \\ &\leq \left(\frac{1}{q_0} \right)^{1/2} \left(\int_0^1 q(t) |u(t)|^2 dt \right)^{1/2} \\ &\quad + \left(\frac{1}{p_0} \right)^{1/2} \left(\int_0^1 p(t) |u'(t)|^2 dt \right)^{1/2} \\ &\leq \sqrt{\frac{2}{M_1}} \|u\|_X. \end{aligned} \tag{7}$$

Define a functional φ_λ as

$$\varphi_\lambda(u) = \Phi(u) - \lambda \Psi(u), \quad u \in X, \tag{8}$$

where

$$\Phi(u) = \frac{1}{2} \|u\|_X^2 + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s) ds, \tag{9}$$

$$\Psi(u) = \int_0^1 F(t, u) dt, \tag{10}$$

with

$$F(t, u) = \int_0^{u(t)} f(t, s) ds. \tag{11}$$

Note that φ_λ is Fréchet differentiable at any $u \in X$, and for any $v \in X$, we have

$$\begin{aligned} \varphi'_\lambda(u)(v) &= \lim_{h \rightarrow 0} \frac{\varphi_\lambda(u + hv) - \varphi_\lambda(u)}{h} \\ &= \int_0^1 (p(t) u'(t) v'(t) + q(t) u(t) v(t)) dt \\ &\quad + \sum_{j=1}^m I_j(u(t_j)) v(t_j) \\ &\quad - \lambda \int_0^1 f(t, u(t)) v(t) dt. \end{aligned} \tag{12}$$

Lemma 2. *If $u \in X$ is a weak solution of problem (1), then u is a classical solution of problem (1).*

Proof. For any function $v \in X$, we get

$$\begin{aligned} 0 &= \int_0^1 (p(t) u'(t) v'(t) + q(t) u(t) v(t)) dt \\ &\quad + \sum_{j=1}^m I_j(u(t_j)) v(t_j). \end{aligned} \tag{13}$$

By the regularity theory, the weak solution u is a classical solution of problem (1). \square

Next we show that a critical point of the functional φ_λ is a solution of problem (1).

Lemma 3. *If $u \in X$ is a critical point of φ_λ , then u is a classical solution of problem (1).*

Proof. If $u \in X$ is a critical point of φ_λ , by (12) and Lemma 2, we have u is a classical solution of problem (1) \square

For $r \in \mathbb{R}$, we define

$$\rho(r) = \inf_{v \in \Phi^{-1}((-\infty, r))} \frac{\sup_{u \in \Phi^{-1}((-\infty, r))} \Psi(u) - \Psi(v)}{r - \Phi(v)}, \tag{14}$$

$$\gamma_1 := \liminf_{r \rightarrow +\infty} \rho(r), \quad \delta := \liminf_{r \rightarrow (\inf_X \Phi)^+} \rho(r). \tag{15}$$

3. Main Results

3.1. Existence of At Least Three Solutions. In this section we derive conditions under which problem (1) admits at least three solutions. For this purpose, we introduce the following assumptions.

(H1) Assume that there exists a positive constant k_1 such that for each $u \in X$

$$0 \leq \sum_{j=1}^m \int_0^{u(t_j)} I_j(s) ds \leq k_1 \max_{j \in \{1, 2, \dots, m\}} |u(t_j)|^2. \tag{16}$$

(H2) Assume that there exist positive constants a, b , and $\gamma \in [0, 1]$, such that

$$0 < f(t, u) \leq a + b|u|^\gamma, \quad \text{for } (t, u) \in [0, 1] \times \mathbb{R}. \quad (17)$$

