Research Article

Free Boundary Value Problem for the One-Dimensional Compressible Navier-Stokes Equations with Density-Dependent Viscosity and Discontinuous Initial Data

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We study the free boundary value problem for one-dimensional isentropic compressible Navier-Stokes equations with density-dependent viscosity coefficient and discontinuous initial data in this paper. For piecewise regular initial density, we show that there exists a unique global piecewise regular solution, the interface separating the flow and vacuum state propagates along particle path and expands outwards at an algebraic time-rate, the flow density is strictly positive from blow for any finite time and decays pointwise to zero at an algebraic time-rate, and the jump discontinuity of density also decays at an algebraic time-rate as the time tends to infinity.

1. Introduction

In the present paper, we consider the free boundary value problem to one-dimensional isentropic compressible Navier-Stokes equations with density-dependent viscosity coefficient for piecewise regular initial data connected with the infinite vacuum via jump discontinuity. In general, the one-dimensional isentropic compressible Navier-Stokes equations with density-dependent viscosity coefficient read

$$\rho_t + (\rho u)_x = 0,$$
$$(\rho u)_t + \left(\rho u^2 + P(\rho)\right)_x = (\mu(\rho) u_x)_x, \quad (x,t) \in R \times [0,T],$$

(1)

where $\rho > 0$ and u denote the flow density and velocity, respectively, the pressure-density function is taken as $P(\rho) = \rho^\gamma$ with $\gamma > 1$, and the viscosity coefficient is $\mu(\rho) = \rho^\alpha$ with $\alpha > 0$. Note here that the case $\gamma = 2$ and $\alpha = 1$ in (1) corresponds to the viscous Saint-Venant system.

There is huge literature on the studies of the global existence of weak solutions and dynamical behaviors of jump discontinuity for the compressible Navier-Stokes equations with discontinuous initial data; for example, as the viscosity coefficients are both constants, the global existence of discontinuous solutions of one-dimensional Navier-Stokes equations was derived by Hoff [1–3]. Hoff investigated the construction of global spherically symmetric weak solutions of compressible Navier-Stokes equations for isothermal flow with large and discontinuous initial data [4]; therein it is also proved that the discontinuities in the density and pressure persist for all time, convecting along particle trajectories and decaying at a rate inversely proportional to the viscosity coefficient. The global existence theorems for the multidimensional Navier-Stokes equations of isothermal compressible flows with the polytropic equation of state $p(\rho) = \rho^\gamma$ ($\gamma \geq 1$) were also showed by Hoff [5, 6]. Chen et al. obtained the global existence of weak solutions for the Navier-Stokes equations for compressible, heat-conducting flow in one space dimension with large, discontinuous initial data [7]. Hoff showed the global existence of weak solutions of the Navier-Stokes equations for compressible, heat-conducting

fluids in two and three space dimensions, when the initial data may be discontinuous across a hypersurface of R^n [8]. The global existence of solutions of the Navier-Stokes equations for compressible, barotropic flow in two space dimensions which exhibit convecting singularity curves was also proved by Hoff [9].

If the viscosity coefficients $\mu(\rho) = \rho^\alpha$, $\lambda(\rho) = 0$, for the case of one space dimension, the global existence of unique piecewise smooth solution to the free boundary value problem was obtained by Fang-Zhang for (1) with $0 < \alpha < 1$, where the initial density is piecewise smooth with possibly large jump discontinuities [10]. Lian et al. considered the initial boundary value problem for (1) with $0 < \alpha \leq 1$ subject to piecewise regular initial data with initial vacuum state included in [11]. Lian et al. also addressed the Cauchy problem for one-dimensional isentropic compressible Navier-Stokes equations with density-dependent viscosity coefficient [12]; in these two cases above, they proved the global existence of unique piecewise regular solution and the finite time vanishing of vacuum state was proved in [11]. In particular, they got that the jump discontinuity of density decays exponentially but never vanishes in any finite time and the piecewise regular solution tends to the equilibrium state as $t \to +\infty$.

Recently, there are also many significant progresses achieved on the compressible Navier-Stokes equations with density-dependent viscosity coefficients. For instance, the mathematical derivations are obtained in the simulation of flow surface in shallow region [13, 14]. The good-posedness of solutions to the free boundary value problem with initial finite mass and the flow density being connected with the infinite vacuum either continuously or via jump discontinuity was considered by many authors; refer to [15–22] and references therein. The global existence of classical solutions is shown by Mellet and Vasseur [23]. The qualitative behaviors of global solutions and dynamical asymptotics of vacuum states are also made, such as the finite time vanishing of finite vacuum or asymptotical formation of vacuum in long-time, the dynamical behaviors of vacuum boundary, the long-time convergence to rarefaction wave with vacuum, and the stability of shock profile with large shock strength; refer to [24–28] and references therein.

In this present paper, we consider the free boundary value problem (FBVP) for one-dimensional isentropic compressible Navier-Stokes equations and focus on the existence, regularities, and dynamical behaviors of global piecewise regular solution, and so forth. As $\gamma > 1$, $0 < \alpha \leq 1$, we show that the free boundary value problem with piecewise regular initial data admits a unique global piecewise regular solution, the interface separating the flow and vacuum state propagates along particle path and expands outwards at an algebraic time-rate, the flow density is strictly positive from blow for any finite time and decays pointwise to zero at an algebraic, and the jump discontinuity of density also decays at an algebraic time-rate as $t \to +\infty$ (refer to Theorem 2 for details).

The rest part of the paper is arranged as follows. In Section 2, the main results about the existence and dynamical behaviors of global piecewise regular solution for compressible Navier-Stokes equations are stated. Then, some important a priori estimates will be given in Section 3. Finally, the theorem is proved in Section 4.

2. Main Results

We are interested in the global existence and dynamics of the free boundary value problem for (1) with following initial data and boundary conditions:

$$(\rho, u)(x, 0) = (\rho_0, u_0), \quad x \in [0, a_0],$$
$$u(0, t) = 0, \quad (\rho^\gamma - \rho^\alpha u_x)(a(t), t) = 0, \quad t > 0, \tag{2}$$

where $x = a(t)$ is the free boundary defined by

$$\frac{d}{dt} a(t) = u(a(t), t), \quad a(0) = a_0, \quad t > 0. \tag{3}$$

Next, we give the definition of weak solution to the free boundary problem (1) and (2).

