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Research Article

New Recursive Representations for the Favard Constants with Application to Multiple Singular Integrals and Summation of Series

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There are obtained integral form and recurrence representations for some Fourier series and connected with them Favard constants. The method is based on preliminary integration of Fourier series which permits to establish general recursion formulas for Favard constants. This gives the opportunity for effective summation of infinite series and calculation of some classes of multiple singular integrals by the Favard constants.

1. Introduction

The Fourier series and related with them Achieser-Krein-Favard constants, often simply called Favard constants, have significant theoretical and practical roles in many areas [1, 2]. These remarkable mathematical constants are introduced firstly in the theory of Fourier series and approximations of functions by trigonometric polynomials [3].

The classical definitions of Favard constants are given by the infinite series [1, 4, 5]

$$K_r = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu(r+1)}}{(2\nu+1)^{r+1}}, \quad r = 0, 1, 2, \ldots, \tag{1}$$

$$\widetilde{K}_r = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu r}}{(2\nu+1)^{r+1}}, \quad r = 1, 2, \ldots. \tag{2}$$

These constants find wide applications in the approximation theory for exact and asymptotic results on the approximation of functions, and especially for the best approximations of trigonometric and other classes of functions in different spaces and related inequalities [1, 5–14]. In particular, many important applications are concerned with the approximation of Euler, cardinal, periodic, and other type of splines [15–17]. It can be noted that Favard constants are connected to approximations that are best in a pointwise sense in comparison, for instance, with the Lebesgue constants which are connected to approximations that are best in a least-squares sense (Fourier series) [5]. The Favard constants play also an important role in estimating optimal quadrature and cubature formulas, calculation of singular integrals, some classes of differential, integrodifferential and integral equations [18–24], and in other areas.

Nevertheless widely used, as a whole, the properties of the Favard constants have not been investigated well enough [14], except for some particular cases.

Different methods for their calculation are given, for instance, in [2, Ch. 5.2]. In general these methods are based on the properties of the well-known constants and special functions as gamma function $\Gamma(z)$, generalized Riemann zeta function $\zeta(z,a)$, the Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n, and the Euler polynomials $E_n(x)$ and the Euler numbers E_n, given by the following expressions [2]:

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt \quad (\operatorname{Re} z > 0),$$

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

$$\zeta(z,a)=\sum_{k=0}^{\infty}\frac{1}{(k+a)^z}=\frac{1}{\Gamma(z)}\int_0^{\infty}\frac{t^{z-1}e^{-at}}{1-e^{-t}}dt\quad(\operatorname{Re}z>1),$$

$$B_n(x):\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!}\quad(|t|<2\pi),$$

$$B_n=B_n(0),\qquad B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k},$$

$$E_n(x):\frac{2e^{xt}}{e^t+1}=\sum_{k=0}^{\infty}E_k(x)\frac{t^k}{k!},$$

$$\left(\frac{1}{\cosh(t)}=\sum_{k=0}^{\infty}E_k\frac{t^k}{k!}\right),\quad(|t|<\pi).\tag{3}$$

From now on we will use the following notations:

$$\widetilde{A}_n(x)=\int_0^x dx\cdots\int_0^x \ln\tan\frac{\theta}{2}d\theta\quad(0\le x\le\pi),$$

$$\widetilde{B}_n(x)=\int_0^x dx\cdots\int_0^x \ln\left(2\sin\frac{\theta}{2}\right)d\theta\quad(0\le x\le 2\pi),$$

$$\widetilde{C}_n(x)=\int_0^x dx\cdots\int_0^x \ln\left(2\cos\frac{\theta}{2}\right)d\theta$$

$$(-\pi\le x\le\pi)\ \ (n=1,2,3,\ldots)\tag{4}$$

$$D_n(x)=\frac{x^n}{n!},\tag{5}$$

$$T_r=\sum_{n=1}^{\infty}\frac{1}{n^r}\quad(r=2,3,\ldots),$$

$$Q_r=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^r}\quad(r=1,2,3,\ldots),\tag{6}$$

where the three multiple singular integrals (4) contain exactly n integral operations.

The main purpose of this paper is to establish new recursive formulas for the Favard constants (1), including only finite number of terms and recursive formulas for (2). They will be further used to obtain new integral representations for the previously stated objects, in particular for calculation of the multiple singular integrals (4) and summation of series.

2. Recursive Representations for K_r and Some Fourier Series

We will prove the following.

Theorem 1. *For the constants K_r $(r=1,2,3,\ldots)$, the following recursive representations hold:*

$$K_{2s+1}=\frac{1}{2}\left[(-1)^sD_{2s+1}(\pi)+\sum_{p=1}^{s}K_{2s+1-2p}\frac{(-1)^{p-1}\pi^{2p}}{(2p)!}\right]$$

$$\left(K_0=1,\ K_1=\frac{\pi}{2}\right),\tag{7a}$$

$$K_{2s}=(-1)^sD_{2s}\left(\frac{\pi}{2}\right)+\sum_{p=1}^{s}K_{2s+1-2p}\frac{(-1)^{p-1}}{(2p-1)!}\left(\frac{\pi}{2}\right)^{2p-1}$$

$$(s=1,2,3,\ldots).\tag{7b}$$

Proof. We will use the method of induction and preliminary integration of appropriate Fourier series. Let us start by the well-known expansion [2]

$$\frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{\sin[(2\nu+1)x]}{2\nu+1}=1\quad(0<x<\pi),\tag{8}$$

where for $x=\pi/2$ we will have $K_0=1$. By integration of both sides of (8) in $[0,x]$, we get

$$-\frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{\cos[(2\nu+1)x]}{(2\nu+1)^2}+\frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+1)^2}=x=D_1(x)$$

$$(0\le x\le\pi).\tag{9}$$

For $x=\pi$, (9) gives us $2K_1=\pi=D_1(\pi)$, and consequently $K_1=D_1(\pi)/2=\pi/2$. The same results for K_0 and K_1 can be achieved starting from the equality [2]

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\frac{\pi}{2}-\frac{x}{2}=\frac{\pi}{2}-\frac{1}{2}D_1(x)\quad(0<x<2\pi).\tag{10}$$

For $x=\pi/2$ we find again that $K_0=1$. After integration of the both sides of (10), we have

$$-\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}+T_2=-\frac{x^2}{4}+\frac{\pi x}{2}=-\frac{1}{2}D_2(x)+\frac{\pi}{2}D_1(x)$$

$$(0\le x\le 2\pi).\tag{11}$$

For $x=\pi$ we find that $K_1=\pi/2$. Next after integration of the both sides of (11), we obtain

$$-\sum_{n=1}^{\infty}\frac{\sin nx}{n^3}+T_2x=-\frac{1}{2}D_3(x)+\frac{\pi}{2}D_2(x)\quad(0\le x\le 2\pi),\tag{12}$$

which for $x=\pi$ gives us

$$T_2=-\frac{1}{2\pi}D_3(\pi)+\frac{1}{2}D_2(\pi)=\frac{\pi^2}{6}.\tag{13}$$

If we put $x = \pi/2$ in the same equality (12) and make a little processing, we will arrive at the value $K_2 = \pi^2/8$.