Let $k_0 = \int_0^1 q(t)dt$ and let $k_2 = (1/2)k_0 + k_1$ with k_1 given in (16). Clearly, $M_1 \leq k_0$. For constants c_1, c , we define

$$\begin{aligned} \lambda_1 &= \frac{4}{M_1} \frac{\int_0^1 \max_{|u| \leq c_1} F(t, u) dt}{c_1^2}, \\ \lambda_2 &= \frac{2}{3} \frac{\int_0^1 F(t, c) dt}{k_2 c^2}. \end{aligned} \quad (18)$$

Theorem 4. Assume that (H1), (H2) are satisfied. If there exist two positive constants c_1, c satisfying $c_1 < c$, and

$$\lambda_1 < \lambda_2, \quad (19)$$

then, for each $\lambda \in (1/\lambda_2, 1/\lambda_1)$, problem (1) admits at least three solutions.

Proof. By Lemma 3, it suffices to show that the functional φ_λ defined in (8) has at least three critical points. We prove this by verifying the conditions given in [21, Theorem 3.2]. Note that Φ defined in (9) is a nonnegative Gâteaux differentiable, coercive, and sequentially weakly lower semicontinuous functional and its Gâteaux derivative admits a continuous inverse on X^* . Moreover, Ψ defined in (10) is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Set

$$r = \frac{M_1 c_1^2}{4}. \quad (20)$$

Note that $u_1(t) = c \in X$. It then follows from (H1) that

$$\begin{aligned} \Phi(u_1) &= \frac{1}{2} \|u_1\|_X^2 + \sum_{j=1}^m \int_0^{u_1(t_j)} I_j(s) ds \\ &= \frac{1}{2} c^2 \int_0^1 q(s) ds + \sum_{j=1}^m \int_0^c I_j(s) ds \\ &\leq \frac{1}{2} k_0 c^2 + k_1 c^2 \\ &\leq k_2 c^2, \\ \Phi(u_1) &\geq \frac{1}{2} \|u_1\|_X^2 = \frac{1}{2} k_0 c^2. \end{aligned} \quad (21)$$

Then, we have

$$2r = \frac{M_1 c_1^2}{2} < \frac{1}{2} k_0 c^2 \leq \Phi(u_1). \quad (22)$$

For $u \in X$ satisfying $\Phi(u) < r$, by Lemma 1, one has

$$\begin{aligned} \|u\|^2 \leq \|u\|_\infty^2 &\leq \frac{2}{M_1} \|u\|_X^2 \leq \frac{4}{M_1} \Phi(u) < \frac{4}{M_1} r = c_1^2, \\ t &\in [0, 1], \end{aligned} \quad (23)$$

which implies that

$$\Psi(u) = \int_0^1 F(t, u(t)) dt \leq \int_0^1 \max_{|u| \leq c_1} F(t, u) dt. \quad (24)$$

Hence

$$\sup_{u \in \Phi^{-1}((-\infty, r))} \Psi(u) \leq \int_0^1 \max_{|u| \leq c_1} F(t, u) dt. \quad (25)$$

So, one has

$$\begin{aligned} 0 &\leq \frac{\sup_{u \in \Phi^{-1}((-\infty, r))} \Psi(u)}{r} \leq \frac{4}{M_1} \frac{\int_0^1 \max_{|u| \leq c_1} F(t, u) dt}{c_1^2}, \\ \frac{2}{3} \frac{\Psi(u_1)}{\Phi(u_1)} &\geq \frac{2}{3} \frac{\int_0^1 F(t, c) dt}{k_2 c^2}. \end{aligned} \quad (26)$$

Making use of (19), we obtain

$$\frac{\sup_{u \in \Phi^{-1}((-\infty, r))} \Psi(u)}{r} < \frac{2}{3} \frac{\Psi(u_1)}{\Phi(u_1)}. \quad (27)$$

By (H1) and (H2), when $\lambda \in (1/\lambda_2, 1/\lambda_1)$, we easily obtain that the functional φ_λ is coercive.