Definition 1 (weak solution). For any $T > 0$, (ρ, u) is said to be a weak solution of the free boundary problems (1) and (2), if $(\rho, \sqrt{\rho}u)$ has the following regularities:

$$0 \leq \rho \in L^\infty\left(0, T; L^1([0, a(t)]) \cap L^\gamma([0, a(t)])\right),$$
$$\sqrt{\rho}u \in L^\infty\left(0, T; L^2([0, a(t)])\right), \tag{4}$$
$$(\rho^\gamma - \rho^\alpha u_x) \in L^2\left(0, T; H^1([0, a(t)])\right),$$

and (1) is satisfied in the sense of distributions. Namely, it holds for all $\varphi \in C_0^\infty([0, a(t)] \times [0, T))$ that

$$\int_0^{a_0} \rho_0 \varphi(x, 0)\, dx + \int_0^T \int_0^{a(t)} \rho \varphi_t\, dx\, dt$$
$$+ \int_0^T \int_0^{a(t)} \sqrt{\rho}\sqrt{\rho}u\varphi_x\, dx\, dt = 0 \tag{5}$$

and for all $\psi \in C_0^\infty([0, a(t)] \times [0, T))$ that

$$\int_0^{a_0} \rho_0 u_0 \psi(x, 0)\, dx$$
$$+ \int_0^T \int_0^{a(t)} \left(\sqrt{\rho}\sqrt{\rho}u\psi_t + (\sqrt{\rho}u)^2 \psi_x\right) dx\, dt \tag{6}$$
$$+ \int_0^T \int_0^{a(t)} (\rho^\gamma - \rho^\alpha u_x)\psi_x\, dx\, dt = 0.$$

For simplicity, we consider the initial data in FBVP (1) and (2) with one discontinuous point $y_0 \in (0, a_0)$; namely, for some constant $\rho_- > 0$

$$\inf_{[0, y_0) \cup (y_0, a_0]} \rho_0 \geq \rho_- > 0,$$
$$(\rho_0, u_0) \in W^{1,\infty}([0, y_0) \cup (y_0, a_0]), \tag{7}$$
$$(\rho_0^\gamma - \rho_0^\alpha u_{0x})(a_0) = 0,$$
$$\rho_0(y_0 - 0) > \rho_0(y_0 + 0),$$

and the compatibility conditions between initial data and boundary conditions hold.

We will give the global existence and time-asymptotic behavior of piecewise regular solution as follows.

Theorem 2. *Let $\gamma > 1$ and $0 < \alpha \leq 1$. Assume that the initial data satisfies (7). Then, there exists a unique global piecewise regular solution (ρ, u, a) to the FBVP (1) and (2) satisfying for $T > 0$*

$$
\begin{aligned}
\rho &\in L^\infty\left(0, T; H^1\left([0, y(t)) \cup (y(t), a(t)]\right)\right) \\
&\quad \cap C^0\left([0, y(t)) \cup (y(t), a(t)] \times [0, T]\right), \\
u &\in L^\infty\left(0, T; H^1\left([0, a(t)]\right)\right) \\
&\quad \cap L^2\left(0, T; H^2\left([0, a(t)]\right)\right), \\
u_t &\in L^2\left(0, T; L^2\left([0, a(t)]\right)\right), \\
a(t) &\in H^1\left([0, T]\right), \\
\left(\rho^\gamma - \rho^\alpha u_x\right) &\in L^\infty\left(0, T; L^2\left([0, y(t)) \cup (y(t), a(t)]\right)\right),
\end{aligned}
\tag{8}
$$

where $x = y(t)$ is a curve defined by

$$
\frac{dy(t)}{dt} = u(y(t), t), \quad y(0) = y_0, \ t > 0, \tag{9}
$$

along which the Rankine-Hugoniot conditions hold

$$
[u(y(t), t)] = 0, \quad [\rho^\gamma(y(t), t)] = [\rho^\alpha u_x(y(t), t)], \tag{10}
$$

where $[f(y(t), t)] := f(y(t) + 0, t) - f(y(t) - 0, t)$, and along the discontinuity $r = \xi(t)$ the jump satisfies

$$
\left|[\rho_0^\alpha(y_0)]\right| e^{-C_0 t} \leq \left|[\rho^\alpha(y(t), t)]\right|, \tag{11}
$$

where C_0 is a positive constant independent of time.

If it further holds that $u_0 \in H^2([0, a_0])$, then (ρ, u, a, b) satisfies

$$
\begin{aligned}
\rho &\in L^\infty\left(0, T; H^1\left([0, y(t)) \cup (y(t), a(t)]\right)\right), \\
\rho_t &\in L^\infty\left(0, T; L^2\left([0, y(t)) \cup (y(t), a(t)]\right)\right), \\
u &\in L^\infty\left(0, T; H^1\left([0, a(t)]\right)\right) \\
&\quad \cap L^2\left(0, T; H^2\left([0, a(t)]\right)\right), \\
u_t &\in L^\infty\left(0, T; L^2\left([0, a(t)]\right)\right) \\
&\quad \cap L^2\left(0, T; H^1\left([0, a(t)]\right)\right), \\
a(t) &\in H^2\left([0, T]\right), \\
\left(\rho^\gamma - \rho^\alpha u_x\right) &\in L^\infty\left(0, T; H^1\left([0, y(t)) \cup (y(t), a(t)]\right)\right).
\end{aligned}
\tag{12}
$$

The domain expands outwards at an algebraic rate in time as

$$
C(1+t)^{\gamma/(\gamma-\alpha)} \geq a(t) \geq
\begin{cases}
c(1+t), & 1 < \gamma < 2\alpha, \\
c(1+t)^{1-\nu}, & \gamma = 2\alpha, \\
c(1+t)^{\alpha/(\gamma-\alpha)}, & \gamma > 2\alpha,
\end{cases}
\tag{13}
$$

and the density decays pointwise to zero for any $x \in [0, y(t)) \cup (y(t), a(t)]$ and $t > 0$ as

$$
\rho(a(t), t) = \left((\gamma - \alpha) t + \rho_0(a_0)^{\alpha - \gamma}\right)^{-1/(\gamma - \alpha)}, \tag{14}
$$

$$
\rho(x, t) \leq C(1+t)^{-1/\gamma - \alpha}
$$

$$
+
\begin{cases}
C(1+t)^{-(\gamma-1)/(3\gamma+2\alpha-1)}, \\
\qquad\qquad 1 < \gamma < 2\alpha, \\
C(1+t)^{-((\gamma-1)/(3\gamma+2\alpha-1))+\nu}, \\
\qquad\qquad \gamma = 2\alpha, \\
C(1+t)^{-\alpha(\gamma-1)/(3\gamma+2\alpha-1)(\gamma-\alpha)}, \\
\qquad\qquad \gamma > 2\alpha,
\end{cases}
\tag{15}
$$

$$
+
\begin{cases}
C(1+t)^{-2/(3\gamma+2\alpha-1)}, \\
\qquad\qquad 1 < \gamma < 2\alpha, \\
C(1+t)^{-(2/(3\gamma+2\alpha-1))+\nu}, \\
\qquad\qquad \gamma = 2\alpha, \\
C(1+t)^{-2\alpha/(3\gamma+2\alpha-1)(\gamma-\alpha)}, \\
\qquad\qquad \gamma > 2\alpha,
\end{cases}
$$

where $C > 0$ and $c > 0$ are positive constants independent of time, and $\nu \in (0, 1)$ is a small constant.

Remark 3. Theorem 2 holds for the Saint-Venant model for shallow water; that is, $\gamma = 2, \alpha = 1$.