On the other hand, after integration of both sides of (9), we obtain

$$-\frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{\sin[(2\nu+1)x]}{(2\nu+1)^3} + K_1 x = D_2(x) \qquad (0 \le x \le \pi). \tag{14}$$

Here for $x = \pi/2$, we get $-K_2 + (\pi/2)K_1 = D_2(\pi/2)$, and consequently

$$K_2 = \frac{\pi}{2}K_1 - D_2\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8}. \tag{15}$$

For the constant K_3, we must now integrate both sides of (12) as

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^4} - T_4 + T_2 D_2(x) = -\frac{1}{2}D_4(x) + \frac{\pi}{2}D_3(x) \qquad (0 \le x \le 2\pi), \tag{16}$$

and put $x = \pi$ here, or integrate both sides of (14) as

$$\frac{4}{\pi}\sum_{\nu=1}^{\infty}\frac{\cos[(2\nu+1)x]}{(2\nu+1)^4} - \frac{4}{\pi}\sum_{\nu=1}^{\infty}\frac{1}{(2\nu+1)^4} + K_1 D_2(x) = D_3(x) \qquad (0 \le x \le \pi), \tag{17}$$

and put again $x = \pi$. Then we will have

$$\begin{aligned} K_3 &= \frac{1}{\pi}D_4(\pi) - \frac{3}{2}D_3(\pi) + \frac{\pi}{2}D_2(\pi) \\ &= \frac{\pi^2}{4}K_1 - \frac{1}{2}D_3(\pi) = \frac{\pi^3}{24}. \end{aligned} \tag{18}$$

Going on the indicated procedure on the base of induction, we easily arrive at the recursive representations (7a) and (7b) and complete the proof. □

Remark 2. The scheme of this proof is valid for the most of the other statements in this paper.

In connection with Theorem 1, we would like to note another representation of K_r (see, e.g., [25]). It can be written in terms of Lerch transcendent [25], or as it is shown in [2, Section 5.1.4]

$$\begin{aligned} K_{2s-1} &= \frac{2}{\pi(2s)!}\left(2^{2s}-1\right)\pi^{2s}\left|B_{2s}\right|, \\ K_{2s} &= \frac{2}{\pi(2s)!}\left(\frac{\pi}{2}\right)^{2s+1}\left|E_{2s}\right| \quad (s = 1,2,3,\ldots), \end{aligned} \tag{19}$$

where the Bernoulli and Euler numbers are specified in (3).

Data for values of magnitudes of K_r using (7a) and (7b) are shown in Table 1.

The equalities (10)–(13) outline a procedure for summing up the numerical series T_{2s}, $(s = 1,2,3,\ldots)$ in (6). It leads to the assertion.

Table 1: Exact and approximate values of the Favard constants K_r, calculated by the recursive formulas (7a) and (7b) using *Mathematica* software package.

r	Exact values of K_r	Approximate values of K_r
1	$\pi/2$	1.5707963267948966192313216916
2	$\pi^2/8$	1.2337005501361698273543113749
3	$\pi^3/24$	1.2919281950124925073115131277
4	$5\pi^4/384$	1.2683475395052400681828168318
5	$\pi^5/240$	1.2750820199386727219280887918
6	$61\pi^6/46080$	1.2726723265645306132561498711
7	$17\pi^7/40320$	1.2734371248066831633864461900
8	$277\pi^8/2064384$	1.2731754806526058136347769671
9	$31\pi^9/725760$	1.2732612424724875463814366656
10	$50521\pi^{10}/3715891200$	1.2732323827293948495082797108
11	$691\pi^{11}/159667200$	1.2732419458721540967715077901
12	$540553\pi^{12}/392398110720$	1.2732387471572495304117396905

Corollary 3. *The following recursive representation holds:*

$$\begin{aligned} T_{2s} &= (-1)^s\left[\frac{1}{2\pi}D_{2s+1}(\pi) - \frac{1}{2}D_{2s}(\pi)\right] \\ &\quad + \sum_{p=1}^{s-1}\frac{(-1)^{p+1}\pi^{2p}}{(2p+1)!}T_{2s-2p} \quad (s = 1,2,3,\ldots), \end{aligned} \tag{20}$$

where for $s = 1$ by definition $T_0 = 0$.

It can be noted that the numbers T_{2s} can be also represented by the well known formula (see, e.g., [2, 5.1.2])

$$T_{2s} = \frac{2^{2s-1}\pi^{2s}}{(2s)!}\left|B_{2s}\right| \quad (s = 1,2,3,\ldots). \tag{21}$$

The same procedure applied on the base of the equality

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n} = -\frac{1}{2}D_1(x) \quad (-\pi < x < \pi), \tag{22}$$

leads to the following.

Corollary 4. *The following recursive representation holds:*

$$Q_{2s} = \frac{(-1)^s}{2\pi}D_{2s+1}(\pi) + \sum_{p=1}^{s-1}\frac{(-1)^{p+1}\pi^{2p}}{(2p+1)!}Q_{2s-2p} \quad (s = 1,2,3,\ldots), \tag{23}$$

where for $s = 1$ by definition $Q_0 = 0$.

In this connection we will note the explicit formula for Q_{2s} represented by the Bernoulli numbers (see [2, 5.2.1])

$$Q_{2s} = \frac{\left(1 - 2^{2s-1}\right)\pi^{2s}}{(2s)!}\left|B_{2s}\right|, \quad (s = 1,2,3,\ldots). \tag{24}$$

It is easy to see that the constants K_r satisfy the following inequalities (see also [1]):

$$1 = K_0 < K_2 < K_4 < \cdots < \frac{4}{\pi} < \cdots < K_5 < K_3 < K_1 = \frac{\pi}{2} \tag{25}$$

and $\lim_{r\to\infty}K_r = 4/\pi$.

The procedure of getting the representations (11), (12), and (16) with the help of (10) gives us an opportunity to lay down the following.

Theorem 5. *The following recursive representations hold:*

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2s-1}} = (-1)^{s}\left[\frac{1}{2}D_{2s-1}(x)-\frac{\pi}{2}D_{2s-2}(x)\right] + \sum_{p=1}^{s-1}(-1)^{p+1}T_{2s-2p}D_{2p-1}(x),$$
$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2s}} = (-1)^{s-1}\left[\frac{1}{2}D_{2s}(x)-\frac{\pi}{2}D_{2s-1}(x)\right] + \sum_{p=0}^{s-1}(-1)^{p}T_{2s-2p}D_{2p}(x) \tag{26}$$

($s = 1, 2, 3, \ldots$, ($0 \le x \le 2\pi$), for $s = 1$ by definition $D_0(x) = 1$ and $T_0 = 0$, $0 < x < 2\pi$).

At the same time both series in (26) have the well-known representations [2, 5.4.2]

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2s+1}} = \frac{(-1)^{s-1}}{2(2s+1)!}(2\pi)^{2s+1}B_{2s+1}\left(\frac{x}{2\pi}\right),$$
$$((0 \le x \le 2\pi);\ (s = 1,2,3,\ldots),\ 0 < x < 2\pi \text{ for } s = 0),$$
$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2s}} = \frac{(-1)^{s-1}}{2(2s)!}(2\pi)^{2s}B_{2s}\left(\frac{x}{2\pi}\right)$$
$$(0 \le x \le 2\pi;\ s = 1,2,3,\ldots). \tag{27}$$

The previously stated procedure for obtaining (26) can now be applied on the strength of (22). This leads to the assertion.