Thus all conditions in [21, Theorem 3.2] are verified, and hence for each $\lambda \in (1/\lambda_2, 1/\lambda_1)$, the functional $\varphi_\lambda = \Phi - \lambda\Psi$ admits at least three critical points. Consequently, problem (1) admits at least three solutions. \square

Example 5. Consider the boundary value problem

$$\begin{aligned} -(e^t u'(t))' + u(t) &= \lambda f(t, u), \quad t \neq t_1, t \in [0, 1], \\ \Delta(e^{t_1} u'(t_1)) &= u(t_1), \quad t_1 = \frac{1}{2}, \end{aligned} \quad (28)$$

$$u'(0) = u'(1) = 0,$$

where

$$f(t, u) = \begin{cases} e^u, & u \leq 16, t \in [0, 1], \\ u^{1/2} + e^{16} - 4, & u > 16, t \in [0, 1]. \end{cases} \quad (29)$$

Here, $I_1(s) = s$, $p(t) = e^t$, $q(t) = 1$, and $m = 1$. Note that (H1), (H2) are satisfied. Moreover, we have $k_0 = 1$, $k_1 = 1/2$, $k_2 = 1$, $M_1 = 1$, and

$$F(t, u) = \begin{cases} e^u - 1, & u \leq 16, t \in [0, 1], \\ \frac{2}{3} u^{3/2} + (e^{16} - 4)u \\ \quad + \frac{61}{3} - 15e^{16}, & u > 16, t \in [0, 1]. \end{cases} \quad (30)$$

Choose $c_1 = 1, c = 16$. Direct calculations give

$$\lambda_1 = \frac{4}{M_1} \frac{\int_0^1 \max_{|u| \leq c_1} F(t, u) dt}{c_1^2} = 4(e - 1) < 8, \quad (31)$$

$$\lambda_2 = \frac{2}{3} \frac{\int_0^1 F(t, c) dt}{k_2 c^2} = \frac{1}{384} (e^{16} - 1) > 100.$$

Therefore it follows from Theorem 4 that (28) admits at least three solutions in X provided that $\lambda \in (0.01, 0.125)$.

3.2. *Existence of Infinitely Many Solutions.* In this section, we derive some conditions under which problem (1) admits infinitely many distinct solutions. To this end, we need the following assumption.

(H3) Assume that

$$F(t, u) \geq 0, \quad (t, u) \in [0, 1] \times \mathbb{R}. \tag{32}$$

Set

$$\begin{aligned} \mu_1 &= \frac{4}{M_1} \liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \max_{|u| \leq \xi} F(t, u) dt}{\xi^2}, \\ \mu_2 &= \frac{1}{k_2} \limsup_{\xi \rightarrow +\infty} \frac{\int_0^1 F(t, \xi) dt}{\xi^2}. \end{aligned} \tag{33}$$

Theorem 6. Assume that (H3) is satisfied. If

$$\mu_1 < \mu_2 \tag{34}$$

holds, then for each $\lambda \in (1/\mu_2, 1/\mu_1)$, problem (1) has an unbounded sequence of solutions in X .

Proof. We apply [22, Theorem 2.1] to show that the functional φ_λ defined in (8) has an unbounded sequence of critical points.

We first show that $\gamma_1 < +\infty$. Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \rightarrow +\infty$ as $n \rightarrow \infty$ and

$$\lim_{n \rightarrow +\infty} \frac{\int_0^1 \max_{|u| \leq \xi_n} F(t, u) dt}{\xi_n^2} = \liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \max_{|u| \leq \xi} F(t, \xi) dt}{\xi^2}. \tag{35}$$

For any positive integer n , we let $r_n = M_1 \xi_n^2/4$. For $u \in X$ satisfying $\Phi(u) < r_n$, similar to the proof of Theorem 4, one can show that

$$\|u\|_\infty^2 \leq \frac{4}{M_1} \Phi(u) < \xi_n^2, \quad t \in [0, 1], \tag{36}$$

which implies that

$$\Psi(u) = \int_0^1 F(t, u) dt \leq \int_0^1 \max_{|u| \leq \xi_n} F(t, u) dt. \tag{37}$$