Remark 4. Fang-Zhang [10] obtained that

$$
\left|[\rho_0^\alpha(y_0)]\right| e^{-C_0 t} \leq \left|[\rho^\alpha(y(t), t)]\right|, \tag{16}
$$

which show that the discontinuity in the density persists for all time. However, in this paper, from (15) we can shows that the discontinuity in the density decays at an algebraic rate in time; namely,

$$
\left|[\rho^\alpha(y(t), t)]\right| \leq C(1+t)^{-\alpha/(\gamma-\alpha)}
$$

$$
+
\begin{cases}
C(1+t)^{-\alpha(\gamma-1)/(3\gamma+2\alpha-1)}, \\
\qquad\qquad 1 < \gamma < 2\alpha, \\
C(1+t)^{-(\alpha(\gamma-1)/(3\gamma+2\alpha-1))+\alpha\nu}, \\
\qquad\qquad \gamma = 2\alpha, \\
C(1+t)^{-\alpha^2(\gamma-1)/(3\gamma+2\alpha-1)(\gamma-\alpha)}, \\
\qquad\qquad \gamma > 2\alpha,
\end{cases}
$$

$$
+
\begin{cases}
C(1+t)^{-2\alpha/(3\gamma+2\alpha-1)}, \\
\qquad\qquad 1 < \gamma < 2\alpha, \\
C(1+t)^{-(2\alpha/(3\gamma+2\alpha-1))+\nu}, \\
\qquad\qquad \gamma = 2\alpha, \\
C(1+t)^{-2\alpha^2/(3\gamma+2\alpha-1)(\gamma-\alpha)}, \\
\qquad\qquad \gamma > 2\alpha,
\end{cases}
\tag{17}
$$

where $C > 0$ is a positive constant independent of time, and $\nu \in (0, 1)$ is a small constant.

3. The A Priori Estimates

According to the analysis made in [29], there is a curve $x = y(t)$ defined by

$$\frac{dy(t)}{dt} = u(y(t),t), \quad y(0) = y_0, \ t > 0, \tag{18}$$

along which the Rankine-Hugoniot conditions hold

$$[u(y(t),t)] = 0, \quad [\rho^\gamma(y(t),t)] = [\rho^\alpha u_x(y(t),t)], \tag{19}$$

where $[f(y(t),t)] := f(y(t)+0,t) - f(y(t)-0,t)$.

It is convenient to make use of the Lagrange coordinates in order to establish the uniformly a-priori estimates. Let (ρ, u, a) be a piecewise regular solution to the FBVP (1) and (2), and take the Lagrange coordinates transform

$$\xi = \int_0^x \rho(z,t)\,dz, \quad \tau = t. \tag{20}$$

Since the conservation of total mass holds

$$\int_0^{a(t)} \rho(z,t)\,dz = \int_0^{a_0} \rho_0(z)\,dz := 1 > 0, \tag{21}$$

the boundaries $x = 0$ and $x = a(t)$ are transformed into $\xi = 0$ and $\xi = 1$, respectively, and the domain $[0, a(t)]$ is transformed into $[0,1]$, the curve $x = y(t)$ in the Eulerian coordinates is changed to a line $\xi = \xi_0$ in the Lagrangian coordinates, where

$$\xi_0 := \int_0^{y(t)} \rho(z,t)\,dz = \int_0^{y_0} \rho_0(z)\,dz, \tag{22}$$

and the jump conditions become

$$[u(\xi_0,\tau)] = 0, \quad [\rho^\gamma(\xi_0,\tau)] = [\rho^{1+\alpha} u_\xi(\xi_0,\tau)]. \tag{23}$$

Meanwhile, the FBVP (1) and (2) is reformulated into

$$\rho_\tau + \rho^2 u_\xi = 0,$$

$$u_\tau + (\rho^\gamma)_\xi = (\rho^{1+\alpha} u_\xi)_\xi,$$

$$(\rho^\gamma - \rho^{1+\alpha} u_\xi)(0,\tau) = (\rho^\gamma - \rho^{1+\alpha} u_\xi)(1,\tau) = 0, \tag{24}$$

$$(\rho_0, u_0) = (\rho_0, u_0)(\xi), \quad \xi \in [0,1],$$

where the initial data satisfies

$$\inf_{[0,\xi_0)\cup(\xi_0,1]} \rho_0 \geq \rho_- > 0,$$

$$(\rho_0, u_0) \in W^{1,\infty}([0,\xi_0) \cup (\xi_0,1]),$$

$$(\rho_0^\gamma - \rho_0^{1+\alpha} u_{0x})(1) = 0, \tag{25}$$

$$\rho_0(\xi_0 - 0) > \rho_0(\xi_0 + 0),$$

for some constant $\rho_- > 0$, and the consistencies between initial data and boundary value hold.

Next, we will give the a-priori estimates for the solution (ρ, u) to the FBVP (24). Similarly to the arguments used in [10, 16, 20], we can establish the following a priori estimates and omit the details here.

Lemma 5. *Let $T > 0$. Under the assumptions of Theorem 2, it holds for any piecewise regular solution (ρ, u) to the FBVP (24) that*

$$\int_0^1 \left(\frac{u^2}{2} + \frac{1}{\gamma - 1}\rho^{\gamma-1} \right) d\xi + \int_0^\tau \int_0^1 \rho^{1+\alpha} u_\xi^2 \, d\xi ds$$
$$= \int_0^1 \left(\frac{u_0^2}{2} + \frac{1}{\gamma - 1}\rho_0^{\gamma-1} \right) d\xi, \quad \tau \in [0,T], \tag{26}$$

$$\rho(\xi,\tau) \leq C, \quad (\xi,\tau) \in [0,\xi_0) \cup (\xi_0,1] \times [0,T], \tag{27}$$

$$\int_0^1 u^{2n} d\xi + n(2n-1) \int_0^\tau \int_0^1 \rho^{1+\alpha} u^{2n-2} u_\xi^2 \, d\xi ds$$
$$\leq C(T), \quad \tau \in [0,T], \tag{28}$$

$$\fint_0^1 (\rho^\alpha)_\xi^{2n} d\xi \leq C(T), \quad \tau \in [0,T], \tag{29}$$

for any positive integer $n \in N$, $C > 0$ denotes a constant independent of time and $C(T) > 0$ denotes a constant dependent on time, where $\fint_0^1 := \int_0^{\xi_0} + \int_{\xi_0}^1$.

Lemma 6. *Let $T > 0$. Under the assumptions of Theorem 2, it holds for any piecewise regular solution (ρ, u) to the FBVP (24) that*

$$\rho(0,\tau) = \left((\gamma - \alpha)\tau + \rho_0(0)^{\alpha-\gamma} \right)^{-1/(\gamma-\alpha)}, \tag{30}$$

$$\rho(\xi_0 - 0,\tau) > \rho(\xi_0 + 0,\tau), \quad \tau \in [0,T]. \tag{31}$$

Proof. From $(24)_1$ and $(24)_3$, we have

$$\rho_\tau(0,\tau) + \rho^{\gamma-\alpha+1}(0,\tau) = 0, \tag{32}$$

which yields (30) similarly, because of (23) and $(24)_1$, it holds that

$$[\rho^\alpha]_\tau + \alpha[\rho^\gamma] = 0, \tag{33}$$

which together with (25) implies

$$[\rho^\alpha] = [\rho_0^\alpha] \exp\left\{ -\alpha \int_0^\tau \frac{[\rho^\gamma]}{[\rho^\alpha]} ds \right\} < 0. \tag{34}$$

$\square$

Lemma 7. *Let $T > 0$. Under the assumptions of Theorem 2, it holds for any piecewise regular solution (ρ, u) to the FBVP (24) that*

$$\frac{1}{2}\fint_0^1 \left(u + \frac{(\rho^\alpha)_\xi}{\alpha} \right)^2 d\xi + \frac{1}{\gamma - 1} \int_0^1 \rho^{\gamma-1} d\xi$$

$$+ \gamma \int_0^\tau \fint_0^1 \rho^{\gamma+\alpha-2} \rho_\xi^2 \, d\xi ds + \rho^\gamma(1,\tau)a(\tau)$$

$$+ \gamma \int_0^\tau \rho^{2\gamma-\alpha}(1,s)\, a(s)\, ds - [\rho^\gamma]\, y(\tau)$$

$$- \gamma \int_0^\tau [\rho^{\gamma+1}] u_\xi y(s)\, ds$$

$$= \frac{1}{2} \int_0^1 \left(u_0 + \frac{(\rho^\alpha)_\xi}{\alpha} \right)^2 d\xi$$

$$+ \frac{1}{\gamma-1} \int_0^1 \rho_0^{\gamma-1} d\xi + \rho_0^\gamma(1)\, a_0 - [\rho_0^\gamma]\, y_0$$

$$\tau \in [0, T], \tag{35}$$

where $a(\tau)$ satisfies $a'(\tau) = u(0, \tau)$ and $a(0) = a_0$, $y(\tau)$ satisfies $y'(\tau) = u(\xi_0, \tau)$ and $y(0) = y_0$.