Theorem 6. *The following recursive representations hold:*

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin nx}{n^{2s-1}} = \frac{(-1)^{s}}{2}D_{2s-1}(x) + \sum_{p=1}^{s-1}(-1)^{p+1}Q_{2s-2p}D_{2p-1}(x),$$
$$\sum_{n=1}^{\infty}(-1)^{n}\frac{\cos nx}{n^{2s}} = \frac{(-1)^{s-1}}{2}D_{2s}(x) + \sum_{p=0}^{s-1}(-1)^{p}Q_{2s-2p}D_{2p}(x), \tag{28}$$

($s = 1, 2, 3, \ldots$, ($-\pi \le x \le \pi$); for $s = 1$ by definition $D_0(x) = 1$ and $Q_0 = 0$, $-\pi < x < \pi$).

At the same time, both series in (28) have the well-known representations [2, 5.4.2]

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin nx}{n^{2s+1}} = \frac{(-1)^{s-1}}{2(2s+1)!}(2\pi)^{2s+1}B_{2s+1}\left(\frac{x+\pi}{2\pi}\right)$$
$$(-\pi < x \le \pi;\ s = 0,1,2,\ldots),$$
$$\sum_{n=1}^{\infty}(-1)^{n}\frac{\cos nx}{n^{2s}} = \frac{(-1)^{s-1}}{2(2s)!}(2\pi)^{2s}B_{2s}\left(\frac{x+\pi}{2\pi}\right)$$
$$(-\pi \le x \le \pi;\ s = 1,2,3,\ldots). \tag{29}$$

By analogy with the previous, the procedure for obtaining (9), (14), and (17) with the help of (8) leads us to the following.

Theorem 7. *The following recursive representations hold:*

$$\sum_{\nu=0}^{\infty}\frac{\sin[(2\nu+1)x]}{(2\nu+1)^{2s-1}} = \frac{\pi(-1)^{s-1}}{4}\left[D_{2s-2}(x)+\sum_{p=1}^{s-1}(-1)^{p}K_{2p-1}D_{2s-2p-1}(x)\right],$$
$$\sum_{\nu=0}^{\infty}\frac{\cos[(2\nu+1)x]}{(2\nu+1)^{2s}} = \frac{\pi(-1)^{s}}{4}\left[D_{2s-1}(x)+\sum_{p=1}^{s}(-1)^{p}K_{2p-1}D_{2s-2p}(x)\right], \tag{30}$$

$s = 1, 2, 3, \ldots$ ($0 \le x \le \pi$); for $s = 1$ by definition $D_0(x) = 1$ ($0 < x < \pi$).

At the same time, both series in (30) have the well-known formulas [2, 5.4.6]

$$\sum_{\nu=0}^{\infty}\frac{\sin[(2\nu+1)x]}{(2\nu+1)^{2s+1}} = \frac{(-1)^{s}\pi^{2s+1}}{4(2s)!}E_{2s}\left(\frac{x}{\pi}\right)$$
$$(0 < x < \pi;\ s = 0,1,2,\ldots),$$
$$\sum_{\nu=0}^{\infty}\frac{\cos[(2\nu+1)x]}{(2\nu+1)^{2s}} = \frac{(-1)^{s}\pi^{2s}}{4(2s-1)!}E_{2s-1}\left(\frac{x}{\pi}\right)$$
$$(0 \le x \le \pi;\ s = 1,2,\ldots). \tag{31}$$

The same procedure applied on the base of the equality

$$\frac{4}{\pi}\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\cos[(2\nu+1)x]}{2\nu+1} = 1 \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right), \tag{32}$$

gives us the next theorem.

Theorem 8. *The following recursive representations hold:*

$$\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\cos[(2\nu+1)x]}{(2\nu+1)^{2s-1}} = \frac{\pi(-1)^{s-1}}{4}\left[D_{2s-2}(x)+\sum_{p=1}^{s-1}(-1)^{p}K_{2p}D_{2s-2p-2}(x)\right],$$
$$\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\sin[(2\nu+1)x]}{(2\nu+1)^{2s}} = \frac{\pi(-1)^{s-1}}{4}\left[D_{2s-1}(x)+\sum_{p=1}^{s-1}(-1)^{p}K_{2p}D_{2s-2p-1}(x)\right] \tag{33}$$

($s=1,2,3,\ldots$, ($-\pi/2\le x\le\pi/2$), for $s=1$: K_0 declines, $D_0(x)=1$, ($-\pi/2<x<\pi/2$)).

At the same time, both series in (33) have the well-known representations ([2, 5.4.6])

$$\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\cos[(2\nu+1)x]}{(2\nu+1)^{2s+1}} = \frac{(-1)^{s}\pi^{2s+1}}{4(2s)!}E_{2s}\left(\frac{x}{\pi}+\frac{1}{2}\right) \quad \left(-\frac{\pi}{2}<x<\frac{\pi}{2};\ s=0,1,\ldots\right),$$
$$\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\sin[(2\nu+1)x]}{(2\nu+1)^{2s}} = \frac{(-1)^{s-1}\pi^{2s}}{4(2s-1)!}E_{2s-1}\left(\frac{x}{\pi}+\frac{1}{2}\right) \quad \left(-\frac{\pi}{2}\le x\le\frac{\pi}{2};\ s=1,2,\ldots\right). \tag{34}$$

Meanwhile it is important to note that the number of addends in our recurrence representations (26), (28), (30), and (33) is two times less than the number of the addends in the corresponding cited formulas from [2]. So our method appears to be more economic and effective.

Moreover, one can get many other representations of the constants K_r ($r=1,2,3,\ldots$) and numerical series Q_{2s} ($s=1,2,\ldots$) from Theorems 5–8 putting, in particular, $x=\pi/2$ or $x=\pi$. For completeness we will note the main results.

From Theorem 5 for $x=\pi/2$ and $x=\pi$ immediately follows the following.

Corollary 9. *For the Favard constants K_r, the following recursive representations hold:*

$$K_{2s-2} = \frac{4}{\pi}\left\{(-1)^{s}\left[\frac{1}{2}D_{2s-1}\left(\frac{\pi}{2}\right)-\frac{\pi}{2}D_{2s-2}\left(\frac{\pi}{2}\right)\right] + \sum_{p=1}^{s-1}(-1)^{p+1}T_{2s-2p}D_{2p-1}\left(\frac{\pi}{2}\right)\right\},$$
$$K_{2s-1} = \frac{2}{\pi}\left\{(-1)^{s}\left[\frac{1}{2}D_{2s}(\pi)-\frac{\pi}{2}D_{2s-1}(\pi)\right] + \sum_{p=1}^{s-1}(-1)^{p+1}T_{2s-2p}D_{2p}(\pi)\right\}, \tag{35}$$

($s=1,2,\ldots$, for $s=1$: $D_0(x)=1, T_0=0$).

For $x=\pi$, one can get (20) too by replacing previously s by $s+1$.

For $x=\pi/2$, we obtain the next corollary.

Corollary 10. *For numbers Q_{2s}, the following recursive representations hold:*

$$Q_{2s} = 4^{s}\left\{(-1)^{s-1}\left[\frac{1}{2}D_{2s}\left(\frac{\pi}{2}\right)-\frac{\pi}{2}D_{2s-1}\left(\frac{\pi}{2}\right)\right] + \sum_{p=0}^{s-1}(-1)^{p}T_{2s-2p}D_{2p}\left(\frac{\pi}{2}\right)\right\} \tag{36}$$

($s=1,2,\ldots$, for $s=1$: $D_0(x)=1$).