Note that $\Psi(0) = \Phi(0) = 0$; thus we have

$$\begin{aligned} &\rho(r_n) \\ &= \inf_{v \in \Phi^{-1}((-\infty, r_n))} \frac{\sup_{u \in \Phi^{-1}((-\infty, r_n))} \Psi(u) - \Psi(v)}{r_n - \Phi(v)} \\ &\leq \frac{\sup_{u \in \Phi^{-1}((-\infty, r_n))} \Psi(u) - \Psi(0)}{r_n - \Phi(0)} \\ &= \frac{\sup_{u \in \Phi^{-1}((-\infty, r_n))} \Psi(u)}{r_n} \\ &\leq \frac{4 \int_0^1 \max_{|u| \leq \xi_n} F(t, u) dt}{M_1 \xi_n^2}, \end{aligned} \tag{38}$$

which, together with (15), gives us

$$\gamma_1 \leq \frac{4}{M_1} \liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \max_{|u| \leq \xi} F(t, u) dt}{\xi^2} = \mu_1 < +\infty. \tag{39}$$

This shows that $(1/\mu_2, 1/\mu_1) \subseteq (0, 1/\gamma_1)$. For any fixed $\lambda \in (1/\mu_2, 1/\mu_1)$, it follows from [22, Theorem 2.1] that either $\varphi_\lambda = \Phi - \lambda\Psi$ has a global minimum or there is a sequence $\{u_n\}$ of critical points (local minima) of φ_λ such that $\lim_{n \rightarrow +\infty} \|u_n\|_X = +\infty$.

Next we show that the functional φ_λ has no global minimum for $\lambda \in (1/\mu_2, 1/\mu_1)$. Since $\lambda > 1/\mu_2 = k_2/\limsup_{\xi \rightarrow +\infty} (\int_0^1 F(t, \xi) dt/\xi^2)$, we can choose a constant M_2 such that, for each $n \in N = \{1, 2, \dots\}$,

$$\sup_{\xi \geq n} \frac{\int_0^1 F(t, \xi) dt}{\xi^2} > M_2 > \frac{k_2}{\lambda}. \tag{40}$$

Thus, there exists $\xi_n \geq n$ such that

$$\frac{\int_0^1 F(t, \xi_n) dt}{\xi_n^2} > M_2. \tag{41}$$

Define $u_n(t)$ as follows:

$$u_n(t) = \xi_n, \quad t \in [0, 1]. \tag{42}$$

This yields

$$\begin{aligned} \Phi(u_n) &= \frac{1}{2} k_0 \xi_n^2 + \sum_{j=1}^m \int_0^{\xi_n} I_j(s) ds \\ &\leq \frac{1}{2} k_0 \xi_n^2 + k_1 \xi_n^2 \\ &= k_2 \xi_n^2. \end{aligned} \tag{43}$$

Then

$$\begin{aligned} \varphi_\lambda(u_n) &= \Phi(u_n) - \lambda\Psi(u_n) \\ &\leq k_2 \xi_n^2 - \lambda \int_0^1 F(t, \xi_n) dt \\ &\leq \xi_n^2 (k_2 - \lambda M_2). \end{aligned} \tag{44}$$

Note that $k_2 - \lambda M_2 < 0$. Thus the functional φ_λ has no lower bound and hence it has no global minimum and the proof is complete. \square

Let

$$\begin{aligned} \mu_3 &= \frac{4}{M_1} \liminf_{\omega \rightarrow 0^+} \frac{\int_0^1 \max_{|u| \leq \omega} F(t, u) dt}{\omega^2}, \\ \mu_4 &= \frac{1}{k_2} \limsup_{\omega \rightarrow 0^+} \frac{\int_0^1 F(t, \omega) dt}{\omega^2}. \end{aligned} \tag{45}$$

Theorem 7. Assume that (H3) is satisfied. If

$$\mu_3 < \mu_4 \tag{46}$$

holds, then, for each $\lambda \in (1/\mu_4, 1/\mu_3)$, problem (1) has a sequence of nonzero solutions in X , which weakly converges to 0.