Proof. Multiplying $(24)_1$ by $\rho^{\alpha-1}$ gives

$$\frac{(\rho^\alpha)_\tau}{\alpha} + \rho^{1+\alpha} u_\xi = 0, \tag{36}$$

which leads to

$$\frac{(\rho^\alpha)_{\tau\xi}}{\alpha} + (\rho^{1+\alpha} u_\xi)_\xi = 0. \tag{37}$$

Summing $(24)_2$ and (37), we have

$$\left(u + \frac{(\rho^\alpha)_\xi}{\alpha} \right)_\tau + (\rho^\gamma)_\xi = 0. \tag{38}$$

Multiplying (38) by $(u + (\rho^\alpha)_\xi/\alpha)$ and integrating the result over $[0,1] \times [0,\tau]$, we get

$$\frac{1}{2} \int_0^1 \left(u + \frac{(\rho^\alpha)_\xi}{\alpha} \right)^2 d\xi + \frac{1}{\gamma-1} \int_0^1 \rho^{\gamma-1} d\xi$$

$$+ \gamma \int_0^\tau \int_0^1 \rho^{\gamma+\alpha-2} \rho_\xi^2 \, d\xi ds + \int_0^\tau \rho^\gamma u|_{\xi=0}^{\xi=\xi_0-0} ds$$

$$+ \int_0^\tau \rho^\gamma u|_{\xi=\xi_0+0}^{\xi=1} ds \tag{39}$$

$$= \frac{1}{2} \int_0^1 \left(u_0 + \frac{(\rho_0^\alpha)_\xi}{\alpha} \right)^2 d\xi + \frac{1}{\gamma-1} \int_0^1 \rho_0^{\gamma-1} d\xi,$$

which together with the fact that

$$\int_0^\tau \rho^\gamma u|_{\xi=0}^{\xi=\xi_0-0} ds$$

$$= \int_0^\tau \rho^\gamma (\xi_0 - 0, s)\, y'(s)\, ds$$

$$= \rho^\gamma (\xi_0 - 0, s)\, y(s)\big|_0^\tau \tag{40}$$

$$+ \gamma \int_0^\tau \rho^{\gamma-1} \rho^2 (\xi_0 - 0, s)\, u_\xi (\xi_0, s)\, y(s)\, ds$$

$$= \rho^\gamma (\xi_0 - 0, \tau)\, y(\tau) - \rho_0^\gamma (\xi_0 - 0)\, y_0$$

$$+ \gamma \int_0^\tau \rho^{\gamma+1} (\xi_0 - 0, s)\, u_\xi (\xi_0, s)\, y(s)\, ds,$$

$$\int_0^\tau \rho^\gamma u|_{\xi=\xi_0+0}^{\xi=1} ds$$

$$= \int_0^\tau \rho^\gamma (1, \tau)\, a'(s)\, ds$$

$$- \int_0^\tau \rho^\gamma (\xi_0 + 0, \tau)\, y'(s)\, ds$$

$$= \rho^\gamma (1, s)\, a(s)\big|_0^\tau$$

$$+ \gamma \int_0^\tau \rho^{2\gamma-\alpha} (1, s)\, a(s)\, ds$$

$$- \rho^\gamma (\xi_0 + 0, s)\, y(s)\big|_0^\tau$$

$$- \gamma \int_0^\tau \rho^{\gamma+1} (\xi_0 + 0, s)\, u_\xi (\xi_0, s)\, y(s)\, ds$$

$$= \rho^\gamma (1, \tau)\, a(\tau) - \rho_0^\gamma (1)\, a_0 \tag{41}$$

$$+ \gamma \int_0^\tau b(s)\, \rho^{2\gamma-\alpha} (1, s)\, ds$$

$$- \rho^\gamma (\xi_0 + 0, \tau)\, y(\tau) + \rho_0^\gamma (\xi_0 + 0)\, y_0$$

$$- \gamma \int_0^\tau \rho^{\gamma+1} (\xi_0 + 0, s)\, u_\xi (\xi_0, s)\, y(s)\, ds,$$

$$[\rho(\xi_0, \tau)] < 0,$$

$$u_\xi (\xi_0, \tau) = \frac{[\rho^\gamma]}{[\rho^{1+\alpha}]} > 0,$$

$$a(\tau) = \int_0^1 \frac{1}{\rho(\zeta, \tau)} d\zeta > 0,$$

$$y(\tau) = \int_0^{\xi_0} \frac{1}{\rho(\zeta, \tau)} d\zeta > 0,$$

gives rise to (35). □

Remark 8. The estimate (35) can be written in the following form in the Eulerian coordinates; that is to say, for all $t \in [0, T]$,

$$\frac{1}{2} \int_0^{a(t)} \rho \left(u + \rho^{-1} (\rho^\alpha)_x \right)^2 dx + \frac{1}{\gamma-1} \int_0^{a(t)} \rho^\gamma dx$$

$$+ \gamma \int_0^t \int_0^{a(t)} \rho^{\gamma+\alpha-3} \rho_x^2 \, dx ds + \rho^\gamma (a(t), t)\, a(t)$$

$$+ \gamma \int_0^t \rho^{2\gamma-\alpha} (a(s), s)\, a(s)\, ds$$

$$- [\rho^\gamma]\, y(t) - \gamma \int_0^t [\rho^\gamma] u_x y(s)\, ds$$

$$= \frac{1}{2} \int_0^{a_0} \rho_0 \left(u_0 + \rho_0^{-1} (\rho_0^\alpha)_x \right)^2 dx$$

$$+ \frac{1}{\gamma-1} \int_0^{a_0} \rho_0^\gamma dx + \rho_0^\gamma (a_0)\, a_0 - [\rho_0^\gamma]\, y_0. \tag{42}$$

Lemma 9. *Let $T > 0$, for $n \in \mathbb{N}$, and $n > (1+\alpha)/4(\gamma-\alpha)$. Under the assumptions of Theorem 2, it holds for any piecewise regular solution (ρ, u) to the FBVP (24) that*

$$\int_0^\tau \left\| (\rho^\gamma)_\xi^{2n} \right\|_{L^\infty([0,\xi_0) \cup (\xi_0, 1])} ds \le C(T), \quad \tau \in [0, T]. \tag{43}$$

Proof. It follows from $(24)_{1,2}$ that

$$(\rho^\alpha)_\xi (\xi, \tau) = (\rho_0^\alpha)_\xi (\xi) - \alpha u (\xi, \tau)$$
$$+ \alpha u_0 (\xi) - \alpha \int_0^\tau (\rho^\gamma)_\xi (\xi, s) \, ds. \tag{44}$$