Similarly, from Theorem 6 for $x=\pi/2$ and $x=\pi$, we obtain, respectively, the following.

Corollary 11. *For the Favard constants K_r, the following recursive representations hold:*

$$K_{2s-2} = \frac{4}{\pi}\left\{\frac{(-1)^{s-1}}{2}D_{2s-1}\left(\frac{\pi}{2}\right) + \sum_{p=1}^{s-1}(-1)^{p}Q_{2s-2p}D_{2p-1}\left(\frac{\pi}{2}\right)\right\},$$
$$K_{2s-1} = \frac{2}{\pi}\left\{\frac{(-1)^{s-1}}{2}D_{2s}(\pi) + \sum_{p=1}^{s-1}(-1)^{p}Q_{2s-2p}D_{2p}(\pi)\right\}, \tag{37}$$

where $s=1,2,\ldots$, and for $s=1$: $D_0(x)=1$, $Q_0=0$.

For $x=\pi$, one can get (23) too by replacing previously s by $s+1$.

For $x=\pi/2$ from Theorem 6, (the second formula in (28)), we will have also the next analogous corollary.

Corollary 12. *For numbers Q_{2s}, the following recursive representations hold:*

$$Q_{2s} = \frac{4^{s}}{4^{s}-1}\left\{\frac{(-1)^{s}}{2}D_{2s}\left(\frac{\pi}{2}\right) + \sum_{p=1}^{s-1}(-1)^{p+1}Q_{2s-2p}D_{2p}\left(\frac{\pi}{2}\right)\right\}, \tag{38}$$

($s=1,2,\ldots$, for $s=1$: $D_0(x)=1$, $Q_0=0$).

By the same manner from Theorem 7 for $x=\pi$ and $x=\pi/2$, we obtain, respectively, the formulas for K_r ($r=1,2,3,\ldots$) different from these in Theorem 1.

Corollary 13. *For the Favard constants K_{2s-3} and K_{2s-1}, the following recursive representations hold:*

$$K_{2s-3} = \frac{(-1)^s}{\pi} \left\{ D_{2s-2}(\pi) + \sum_{p=1}^{s-2} (-1)^p K_{2p-1} D_{2s-2p-1}(\pi) \right\}, \quad (39)$$

($s = 2, 3, \ldots$, for $s = 2$, $K_1 D_1(\pi)$ must be canceled) and

$$K_{2s-1} = (-1)^{s-1} \left\{ D_{2s-1}\left(\frac{\pi}{2}\right) + \sum_{p=1}^{s-1} (-1)^p K_{2p-1} D_{2s-2p}\left(\frac{\pi}{2}\right) \right\}, \quad (40)$$

($s = 1, 2, 3, \ldots$, for $s = 1$, $K_1 D_0(\pi/2)$ must be canceled).

The remaining cases for $x = \pi/2$ and $x = \pi$ immediately lead to Theorem 1 after replacing s by $s+1$.

From Theorem 8 for $x = \pi/2$, one can get, respectively, other representations for K_r ($r = 1, 2, 3, \ldots$), different from these in Theorem 1.

Corollary 14. *For the Favard constants K_r, the following recursive representations hold:*

$$K_{2s-2} = (-1)^s \left\{ D_{2s-2}\left(\frac{\pi}{2}\right) + \sum_{p=1}^{s-2} (-1)^p K_{2p} D_{2s-2p-2}\left(\frac{\pi}{2}\right) \right\},$$

$$K_{2s-1} = (-1)^{s-1} \left\{ D_{2s-1}\left(\frac{\pi}{2}\right) + \sum_{p=1}^{s-1} (-1)^p K_{2p} D_{2s-2p-1}\left(\frac{\pi}{2}\right) \right\}, \quad (41)$$

where $s = 1, 2, 3, \ldots$, for $s = 1$: $D_0(\pi/2) = -1$, and for $s = 2$: $K_2 D_0(\pi/2)$ must be canceled.

Corollary 15. *From the difference $T_{2s} - Q_{2s} = (\pi/2) K_{2s-1}$ ($s = 1, 2, \ldots$) and after replacing s by $s+1$ in the obtained expression, one gets the following formula:*

$$K_{2s+1} = \frac{(-1)^s}{\pi} D_{2s+2}(\pi) + \sum_{p=1}^{s} \frac{(-1)^{p+1} \pi^{2p}}{(2p+1)!} K_{2s-2p+1} \qquad (s = 0, 1, \ldots). \quad (42)$$

This is somewhat better than the corresponding formula in Theorem 1, because $(2s+1)! > 2(2s)!$ for $s = 1, 2, \ldots$.

3. Recursive Representations for $\widetilde{K}_r$ and Some Fourier Series

Here we will get down to the integral representation and recursive formulas for the constants $\widetilde{K}_r$, defined in (2). As one can see they are closely linked with the approximation of the conjugate classes of functions obtained on the base of the Hilbert transform [1, 26]. First of all let us note their representations easily obtained by means of special functions in (3) as it is shown in [2, 5.1.4] for the Catalan constant as follows:

$$\widetilde{K}_r = \begin{cases} \dfrac{4}{\pi}\left(1 - 2^{-(2s+1)}\right)\zeta(2s+1), & r = 2s,\ s = 1, 2, 3, \ldots, \\ \dfrac{2^{2-4s}}{\pi}\left(\zeta\left(2s, \dfrac{1}{4}\right) - \zeta\left(2s, \dfrac{3}{4}\right)\right) & \\ \quad = \dfrac{2}{\pi\Gamma(2s)} \displaystyle\int_0^\infty \frac{t^{2s-1}}{\operatorname{ch} t} dt, & r = 2s-1,\ s = 1, 2, 3, \ldots. \end{cases} \quad (43)$$

In the beginning we will prove the following assertion.

Theorem 16. *For the constants $\widetilde{K}_r$ ($r = 1, 2, \ldots$), the following recursive representations hold:*

$$\widetilde{K}_{2s-1} = \frac{2(-1)^s}{\pi} \widetilde{A}_{2s-1}\left(\frac{\pi}{2}\right) + \sum_{p=1}^{s-1} \widetilde{K}_{2s-2p} \frac{(-1)^{p-1}}{(2p-1)!}\left(\frac{\pi}{2}\right)^{2p-1},$$

$$\widetilde{K}_{2s} = \frac{2(-1)^s}{\pi} \widetilde{A}_{2s}\left(\frac{\pi}{2}\right) + \sum_{p=1}^{s-1} \widetilde{K}_{2s-2p} \frac{(-1)^{p-1}}{(2p)!}\left(\frac{\pi}{2}\right)^{2p}$$

$$\left(s = 1, 2, 3, \ldots;\ \widetilde{K}_0 \stackrel{\text{def}}{=} 0\right). \quad (44)$$

Proof. The proof of this theorem is based on induction again. However, for completeness we must give somewhat more detailed considerations at first steps, which underline further discussions.