Proof. The proof is similar to that of Theorem 6 by showing that $\delta < +\infty$ and 0 is not a local minimum of the functional $\varphi_\lambda = \Phi - \lambda\Psi$. \square

Example 8. Consider the boundary value problem

$$\begin{aligned} -(e^t u'(t))' + u(t) &= \lambda f(t, u), \quad t \neq t_1, t \in [0, 1], \\ \Delta(e^{t_1} u'(t_1)) &= u(t_1), \quad t_1 = \frac{1}{2}, \\ u'(0) = u'(1) &= 0, \end{aligned} \tag{47}$$

where $f(t, u) = 8tu(1 - \sin u) - 4tu^2 \cos u$.

Here, $I_1(s) = s$, $p(t) = e^t$, $q(t) = 1$, and $m = 1$. Hence we have $k_0 = 1, k_1 = 1/2, k_2 = 1, M_1 = 1$, and

$$F(t, u) = 4t(1 - \sin u)u^2, \tag{48}$$

so (H3) is satisfied. Moreover, we have

$$\begin{aligned} \lim_{\xi \rightarrow +\infty} \inf \frac{\int_0^1 \max_{|u| \leq \xi} F(t, u) dt}{\xi^2} &= 0, \\ \lim_{\xi \rightarrow +\infty} \sup \frac{\int_0^1 F(t, \xi) dt}{\xi^2} &= 2. \end{aligned} \tag{49}$$

Therefore, condition (34) holds and Theorem 6 applies: for $\lambda \in (0.5, +\infty)$, problem (47) admits an unbounded sequence of solutions in X .

Example 9. Consider the boundary value problem

$$\begin{aligned} -(e^t u'(t))' + u(t) &= \lambda f(t, u), \\ t &\neq t_1, t \in [0, 1], \\ \Delta(e^{t_1} u'(t_1)) &= u(t_1), \quad t_1 = \frac{1}{2}, \\ u'(0) = u'(1) &= 0, \end{aligned} \tag{50}$$

where

$$\begin{aligned} f(t, u(t)) &= \begin{cases} tu \left(0.35 + \frac{1}{4} \sin(\ln |u|) + \frac{1}{8} \cos(\ln |u|) \right) \\ 0, \end{cases} \\ & \quad u \in \mathbb{R} \setminus \{0\}, t \in [0, 1] \\ & \quad u = 0. \end{aligned} \tag{51}$$

In this example, $I_1(s) = s$, $p(t) = e^t$, $q(t) = 1$, and $m = 1$. Hence we have $k_0 = 1, k_1 = 1/2, k_2 = 1, M_1 = 1$, and

$$F(t, u(t)) = \begin{cases} \frac{1}{2}tu^2 \left(0.35 + \frac{1}{4} \sin(\ln |u|) \right), \\ u \in \mathbb{R} \setminus \{0\}, t \in [0, 1], \\ 0, \quad u = 0. \end{cases} \tag{52}$$

Hence, one has

$$\frac{4}{M_1} \liminf_{w \rightarrow 0^+} \frac{\int_0^1 \max_{|u| \leq w} F(t, u) dt}{w^2} \tag{53}$$

$$= 0.1 < \frac{1}{k_2} \limsup_{w \rightarrow 0^+} \frac{\int_0^1 F(t, w) dt}{w^2} = 0.15.$$

Therefore (46) holds. Owing to Theorem 7, when $\lambda \in (6.67, 10)$ problem (50) admits a sequence of pairwise distinct classical solutions strongly converging at 0 in X .

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