By means of (25), (26), and (44), we have

$$\int_0^\tau \left\| (\rho^\gamma)_\xi^{2n} \right\|_{L^\infty([0,\xi_0) \cup (\xi_0, 1])} ds$$
$$= \frac{\gamma^{2n}}{\alpha^{2n}} \int_0^\tau \left\| \rho^{2n(\gamma-\alpha)} (\rho^\alpha)_\xi^{2n} \right\|_{L^\infty([0,\xi_0) \cup (\xi_0, 1])} ds$$
$$\leq C(T) + C(T) \int_0^\tau \left\| \rho^{2n(\gamma-\alpha)} u^{2n} \right\|_{L^\infty([0,\xi_0) \cup (\xi_0, 1])} ds \tag{45}$$
$$+ C(T) \int_0^\tau \int_0^s \left\| (\rho^\gamma)_\xi^{2n} \right\|_{L^\infty([0,\xi_0) \cup (\xi_0, 1])} dl \, ds$$
$$\leq C(T) + C(T) \int_0^\tau \int_0^s \left\| (\rho^\gamma)_\xi^{2n} \right\|_{L^\infty([0,\xi_0) \cup (\xi_0, 1])} dl \, ds,$$

where we have used

$$\int_0^\tau \left\| \rho^{2n(\gamma-\alpha)} u^{2n} \right\|_{L^\infty([0,\xi_0) \cup (\xi_0, 1])} ds$$
$$\leq \int_0^\tau \int_0^1 \rho^{2n(\gamma-\alpha)} u^{2n} d\xi \, ds$$
$$+ \int_0^\tau \int_0^1 \left| (\rho^{2n(\gamma-\alpha)} u^{2n})_\xi \right| d\xi \, ds$$
$$\leq C(T) \tag{46}$$
$$+ C \int_0^\tau \int_0^1 \left(\rho^{2(2n(\gamma-\alpha)-\alpha)} \rho^{2\alpha-2} \rho_\xi^2 + u^{4n} \right.$$
$$+ \rho^{4n(\gamma-\alpha)-(1+\alpha)} u^{2n}$$
$$\left. + \rho^{1+\alpha} u^{2n-2} u_\xi^2 \right) d\xi \, ds$$
$$\leq C(T),$$

which can be deduced from (26) and (27). Making use of Gronwall's inequality to (46), we obtain (43). □

Lemma 10. *Let $T > 0$. Under the assumptions of Theorem 2, it holds for any piecewise regular solution (ρ, u) to the FBVP (24) that*

$$\rho(\xi, \tau) \geq C(T), \quad (\xi, \tau) \in [0, \xi_0) \cup (\xi_0, 1] \times [0, T]. \tag{47}$$

Proof. Denote

$$v(\xi, \tau) = \frac{1}{\rho(\xi, \tau)}. \tag{48}$$

By $(24)_1$, we have

$$v_\tau = u_\xi. \tag{49}$$

Multiplying (49) by $\beta v^{\beta-1}$, integrating the result over $[0, 1] \times [0, \tau]$, and using (43), (44), we can obtain that for $\beta \in (1, 2)$

$$\int_0^1 v^\beta d\xi + \beta(\beta-1) \int_0^\tau \int_0^1 v^{\alpha+\beta-1} u^2 d\xi \, ds$$
$$= \int_0^1 v_0^\beta d\xi + \beta \int_0^\tau v^{\beta-1} u ds \Big|_{\xi=0}^{\xi=\xi_0-0}$$

$$+ \beta \int_0^\tau v^{\beta-1} u ds \Big|_{\xi_0+0}^{\xi=1} + \frac{\beta(\beta-1)}{\alpha} \int_0^\tau \int_0^1 v^{\alpha+\beta-1} u(\rho_0^\alpha)_\xi d\xi \, ds$$
$$+ \beta(\beta-1) \int_0^\tau \int_0^1 v^{\alpha+\beta-1} u u_0 d\xi \, ds - \beta(\beta-1)$$
$$\times \int_0^\tau \int_0^1 v^{\alpha+\beta-1} u \int_0^s (\rho^\gamma)_\xi dl \, d\xi \, ds$$
$$\leq C(T) + C \int_0^\tau v^{\beta-1} u(1, s) \, ds + \frac{\beta(\beta-1)}{2}$$
$$\times \int_0^\tau \int_0^1 v^{\alpha+\beta-1} u^2 d\xi \, ds + C(T) \int_0^\tau \int_0^1 v^\beta d\xi \, ds, \tag{50}$$

where we use the fact that

$$\int_0^\tau v^{\beta-1} (\xi_0 - 0, s) u(\xi_0, s) \, ds$$
$$- \int_0^\tau v^{\beta-1} (\xi_0 + 0, s) u(\xi_0, s) \, ds$$
$$= \int_0^\tau \rho^{1-\beta} (\xi_0 - 0, s) y'(s) \, ds$$
$$- \int_0^\tau \rho^{1-\beta} (\xi_0 + 0, s) y'(s) \, ds$$
$$= \rho^{1-\beta} (\xi_0 - 0, s) y(s) \Big|_0^\tau - \rho^{1-\beta} (\xi_0 + 0, s) y(s) \Big|_0^\tau$$
$$+ (1 - \beta) \int_0^\tau \rho^{2-\beta} (\xi_0 - 0, s) u_\xi (\xi_0, s) y(s) \, ds$$
$$- (1 - \beta) \int_0^\tau \rho^{2-\beta} (\xi_0 + 0, s) u_\xi (\xi_0, s) y(s) \, ds$$
$$= - \left[\rho^{1-\beta} \right] y(\tau) + \left[\rho_0^{1-\beta} \right] y_0$$
$$+ (\beta - 1) \int_0^\tau \left[\rho^{2-\beta} \right] u_\xi (\xi_0, s) y(s) \, ds$$
$$\leq \left[\rho_0^{1-\beta} \right] y_0 \leq C. \tag{51}$$

Since it holds that

$$\rho(1, \tau) = \left((\gamma - \alpha) \tau + \rho_0(1)^{\alpha-\gamma} \right)^{-1/(\gamma-\alpha)} \geq C(T), \tag{52}$$

we have from (25) that

$$\int_0^\tau v^{\beta-1} u(1, s) \, d\xi$$
$$\leq C(T) \int_0^\tau \left(\left(\int_0^1 u^2 d\xi \right)^{1/2} \right.$$
$$\left. + \left(\int_0^1 \rho^{1+\alpha} u_\xi^2 d\xi \right)^{1/2} \left(\int_0^1 v^{1+\alpha} d\xi \right)^{1/2} \right) ds$$
$$\leq C(T) + C(T) \int_0^\tau \int_0^1 v^\beta d\xi \, ds. \tag{53}$$