Let us start by the well-known Fourier expansions (see, e.g., [2, 5.4])

$$\sum_{\nu=0}^{\infty} \frac{\sin[(2\nu+1)x]}{(2\nu+1)^2} = -\frac{1}{2}\widetilde{A}_1(x) \quad (0 \le x \le \pi),$$
$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2} = -\widetilde{B}_1(x) \quad (0 \le x \le 2\pi). \quad (45)$$

For $x = \pi/2$, the first equality gives us immediately $(\pi/4)\widetilde{K}_1 = -(1/2)\widetilde{A}_1(\pi/2)$. At the same time, the second equality in (45) leads to $(\pi/4)\widetilde{K}_1 = -\widetilde{B}_1(\pi/2)$. So we obtain the integral representation of $\widetilde{K}_1$ in the form

$$\widetilde{K}_1 = -\frac{4}{\pi}\widetilde{B}_1\left(\frac{\pi}{2}\right) = -\frac{2}{\pi}\widetilde{A}_1\left(\frac{\pi}{2}\right) = 1.166243616123275\cdots. \quad (46)$$

We have another integral representation of the same constant $\widetilde{K}_1$ in our paper [8].

Further by integration of both sides of the first equality in (45), we get

$$\sum_{\nu=0}^{\infty} \frac{\cos[(2\nu+1)x]}{(2\nu+1)^3} = \frac{1}{2}\widetilde{A}_2(x) + \frac{\pi}{4}\widetilde{K}_2 \quad (0 \le x \le \pi), \quad (47)$$

from where for $x = \pi/2$ and $x = \pi$ we have simultaneously $0 = (1/2)\widetilde{A}_2(\pi/2) + (\pi/4)\widetilde{K}_2$ and $-(\pi/4)\widetilde{K}_2 = (1/2)\widetilde{A}_2(\pi) + (\pi/4)\widetilde{K}_2$, and consequently

$$\widetilde{K}_2 = -\frac{2}{\pi}\widetilde{A}_2\left(\frac{\pi}{2}\right) = -\frac{1}{\pi}\widetilde{A}_2(\pi). \tag{48}$$

Then, after the integration of both sides of the second equality in (45), we obtain

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^3} = \widetilde{B}_2(x) + T_3 \quad (0 \le x \le 2\pi), \tag{49}$$

from where for $x = \pi$ we have $(\pi/2)\widetilde{K}_2 = -\widetilde{B}_2(\pi)$. If we correspond this with (48), we find

$$\begin{aligned}\widetilde{K}_2 &= -\frac{2}{\pi}\widetilde{B}_2(\pi) = -\frac{2}{\pi}\widetilde{A}_2\left(\frac{\pi}{2}\right)\\ &= -\frac{1}{\pi}\widetilde{A}_2(\pi) = 1.339193086109090\cdots.\end{aligned} \tag{50}$$

In order to obtain the integral representation of $\widetilde{K}_3$, we must, at first, integrate the both sides of (47) as

$$\sum_{\nu=0}^{\infty}\frac{\sin[(2\nu+1)x]}{(2\nu+1)^4} = \frac{1}{2}\widetilde{A}_3(x) + \frac{\pi}{4}\widetilde{K}_2 D_1(x) \quad (0 \le x \le \pi). \tag{51}$$

Next it remains to put $x = \pi/2$ as

$$\widetilde{K}_3 = \frac{2}{\pi}\widetilde{A}_3\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\widetilde{K}_2 = \frac{2}{\pi}\widetilde{A}_3\left(\frac{\pi}{2}\right) - \widetilde{A}_2\left(\frac{\pi}{2}\right). \tag{52}$$

As another integral representation of $\widetilde{K}_3$, we can get after integration of both sides of (49)

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n^4} = \widetilde{B}_3(x) + T_3 D_1(x) \quad (0 \le x \le 2\pi). \tag{53}$$

On one hand, for $x = \pi$, (53) gives

$$T_3 = -\frac{1}{\pi}\widetilde{B}_3(\pi) = 1.20205690\cdots. \tag{54}$$

On the other hand, the same equality (53) for $x = \pi/2$ gives $\sum_{\nu=0}^{\infty}\sin[(2\nu+1)\pi/2]/(2\nu+1)^4 = \widetilde{B}_3(\pi/2) - (1/2)\widetilde{B}_3(\pi)$. Then

$$\begin{aligned}\widetilde{K}_3 &= \frac{4}{\pi}\widetilde{B}_3\left(\frac{\pi}{2}\right) - \frac{2}{\pi}\widetilde{B}_3(\pi) = \frac{2}{\pi}\widetilde{A}_3\left(\frac{\pi}{2}\right) - \widetilde{A}_2\left(\frac{\pi}{2}\right)\\ &= 1.259163310827165\cdots.\end{aligned} \tag{55}$$

For the constant $\widetilde{K}_4$ there are also different ways to receive its integral representations. One of them is based on the integration of both sides of (53) as

$$-\sum_{n=1}^{\infty}\frac{\cos nx}{n^5} + T_5 = \widetilde{B}_4(x) + T_3 D_2(x) \quad (0 \le x \le 2\pi). \tag{56}$$

For $x = \pi$, (56) gives $(\pi/2)\widetilde{K}_4 = \widetilde{B}_4(\pi) + (\pi^2/2)T_3$. Admitting (54) we will have

$$\widetilde{K}_4 = \frac{2}{\pi}\widetilde{B}_4(\pi) - \widetilde{B}_3(\pi) = 1.278999378416936\cdots. \tag{57}$$

Another integral representation of $\widetilde{K}_4$ can be obtained by integration of both sides of (51) as

$$-\sum_{\nu=0}^{\infty}\frac{\cos[(2\nu+1)x]}{(2\nu+1)^5} + \frac{\pi}{4}\widetilde{K}_4 = \frac{1}{2}\widetilde{A}_4(x) + \frac{\pi}{4}\widetilde{K}_2 D_2(x) \quad (0 \le x \le \pi). \tag{58}$$

For $x = \pi/2$ and $x = \pi$, we get, respectively,

$$\begin{aligned}\widetilde{K}_4 &= \frac{2}{\pi}\widetilde{A}_4\left(\frac{\pi}{2}\right) - \frac{\pi}{4}\widetilde{A}_2\left(\frac{\pi}{2}\right) = \frac{2}{\pi}\widetilde{A}_4\left(\frac{\pi}{2}\right) + \frac{\pi^2}{8}\widetilde{K}_2\\ &= \frac{1}{\pi}\widetilde{A}_4(\pi) + \frac{\pi^2}{4}\widetilde{K}_2.\end{aligned} \tag{59}$$

Further after integration of both sides of (56), we get

$$-\sum_{\nu=0}^{\infty}\frac{\sin nx}{n^6} + T_5 D_1(x) = \widetilde{B}_5(x) + T_3 D_3(x) \quad (0 \le x \le 2\pi). \tag{60}$$

On one hand, if we put here $x = \pi$ and admit (54), we will have

$$\begin{aligned}T_5 &= \frac{1}{\pi}\widetilde{B}_5(\pi) + \frac{\pi^2}{3!}T_3 = \frac{1}{\pi}\widetilde{B}_5(\pi) - \frac{\pi}{6}\widetilde{B}_3(\pi)\\ &= 1.03692776\cdots.\end{aligned} \tag{61}$$

On the other hand, the same formula (60) for $x = \pi/2$ gives us $-(\pi/4)\widetilde{K}_5 + (\pi/2)T_5 = \widetilde{B}_5(\pi/2) + (\pi^3/2^3 3!)T_3$. Then admitting (54) and (61), we obtain

$$\begin{aligned}\widetilde{K}_5 &= -\frac{\pi}{4}\widetilde{B}_5\left(\frac{\pi}{2}\right) + \frac{2}{\pi}\widetilde{B}_5(\pi) - \frac{\pi}{4}\widetilde{B}_3(\pi)\\ &= 1.271565517671139\cdots.\end{aligned} \tag{62}$$

Next, in order to receive another integral representation of $\widetilde{K}_5$, we must integrate both sides of (58). So we will have

$$-\sum_{\nu=0}^{\infty}\frac{\sin[(2\nu+1)x]}{(2\nu+1)^6} + \frac{\pi}{4}\widetilde{K}_4 D_1(x) = \frac{1}{2}\widetilde{A}_2(x) + \frac{\pi}{4}\widetilde{K}_2 D_3(x) \quad (0 \le x \le \pi). \tag{63}$$

Putting $x = \pi/2$, here, we get

$$\begin{aligned}\widetilde{K}_5 &= -\frac{2}{\pi}\widetilde{A}_5\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\widetilde{K}_4 - \frac{\pi^3}{2^3 3!}\widetilde{K}_2\\ &= -\frac{2}{\pi}\widetilde{A}_5\left(\frac{\pi}{2}\right) + \widetilde{A}_4\left(\frac{\pi}{2}\right) - \frac{\pi^2}{2^2\cdot 3}\widetilde{A}_2\left(\frac{\pi}{2}\right).\end{aligned} \tag{64}$$

Now after this preparatory work, we can go on the indicated procedure which leads us to the general recursive representations (43) and so complete the proof. □

The previously stated procedure gives us the opportunity to obtain recursive formulas for the multiple integrals $\widetilde{A}_r$, $\widetilde{B}_r$, and $\widetilde{C}_r$ in (4) for a given x.

First of all we will note that on the base of induction and with the help of the first formula in (45), (47), (51), (58), and (63), we can lay down the following.

Theorem 17. *For the multiple singular integrals $\widetilde{A}_r$, the following recursive representations hold:*

$$\widetilde{A}_{2s-1}(x) = 2(-1)^s \sum_{\nu=0}^{\infty} \frac{\sin[(2\nu+1)x]}{(2\nu+1)^{2s}} + \frac{\pi}{2}\sum_{p=1}^{s-1}(-1)^p \widetilde{K}_{2p} D_{2s-2p-1}(x),$$
$$\widetilde{A}_{2s}(x) = -2(-1)^s \sum_{\nu=0}^{\infty} \frac{\cos[(2\nu+1)x]}{(2\nu+1)^{2s+1}} + \frac{\pi}{2}\sum_{p=1}^{s}(-1)^p \widetilde{K}_{2p} D_{2s-2p}(x), \tag{65}$$

$s = 1, 2, 3, \ldots$ $(0 \le x \le \pi)$, $D_0 = 1$, *for* $s = 0$: $\widetilde{A}_0(x) = \ln(\tan(x/2))$ $(0 < x < \pi)$, *and* $\widetilde{K}_0 = 0$, *where*

$$\sum_{\nu=0}^{\infty} \frac{\sin[(2\nu+1)x]}{(2\nu+1)^{2s}} = \frac{1}{2}\left\{\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2s}} - \sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n^{2s}}\right\} = \frac{1}{\Gamma(2s)}\int_0^{\infty}\frac{t^{2s-1}e^t\left(e^{2t}+1\right)\sin x}{\left(e^{2t}+1\right)^2 - 4e^{2t}\cos^2 x}dt,$$
$$\sum_{\nu=0}^{\infty} \frac{\cos[(2\nu+1)x]}{(2\nu+1)^{2s+1}} = \frac{1}{2}\left\{\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2s+1}} - \sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^{2s+1}}\right\} = \frac{1}{\Gamma(2s+1)}\int_0^{\infty}\frac{t^{2s}e^t\left(e^{2t}-1\right)\cos x}{\left(e^{2t}+1\right)^2 - 4e^{2t}\cos^2 x}dt. \tag{66}$$

By analogy with the previous and by means of the second formula in (45), (49), (53), (56), and (60), one can get the following.

Theorem 18. *For the multiple singular integrals $\widetilde{B}_r$, the following recursive representations hold:*

$$\widetilde{B}_{2s-1}(x) = (-1)^s\left\{\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2s}} + \sum_{p=1}^{s-1}(-1)^p T_{2s-2p+1} D_{2p-1}(x)\right\},$$
$$\widetilde{B}_{2s-2}(x) = (-1)^s\left\{\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2s-1}} + \sum_{p=1}^{s-1}(-1)^p T_{2s-2p+1} D_{2p-2}(x)\right\}, \tag{67}$$

$s = 1, 2, 3, \ldots$ $(0 \le x \le 2\pi)$, $D_0 = 1$, *for* $s = 1$: $T_1 D_k$ $(k = 0, 1)$ *declines, by definition* $\widetilde{B}_0(x) = \ln(2\sin(x/2))$ $(0 < x < 2\pi)$, *where (see [2])*

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2s}} = \frac{1}{\Gamma(2s)}\int_0^{\infty}\frac{t^{2s-1}e^t\sin x}{1-2e^t\cos x + e^{2t}}dt \quad (x \ne 0,\ s = 1, 2, 3, \ldots),$$
$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2s-1}} = \frac{1}{\Gamma(2s-1)}\int_0^{\infty}\frac{t^{2s-2}\left(e^t\cos x - 1\right)}{1-2e^t\cos x + e^{2t}}dt \quad (x \ne 0,\ s = 1, 2, 3, \ldots). \tag{68}$$

As similar representations of $\widetilde{C}_r(x)$, one can get on the base of the expansion (see [2, 5.4.2])

$$\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n} = -\ln\left(2\cos\left(\frac{x}{2}\right)\right) \quad (-\pi < x < \pi),$$
$$Q_1 = -\ln 2. \tag{69}$$

Theorem 19. *For the multiple singular integrals $\widetilde{C}_r$, the following recursive representations hold:*

$$\widetilde{C}_{2s-1}(x) = (-1)^s\left\{\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n^{2s}} + \sum_{p=1}^{s-1}(-1)^p Q_{2s-2p+1} D_{2p-1}(x)\right\},$$
$$\widetilde{C}_{2s-2}(x) = (-1)^s\left\{\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^{2s-1}} + \sum_{p=1}^{s-1}(-1)^p Q_{2s-2p+1} D_{2p-2}(x)\right\}, \tag{70}$$

$s = 1, 2, 3, \ldots$ $(-\pi \le x \le \pi)$, *for* $s = 1$: $\widetilde{C}_0(x) = \ln(2\cos(x/2))$ $(-\pi < x < \pi)$, $Q_1 D_k$ $(k = 0, 1)$ *declines, where (see [2])*

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n^{2s}} = \frac{-1}{\Gamma(2s)}\int_0^{\infty}\frac{t^{2s-1}e^t\sin x}{1+2e^t\cos x + e^{2t}}dt \quad (s = 1, 2, 3, \ldots),$$
$$\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^{2s-1}} = \frac{-1}{\Gamma(2s-1)}\int_0^{\infty}\frac{t^{2s-2}\left(e^t\cos x + 1\right)}{1+2e^t\cos x + e^{2t}}dt \quad (s = 1, 2, \ldots). \tag{71}$$

By analogy with the previous, if we start from the expansion [2, 5.4.6]

$$\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\sin[(2\nu+1)x]}{2\nu+1} = \frac{-1}{2}\ln\left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right) \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right), \tag{72}$$

we can obtain the next theorem.