Substituting (53) in (50), we have

$$\int_0^1 v^\beta d\xi + \frac{\beta(\beta-1)}{2} \int_0^\tau \int_0^1 \rho^{\alpha+\beta-1} u^2 d\xi ds$$

$$\leq C(T) + C(T) \int_0^\tau \int_0^1 v^\beta d\xi ds. \tag{54}$$

Using Gronwall's inequality, we get from (54) that

$$\int_0^1 v^\beta d\xi \leq C(T). \tag{55}$$

It follows from (29) and (55) that

$$v^\beta(\xi,\tau) \leq \int_0^1 v^\beta d\xi + \oint_0^1 \left| v_\xi^\beta \right| d\xi$$

$$\leq C(T) + C\left(\oint_0^1 \rho^{-\beta-1} \left| \rho_\xi \right| d\xi \right)$$

$$\leq C(T) + C\left(\int_0^1 \rho^{-2n(\alpha+\beta)/(2n-1)} d\xi \right)^{(2n-1)/2n}$$

$$\times \left(\oint_0^1 (\rho^\alpha)_\xi^{2n} d\xi \right)^{1/2n}$$

$$\leq C(T) + C(T) \sup_{[0,\xi_0)\cup(\xi_0,1]} v^{\alpha+\beta/2n} \left(\int_0^1 v^\beta d\xi \right)^{(2n-1)/2n}$$

$$\leq C(T) + C(T) \sup_{[0,\xi_0)\cup(\xi_0,1]} \left(v^\beta(\xi,\tau) \right)^{\alpha/\beta+1/2n}, \tag{56}$$

as $\alpha \in (0,1]$, $\beta \in (1,2)$; for some $n \in N$ large enough, we have

$$\frac{\alpha}{\beta} + \frac{1}{2n} < 1, \tag{57}$$

which implies (47). □

We also have the regularity estimates for the solution (ρ, u) to the FBVP (24) as follows.

Lemma 11. *Let $T > 0$. Under the assumptions of Theorem 2, it holds for any piecewise regular solution (ρ, u) to the FBVP (24) that*

$$\rho \in L^\infty\left(0,T; H^1\left([0,\xi_0) \cup (\xi_0,1]\right)\right)$$

$$\cap C^0\left([0,\xi_0) \cup (\xi_0,1] \times [0,T]\right),$$

$$u \in L^\infty\left(0,T; H^1([0,1])\right) \cap L^2\left(0,T; H^2([0,1])\right), \tag{58}$$

$$u_\tau \in L^2\left(0,T; L^2([0,1])\right), \qquad a(\tau) \in H^1([0,T]),$$

$$\left(\rho^\gamma - \rho^{1+\alpha} u_\xi\right) \in L^\infty\left(0,T; L^2\left([0,\xi_0) \cup (\xi_0,1]\right)\right).$$

If it is also satisfied that

$$u_0 \in H^2([0,1]); \tag{59}$$

then the piecewise regular solution (ρ, u) has the regularities

$$\rho \in L^\infty\left(0,T; H^1\left([0,\xi_0) \cup (\xi_0,1]\right)\right),$$

$$\rho_\tau \in L^\infty\left(0,T; L^2\left([0,\xi_0) \cup (\xi_0,1]\right)\right),$$

$$u \in L^\infty\left(0,T; H^1([0,1])\right) \cap L^2\left(0,T; H^2([0,1])\right),$$

$$u_\tau \in L^\infty\left(0,T; L^2([0,1])\right) \cap L^2\left(0,T; H^1([0,1])\right), \tag{60}$$

$$a(\tau) \in H^2([0,T]),$$

$$\left(\rho^\gamma - \rho^{1+\alpha} u_\xi\right) \in L^\infty\left(0,T; H^1\left([0,\xi_0) \cup (\xi_0,1]\right)\right).$$

Proof. Multiplying $(24)_2$ by u_τ, integrating the result over $[0,1]$, and making use of the boundary condition $(24)_3$, after a direct computation and recombination, we deduce

$$\frac{1}{2} \int_0^1 \rho^{1+\alpha} u_\xi^2 d\xi + \int_0^\tau \int_0^1 u_s^2 d\xi ds$$

$$= -\int_0^1 \rho_0^\gamma u_{0\xi} d\xi + \frac{1}{2} \int_0^1 \rho^{1+\alpha} u_{0\xi}^2 d\xi + \int_0^1 \rho^\gamma u_\xi d\xi$$

$$+ \gamma \int_0^\tau \int_0^1 \rho^{1+\gamma} u_\xi^2 d\xi ds - \frac{1+\alpha}{2} \int_0^\tau \int_0^1 \rho^{2+\alpha} u_\xi^3 d\xi ds$$

$$\leq C + \frac{1}{4} \int_0^1 \rho^{1+\alpha} u_\xi^2 d\xi + \int_0^\tau \int_0^1 \left| u_\xi \right|^3 d\xi ds, \tag{61}$$

where we have used (26) and (27). On the other hand, integrating $(24)_2$ over $[\xi, 1]$ and making use of (23) and the boundary conditions $(24)_3$, it holds that

$$\rho^{1+\alpha} u_\xi = \rho^\gamma - \int_\xi^1 u_\tau d\xi \leq C + C\left(\int_0^1 u_\tau^2 d\xi \right)^{1/2}, \tag{62}$$

which implies

$$\sup_{[0,\xi_0)\cup(\xi_0,1]} u_\xi \leq C(T) + C(T) \left(\int_0^1 u_\tau^2 d\xi \right)^{1/2}. \tag{63}$$

It holds from (61) and (63) that

$$\frac{1}{2} \int_0^1 u_\xi^2 d\xi + \int_0^\tau \int_0^1 u_s^2 d\xi ds$$

$$\leq C(T) + C(T) \int_0^\tau \sup_{[0,\xi_0)\cup(\xi_0,1]} \left| u_\xi \right| \int_0^1 u_\xi^2 d\xi ds$$

$$\leq C(T) + C(T) \int_0^\tau \left(1 + \int_0^1 u_\tau^2 d\xi \right)^{1/2} \int_0^1 u_\xi^2 d\xi ds \tag{64}$$

$$\leq \frac{1}{2} \int_0^\tau \int_0^1 u_s^2 d\xi ds + \int_0^\tau \left\| u_\xi \right\|_{L^2}^2 \int_0^1 u_\xi^2 d\xi ds;$$

using Gronwall's inequality, (26), and (47), we have

$$\int_0^1 u_\xi^2 d\xi + \int_0^\tau \int_0^1 u_s^2 d\xi ds \leq C(T), \tag{65}$$

where $C(T) > 0$ denotes a constant dependent of time.

Differentiating $(24)_2$ with respect to τ, we get

$$u_{\tau\tau} + (\rho)^{\gamma}_{\xi\tau} = (\rho^{1+\alpha}u_{\xi})_{\xi\tau}. \tag{66}$$

Taking inner product between (66) and u_τ, integrating the results over $[0, 1]$, and using the boundary conditions $(24)_3$, it holds that

$$\frac{1}{2}\frac{d}{d\tau}\int_0^1 u_\tau^2 d\xi = \int_0^1 (\rho^\gamma)_\tau u_{\xi\tau}d\xi - \int_0^1 (\rho^{1+\alpha}u_\xi)_\tau u_{\xi\tau}d\xi. \tag{67}$$