Theorem 20. *The following recursive formulas hold:*

$$\int_0^{\pi/2-x} \widetilde{A}_{2s-1}(u)\,du = -2(-1)^s \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\sin[(2\nu+1)x]}{(2\nu+1)^{2s+1}} + \frac{\pi}{2}\sum_{p=1}^{s}(-1)^p \widetilde{K}_{2p} D_{2s-2p}\left(\frac{\pi}{2}-x\right),$$

$$\int_0^{\pi/2-x} \widetilde{A}_{2s-2}(u)\,du = 2(-1)^s \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\cos[(2\nu+1)x]}{(2\nu+1)^{2s}} - \frac{\pi}{2}\sum_{p=1}^{s-1}(-1)^p \widetilde{K}_{2p} D_{2s-2p-1}\left(\frac{\pi}{2}-x\right), \quad (73)$$

$s = 1,2,3,\ldots$ *(*$-\pi/2 \le x \le \pi/2$*),* $D_0(x) = 1$*, for* $s = 1$*:* $\widetilde{A}_0(u) = \ln\tan(u/2)$ *(*$-\pi/2 < u < \pi/2$*), where*

$$\sum_{\nu=0}^{\infty}(-1)^\nu \frac{\sin[(2\nu+1)x]}{(2\nu+1)^{2s+1}} = \frac{1}{\Gamma(2s+1)}\int_0^\infty \frac{t^{2s}e^t\left(e^{2t}-1\right)\sin x}{\left(e^{2t}+1\right)^2 - 4e^{2t}\sin^2 x}dt \quad (s=1,2,\ldots),$$

$$\sum_{\nu=0}^{\infty}(-1)^\nu \frac{\cos[(2\nu+1)x]}{(2\nu+1)^{2s}} = \frac{1}{\Gamma(2s)}\int_0^\infty \frac{t^{2s-1}e^t\left(e^{2t}+1\right)\cos x}{\left(e^{2t}+1\right)^2 - 4e^{2t}\sin^2 x}dt \quad (s=1,2,\ldots). \quad (74)$$

The proof of this theorem requires to take integration from $\pi/2$ to x and then to make the substitution $t = \pi/2 - x$.

From Table 2 one can see that $\widetilde{A}_r(x) = \widetilde{B}_r(x) - \widetilde{C}_r(x)$ $(0 \le x \le \pi)$, $r = 1,2,\ldots$. The constants $\widetilde{K}_r$ satisfy the following inequalities [1]:

$$1 < \widetilde{K}_1 < \widetilde{K}_3 < \widetilde{K}_5 < \cdots < \frac{4}{\pi} < \cdots < \widetilde{K}_6 < \widetilde{K}_4 < \widetilde{K}_2 < \frac{\pi}{2}. \quad (75)$$

Remark 21. The equalities (49), (53), (56), and (60) outline a procedure for summing up the numerical series T_{2s+1} $(s = 1,2,\ldots)$ in (2). It leads us to the following.

Corollary 22. *The following recursive representation holds:*

$$T_{2s+1} = \frac{(-1)^s}{\pi}\widetilde{B}_{2s+1}(\pi) + \sum_{p=1}^{s-1}(-1)^{p+1}\frac{\pi^{2p}}{(2p+1)!}T_{2s-2p+1}, \quad (76)$$

$s = 1,2,\ldots$ *for* $s = 1$ *the term* $(\pi^2/6)\ T_1$ *declines.*

For comparison we will note also the well-known formula [2, 5.1.2]

$$T_{2s+1} = \zeta(2s+1) = \frac{1}{\Gamma(2s+1)}\int_0^\infty \frac{t^{2s}}{e^t - 1}dt = \frac{-1}{(2s)!}\Psi^{(2s)}(1) \quad (s = 1,2,\ldots). \quad (77)$$

The same procedure based on induction and preliminary multiple integration of both sides of (69) leads us to the next.

Corollary 23. *The following recursive representation holds:*

$$Q_{2s+1} = \frac{(-1)^s}{\pi}\widetilde{C}_{2s+1}(\pi) + \sum_{p=1}^{s-1}(-1)^{p+1}\frac{\pi^{2p}}{(2p+1)!}Q_{2s-2p+1}, \quad (78)$$

$s = 1,2,\ldots$*, for* $s = 1$ *the term* $(\pi^2/6)\ Q_1$ *declines.*

As in previous, let us give the alternate formula [2, 5.1.3]

$$Q_{2s+1} = \left(2^{-2s} - 1\right)\zeta(2s+1) = \frac{-1}{\Gamma(2s+1)}\int_0^\infty \frac{t^{2s}}{e^t+1}dt \quad (s = 1,2,\ldots). \quad (79)$$

At the end we would like to note also that one can get many other representations (through multiple integrals) of the constants $\widetilde{K}_r$ $(r = 1,2,\ldots)$ from Theorems 17–20 setting, in particular, $x = \pi/2$ or $x = \pi$, as it is made for the constants K_r.

From the difference $T_{2s+1} - Q_{2s+1} = (\pi/2)\widetilde{K}_{2s}$ $(s = 1,2,\ldots)$, one can get the following formula:

$$\widetilde{K}_{2s} = \frac{2(-1)^s}{\pi^2}\widetilde{A}_{2s+1}(\pi) + \sum_{p=1}^{s-1}(-1)^{p+1}\frac{\pi^{2p}}{(2p+1)!}\widetilde{K}_{2s-2p} \quad (s = 1,2,\ldots), \quad (80)$$

which is somewhat inferior to the similar recursion formula in Theorem 16 because $(2s-1)! \le 2^{2s-2}(2s-2)!$ for $s = 1,2,\ldots$.

As an exception we will give the inverse formula of (80) as

$$\widetilde{A}_{2s+1}(\pi) = (-1)^s\sum_{p=1}^{s}(-1)^{p+1}\frac{\pi^{2p}}{2(2p-1)!}\widetilde{K}_{2s-2p+2} \quad (s = 1,2,\ldots). \quad (81)$$

If we, by definition, lay $\widetilde{K}_0 = 0$, then the equality (80) is valid for $s = 0$ too.

4. Some Notes on Numerical and Computer Implementations of the Derived Formulas

We will consider some aspects of numerical and symbolic calculations of the Favard constants K_r, singular integrals (4), and summation of series.

Table 2: Calculated values of the magnitudes $\widetilde{A}_r(x)$, $\widetilde{B}_r(x)$, and $\widetilde{C}_r(x)$ for $x = \pi/2, \pi$, by using formulas (65), (67), and (70), respectively.