The terms on the right-hand side of (67) can be bounded, respectively, as described below:

$$\int_0^1 (\rho^\gamma)_\tau u_{\xi\tau}d\xi$$

$$= -\int_0^1 \gamma\rho^{\gamma+1}u_\xi u_{\xi\tau}d\xi$$

$$\leq -\frac{\gamma}{2}\frac{d}{d\tau}\int_0^1 \rho^{\gamma+1}u_\xi^2 d\xi$$

$$+ C\int_0^1 \left(\rho^{1+\alpha}u_\xi^2 + \rho^{2\gamma-\alpha+3}u_\xi^4\right)d\xi, \tag{68}$$

$$-\int_0^1 \left(\rho^{1+\alpha}u_\xi\right)_\tau u_{\xi\tau}d\xi$$

$$= -\int_0^1 \left((1+\alpha)\rho^\alpha \rho_\tau u_\xi + \rho^{1+\alpha}u_{\xi\tau}\right)u_{\xi\tau}d\xi$$

$$\leq -\frac{1}{2}\int_0^1 \rho^{1+\alpha}u_{\xi\tau}^2 d\xi + C\int_0^1 \rho^{3+\alpha}u_\xi^4 d\xi.$$

Summing (67) and (68) together and making use of (27) and (65), we obtain

$$\frac{1}{2}\frac{d}{d\tau}\int_0^1 u_\tau^2 d\xi + \frac{\gamma}{2}\frac{d}{d\tau}\int_0^1 \rho^{\gamma+1}u_\xi^2 d\xi + \frac{1}{2}\int_0^1 \rho^{1+\alpha}u_{\xi\tau}^2 d\xi$$

$$\leq C\int_0^1 \left(\rho^{1+\alpha}u_\xi^2 + \rho^{2\gamma-\alpha+3}u_\xi^4\right)d\xi + \int_0^1 \rho^{3+\alpha}u_\xi^4 d\xi \tag{69}$$

$$\leq C(T) + C\left\|\rho^{1+\alpha}u_\xi\right\|_{L^\infty([0,1])}^2 \int_0^1 \rho^{1-\alpha}u_\xi^2 d\xi.$$

Substituting (62) into (69), it follows from (27), (47), and (59) that

$$\frac{1}{2}\int_0^1 u_\tau^2 d\xi + \frac{\gamma}{2}\int_0^1 \rho^{\gamma+1}u_\xi^2 d\xi + \frac{1}{2}\int_0^\tau \int_0^1 \rho^{1+\alpha}u_{\xi\tau}^2 d\xi ds$$

$$\leq C(T) + C(T)\int_0^\tau \int_0^1 u_s^2 d\xi ds, \tag{70}$$

which together with Gronwall's inequality, (27), (47), and (65) yields

$$\int_0^1 u_\xi^2 d\xi + \int_0^1 u_\tau^2 d\xi + \int_0^\tau \int_0^1 u_s^2 d\xi ds + \int_0^\tau \int_0^1 u_{\xi s}^2 d\xi ds \leq C(T), \tag{71}$$

which implies $(\rho^\gamma - \rho^{1+\alpha}u_\xi) \in L^\infty(0, T; H^1([0, 1]))$, and it follows from the definition of $a'(\tau) = u(0, \tau)$ that $a(\tau) \in H^2([0, T])$. The proof of this Lemma is completed. $\qquad\square$

Lemma 12. *Let $T > 0$. Under the assumptions of Theorem 2, it holds for any piecewise regular solution (ρ, u) to the FBVP (24) that*

$$\left|\left[\rho_0^\alpha(\xi_0)\right]\right|e^{-C_0\tau} \leq \left|\left[\rho^\alpha(\xi_0, \tau)\right]\right|, \quad \tau \in [0, T], \tag{72}$$

where C_0 is a positive constant independent of time.

Proof. From (27) and (34), we can obtain (72). $\qquad\square$

Finally, we will give the large time behaviors of the interface and decay rate of the density as follows.

Lemma 13. *Let (ρ, u, a) be any piecewise regular solution to the FBVP (1) and (2). Under the assumptions of Theorem 2, it holds for $\alpha \in (0, 1]$ and time $t > 0$ large enough that*

$$C(1+t)^{\gamma/(\gamma-\alpha)} \geq a(t) \geq \begin{cases} c(1+t), & 1 < \gamma < 2\alpha, \\ c(1+t)^{1-\gamma}, & \gamma = 2\alpha, \\ c(1+t)^{\alpha/(\gamma-\alpha)}, & \gamma > 2\alpha, \end{cases} \tag{73}$$

and the density decays pointwise to zero for any $x \in [0, a(t)]$ and $t > 0$ as

$$\rho(a(t), t) = \left((\gamma-\alpha)t + \rho_0(a_0)^{\alpha-\gamma}\right)^{-1/(\gamma-\alpha)}, \tag{74}$$

$$\rho(x, t) \leq C(1+t)^{-1/(\gamma-\alpha)}$$

$$+ \begin{cases} C(1+t)^{-(\gamma-1)/(3\gamma+2\alpha-1)}, \\ \qquad 1 < \gamma < 2\alpha, \\ C(1+t)^{-((\gamma-1)/(3\gamma+2\alpha-1))+\nu}, \\ \qquad \gamma = 2\alpha, \\ C(1+t)^{-\alpha(\gamma-1)/(3\gamma+2\alpha-1)(\gamma-\alpha)}, \\ \qquad \gamma > 2\alpha, \end{cases} \tag{75}$$

$$+ \begin{cases} C(1+t)^{-2/(3\gamma+2\alpha-1)}, \\ \qquad 1 < \gamma < 2\alpha, \\ C(1+t)^{-(2/(3\gamma+2\alpha-1))+\nu}, \\ \qquad \gamma = 2\alpha, \\ C(1+t)^{-2\alpha/(3\gamma+2\alpha-1)(\gamma-\alpha)}, \\ \qquad \gamma > 2\alpha, \end{cases}$$

where $C > 0$ and $c > 0$ are positive constants independent of time and $\nu \in (0, 1)$ is a small constant.

Proof. We introduce the following functional $H(t)$ in the Eulerian form as [22, 28]:

$$H(t)$$

$$= \int_0^{a(t)} (x - (1+t)u(x, t))^2 \rho(x, t)dx$$

$$+ \frac{2}{\gamma-1}(1+t)^2 \int_0^{a(t)} \rho^\gamma(x, t)dx$$

$$= \int_0^{a(t)} x^2 \rho(x, t)dx - 2(1+t)$$

$$\times \int_0^{a(t)} x\rho u\,dx + (1+t)^2 \int_0^{a(t)} \left(\rho u^2 + \frac{2}{\gamma - 1}\rho^\gamma\right) dx$$

$$:= I_1 + I_2 + I_3. \tag{76}$$

Differentiating (76) with respect to t, using (1), (2), and $a'(t) = u(a(t), t)$, we have

$$I_1' = \int_0^{a(t)} x^2 \rho_t\,dx + a^2(t)\,\rho(a(t),t)\,a'(t)$$

$$= 2\int_0^{a(t)} x\rho u\,dx + y^2(t)\,y'(t)\,[\rho],$$

$$I_2' = -2\int_0^{a(t)} x\rho u\,dx - 2(1+t)$$

$$\times \int_0^{a(t)} \left(\rho u^2 + \rho^\gamma\right) dx + 2(1+t)\int_0^{a(t)} \rho^\alpha u_x\,dx$$

$$- 2(1+t)\,y(t)\,(y'(t))^2\,[\rho],$$

$$I_3' = 2(1+t)\int_0^{a(t)} \left(\rho u^2 + \frac{2}{\gamma-1}\rho^\gamma\right) dx$$

$$- 2(1+t)^2 \int_0^{a(t)} \rho^\alpha u_x^2\,dx$$

$$+ (1+t)^2 (y'(t))^3\,[\rho] + \frac{2}{\gamma-1}(1+t)^2 y'(t)\,[\rho^\gamma]. \tag{77}$$

Combining (77), we deduce

$$H'(t) = \frac{2(3-\gamma)}{\gamma-1}(1+t)\int_0^{a(t)} \rho^\gamma\,dx + 2(1+t)$$

$$\times \int_0^{a(t)} \rho^\alpha u_x\,dx - 2(1+t)^2 \int_0^{a(t)} \rho^\alpha u_x^2\,dx$$

$$+ (y(t) - (1+t)\,y'(t))^2\,y'(t)\,[\rho] \tag{78}$$

$$+ \frac{2}{\gamma-1}(1+t)^2 y'(t)\,[\rho^\gamma]$$

$$\leq \frac{2(3-\gamma)}{\gamma-1}(1+t)\int_0^{a(t)} \rho^\gamma\,dx + \frac{1}{2}\int_0^{a(t)} \rho^\alpha\,dx,$$

where we use the fact that as $t > 0$ becomes large enough; it holds from (31) and

$$y(t) = \int_0^{\xi_0} \frac{1}{\rho(\zeta,t)}\,d\zeta$$

$$= \int_0^t u(y(s),s)\,ds + y_0 > 0, \quad t \in [0, +\infty), \tag{79}$$

that

$$y'(t) = u(y(t),t) \geq 0. \tag{80}$$

If $\gamma \geq 3$, we have from (78) and the conservation of mass that

$$H'(t) \leq \int_0^{a(t)} \rho^\alpha\,dx$$

$$\leq \left(\int_0^{a(t)} \rho\,dx\right)^\alpha \left(\int_0^{a(t)} 1\,dx\right)^{1-\alpha} \tag{81}$$

$$\leq Ca(t)^{1-\alpha}.$$

Hence, it holds that

$$H(t) \leq H(0) + C\int_0^t a(s)^{1-\alpha}\,ds \leq C\left(1 + \int_0^t a(s)^{1-\alpha}\,ds\right), \tag{82}$$

$$\int_0^{a(t)} \rho^\gamma\,dx \leq C\left(1 + \int_0^t a(s)^{1-\alpha}\,ds\right)(1+t)^{-2}. \tag{83}$$

From (30) and (35), we obtain

$$a(t) \leq C\rho^{-\gamma}(a(t),t) \leq C(1+t)^{\gamma/(\gamma-\alpha)}, \tag{84}$$

and then

$$\int_0^t a(s)^{1-\alpha}\,ds \leq C(1+t)^{\gamma(1-\alpha)/(\gamma-\alpha)+1}, \tag{85}$$

which with (83) implies

$$\int_0^{a(t)} \rho^\gamma\,dx \leq C(1+t)^{-\alpha(\gamma-1)/(\gamma-\alpha)}, \quad \gamma \geq 3. \tag{86}$$

If $1 < \gamma < 3$, we deduce from (76), (78), and the conservation of mass that

$$H'(t) = (3-\gamma)(1+t)^{-1}H(t) + 2\int_0^{a(t)} \rho^\alpha\,dx$$

$$\leq (3-\gamma)(1+t)^{-1}H(t) + Ca(t)^{1-\alpha}, \tag{87}$$

to which the application of Gronwall's inequality gives

$$H(t) \leq C\left(H(0) + \int_0^t a(s)^{1-\alpha}(1+s)^{\gamma-3}\,ds\right)(1+t)^{3-\gamma}, \tag{88}$$

$$\int_0^{a(t)} \rho^\gamma\,dx \leq C\left(1 + \int_0^t a(s)^{1-\alpha}(1+s)^{\gamma-3}\,ds\right)(1+t)^{1-\gamma}. \tag{89}$$

We get from (84) that

$$\int_0^t a(s)^{1-\alpha}(1+s)^{\gamma-3}\,ds$$

$$\leq \begin{cases} C(1+t)^{\gamma(1-\alpha)/(\gamma-\alpha)+\gamma-2}, & \gamma \in (1,3),\ \gamma \neq 2\alpha, \\ C\ln(1+t), & \gamma = 2\alpha, \end{cases} \tag{90}$$

which together with (89) yields

$$\int_0^{a(t)} \rho^\gamma\,dx \leq \begin{cases} C(1+t)^{-(\gamma-1)}, & \gamma \in (1, 2\alpha), \\ C(1+t)^{-(\gamma-1)}\ln(1+t), & \gamma = 2\alpha, \\ C(1+t)^{-\alpha(\gamma-1)/(\gamma-\alpha)}, & \gamma \in (2\alpha, 3). \end{cases} \tag{91}$$

Also, it follows from the conservation of mass and Hölder's inequality that

$$0 < \left(\int_0^{a_0} \rho_0(x)\,dx \right)^\gamma$$
$$= \left(\int_0^{a(t)} \rho(x,t)\,dx \right)^\gamma \le a(t)^{\gamma-1} \int_0^{a(t)} \rho^\gamma dx, \tag{92}$$

which together with (84), (86), and (91) gives (73). Similarly, we have

$$0 < \left(\int_0^{y_0} \rho_0(x)\,dx \right)^\gamma$$
$$= \left(\int_0^{y(t)} \rho(x,t)\,dx \right)^\gamma \le y(t)^{\gamma-1} \int_0^{y(t)} \rho^\gamma dx, \tag{93}$$

which together with (35) implies that

$$y(t) \ge \begin{cases} c(1+t), & 1 < \gamma < 2\alpha, \\ c(1+t)^{1-\gamma}, & \gamma = 2\alpha, \\ c(1+t)^{\alpha/(\gamma-\alpha)}, & \gamma > 2\alpha, \end{cases} \tag{94}$$

$$-[\rho^\gamma] \le \begin{cases} c(1+t)^{-1}, & 1 < \gamma < 2\alpha, \\ c(1+t)^{-1+\gamma}, & \gamma = 2\alpha, \\ c(1+t)^{-\alpha/(\gamma-\alpha)}, & \gamma > 2\alpha. \end{cases} \tag{95}$$

Finally, it follows from (30) that

$$\rho(a(t),t) = \left((\gamma - \alpha)t + \rho_0^{\alpha-\gamma}(a_0) \right)^{-1/(\gamma-\alpha)}, \tag{96}$$

and as t is large enough, it holds due to (35) that

$$\rho^{(3\gamma+2\alpha-1)/2}(x,t)$$
$$\le \rho^{(3\gamma+2\alpha-1)/2}(a(t),t)$$
$$+ \int_0^{a(t)} \left| \left(\rho^{(3\gamma+2\alpha-1)/2} \right)_x \right| dx - \left[\rho^{(3\gamma+2\alpha-1)/2} \right]$$
$$\le \rho^{(3\gamma+2\alpha-1)/2}(a(t),t)$$
$$+ C \left(\int_0^{a(t)} \rho^\gamma dx \right)^{1/2} - C[\rho^\gamma], \tag{97}$$

which together with (86), (91), (95), and (96) gives (75). □

4. Proof of the Main Results

Proof. The global existence of unique piecewise regular solution to the FBVP (1) and (2) can be established in terms of the short-time existence carried out as in [1, 2, 4], the uniform a-priori estimates, and the analysis of regularities, which indeed follow from Lemmas 5–11. We omit the details. The large time behaviors follow from Lemmas 12–13 directly. The proof of Theorem 2 is completed. □

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