r	$\widetilde{A}_r(\pi/2)$	$\widetilde{B}_r(\pi/2)$	$\widetilde{C}_r(\pi/2)$
0		0.34657359027997265 4709	0.34657359027997265471
1	−1.831931188354438030	−0.91596559417721901505	0.91596559417721901506
2	−2.103599580529289999	−1.31474973783080624966	0.78884984269848374979
3	−1.326437390660483389	−0.89924201634043412749	0.42719537432004926159
4	−0.586164434174921923	−0.41567176475313624651	0.17049266942178567636
5	−0.200415775434410589	−0.14636851588819525831	0.05404725954621533063
6	−0.055999130148030908	−0.04177075772178286763	0.01422837242624804042
7	−0.013242890305882861	−0.01003832077074076074	0.00320456953514210000
8	−0.002716221784223860	−0.00208540018388810697	0.00063082160033575329
9	−0.000492033289602442	−0.00038173093475665008	0.00011030235484579206
10	−0.000079824623761109	−0.00006247448634651347	0.0000173501374145 9518
11	−0.000011727851308441	−0.00000924763886476940	0.00000248021244367167
12	−0.000001574609462951	−0.00000124968150024699	0.00000032492796270360
r	$\widetilde{A}_r(\pi)$	$\widetilde{B}_r(\pi)$	$\widetilde{C}_r(\pi)$
0		0.69314718055994530941 7	
1	0	0	0
2	−4.207199161058579998 9	−2.10359958052928999945	2.10359958052928999945
3	−6.60865298828538812 26	−3.77637313616307892720	2.83227985212230919540
4	−6.36275278788024615 60	−3.92286552530160912232	2.43988726257863703368
5	−4.55918948930179465 64	−2.95428020294335768590	1.60490928635843697049
6	−2.62553917094720961 41	−1.76271265891422894113	0.86282651203298067295
7	−1.26849156559002743 47	−0.87472015815662857450	0.39377140743339886016
8	−0.52876566480683091 89	−0.37237211738940254175	0.15639354741742837714
9	−0.19400315595064486 17	−0.13897143291435930434	0.05503172303628555739
10	−0.06359857329181458 06	−0.04620733330628432785	0.01739123998553025277
11	−0.01884893900191138 04	−0.01385968931040814707	0.00498924969150323336
12	−0.00509853278859749 09	−0.00378779048977775268	0.00131074229881973820

Let us take only the first $m-1$, $m \le s+1$ terms in the final sums in the right-hand side of the formulas (7a) and (7b) and denote the remaining truncation sums by

$$S_{1,m} = \sum_{j=m}^{s} \frac{(-1)^{j-1}\pi^{jp}}{(2j)!} K_{2s-2j+1},$$
$$S_{2,m} = \sum_{j=m}^{s} \frac{(-1)^{j-1}}{(2j-1)!} \left(\frac{\pi}{2}\right)^{2j-1} K_{2s-2j+1}, \tag{82}$$

respectively.

Theorem 24. *For $m \ge 2$ and any $s > m$, the following estimates for the truncation errors (82) hold:*

$$|S_{1,m}| = O\left(\frac{\pi^{2m}}{(2m)!}\right), \qquad |S_{2,m}| = O\left(\left(\frac{\pi}{2}\right)^{2m-1} \frac{1}{(2m-1)!}\right), \tag{83}$$

where Landau big O notation is used. For the Landau notation see [27].

Proof. By means of the inequalities (25), it is easy to obtain

$$\begin{aligned} |S_{1,m}| &= \left| \sum_{j=m}^{s} \frac{(-1)^{j-1}\pi^{jp}}{(2j)!} K_{2s-2j+1} \right| \\ &\le \frac{\pi}{2} \sum_{j=m}^{s} \frac{\pi^{2j}}{(2j)!} = \frac{\pi}{2} \frac{\pi^{2m}}{(2m)!} \\ &\quad \times \left(1 + \frac{\pi^2}{(2m+1)(2m+2)} + \cdots + \frac{\pi^{2s-2m}}{(2m+1)\cdots(2s)}\right) \\ &\le \frac{\pi^{2m+1}}{2(2m)!} \left(1 + \frac{10}{5.6} + \frac{10^2}{5^2.6^2} + \cdots\right) \\ &\le \frac{\pi^{2m+1}}{2(2m)!} \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right) \\ &= \frac{\pi^{2m+1}}{2(2m)!} \frac{3}{2} \le \frac{3\pi^{2m}}{(2m)!}. \end{aligned} \tag{84}$$

```
m = 12; Array [K,m]; K_1 = π/2; d[n_, x_] := x^n/n!
For [s = 1, s ≤ m/2, s ++,
    K_{2s+1} = 1/2 ((-1)^s d[2s + 1, π] + Σ_{p=1}^{s} K_{2s+1-2p} ((-1)^{p-1} π^{2p})/(2p)!);
    K_{2s} = (-1)^s d[2s, π/2] + Σ_{p=1}^{s} K_{2s+1-2p} ((-1)^{p-1} (π/2)^{2p-1})/(2p - 1)!;
  ]
For [s = 1, s ≤ m, s ++, Print [K_s, "   ", N[K_s, 30]]];
```

ALGORITHM 1: *Mathematica* code for exact symbolic and approximate computation with optional 30 digits accuracy of the Favard constants by recursive representations (7a) and (7b).

Consequently, $|S_{1,m}| = O(\pi^{2m}/(2m)!)$. In the same way for the second truncation sum, we have

$$|S_{2,m}| \le 2\left(\frac{\pi}{2}\right)^{2m-1}\frac{1}{(2m-1)!},$$
$$\text{so } |S_{2,m}| = O\left(\left(\frac{\pi}{2}\right)^{2m-1}\frac{1}{(2m-1)!}\right). \tag{85}$$

□

By the details of the proof it follows that the obtained representations did not depend on s, for any $m < s$. The a priori estimates (83) can be used for calculation of K_r with a given numerical precision ε in order to decrease the number of addends in the sums for larger s at the condition $m < s$.

Remark 25. The error estimates, similar to (83), can be established for other recurrence representations in this paper too.

In Algorithm 1, we provide the simplest *Mathematica* code for exact symbolic and numerical calculation with optional 30 digits accuracy of the Favard constants K_r for $r = 1, 2, 3, \ldots, m$, based on recursive representations (7a) and (7b) for a given arbitrary integer $m > 0$. The obtained results are given in Table 1. We have to note that this code is not the most economic. It can be seen that the thrifty code will take about $2m^2 + 8m$ or $O(m^2)$ arithmetic operations in (7a) and (7b).

The basic advantage of using formulas (7a) and (7b) is that they contain only finite number of terms (i.e., finite number of arithmetic operations) in comparison with the initial formula $K_r = (4/\pi)\sum_{\nu=0}^{\infty}((-1)^{\nu(r+1)}/(2\nu+1)^{r+1})$ $(r = 0, 1, 2, \ldots)$, which needs the calculation of the slowly convergent infinite sum. It must be also mentioned that in *Mathematica*, *Maple*, and other powerful mathematical software packages, the Favard constants are represented by sums of `Zeta` and related functions, which are calculated by the use of Euler-Maclaurin summation and functional equations. Near the critical strip they also use the Riemann-Siegel formula (see, e.g., [28, A.9.4]).

Finally, we will note that the direct numerical integration in (4) is very difficult. This way a big opportunity is given by Theorems 17, 18, and 19 (formulas (65), (67) and (70)) for effective numerical calculation of the classes of multiple singular integrals $\widetilde{A}_r(x)$, $\widetilde{B}_r(x)$ and $\widetilde{C}_r(x)$ for any x. Their computation is reduced to find the finite number of the Favard constants $\widetilde{K}_\nu$ for all $\nu \le r$ and the calculation of one additional numerical sum. Numerical values for some of the singular integrals are presented in Table 2.

5. Concluding Remarks

As it became clear, there are many different ways and well-developed computer programs at present, for calculation of the significant constants in mathematics (in particular Favard constants) and for summation of important numerical series. Most of them are based on using of generalized functions as one can see in [2–5]. Nonetheless, we hope that our previously stated approach through integration of Fourier series appears to be more convenient and has its theoretical and practical meanings in the scope of applications, in particular for computing of the pointed special types of multiple singular integrals. The basic result with respect to multiple integrals reduces their calculation to finite number of numerical sums.

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