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Research Article

A Note on Scalar-Valued Gap Functions for Generalized Vector Variational Inequalities

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This paper is concerned with gap functions of generalized vector variational inequalities (GVVI). By using scalarization approach, scalar-valued variational inequalities of (GVVI) are introduced. Some relationships between the solutions of (GVVI) and its scalarized versions are established. Then, by using these relationships and some mild conditions, scalar-valued gap functions for (GVVI) are established.

1. Introduction

The concept of vector variational inequalities was firstly introduced by Giannessi [1] in a finite-dimensional space. Since then, extensive study of vector variational inequalities has been done by many authors in finite- or infinite-dimensional spaces under generalized monotonicity and convexity assumptions. See [2–10] and the references therein. Among solution approaches for vector variational inequalities, scalarization is one of the most analyzed topics at least from the computational point of view; see [8–10].

Gap functions are very useful for solving vector variational inequalities. One advantage of the introduction of gap functions in vector variational inequalities is that vector variational inequalities can be transformed into optimization problems. Then, powerful optimization solution methods and algorithms can be applied for finding solutions of vector variational inequalities. Recently, some authors have investigated the gap functions for vector variational inequalities. Yang and Yao [11] introduced gap functions and established necessary and sufficient conditions for the existence of a solution of vector variational inequalities. Chen et al. [12] extended the theory of gap function for scalar variational inequalities to the case of vector variational inequalities. They also obtained the set-valued gap functions for vector variational inequalities. Li and Chen [13] introduced set-valued gap functions for a vector variational inequality and obtained some related properties. Li et al. [14] investigated differential and sensitivity properties of set-valued gap functions for vector variational inequalities and weak vector variational inequalities. Meng and Li [15] also investigated the differential and sensitivity properties of set-valued gap functions for Minty vector variational inequalities and Minty weak vector variational inequalities.

The purpose of this paper is to define a single variable gap function for generalized vector variational inequalities by using the scalarization approach. To this end, we first transform the generalized vector variational inequality into an equivalent scalar variational inequality by using the scalarization approach of [9]. Then, we establish the relations between vector variational inequalities and variational inequalities. Finally, we apply the results to obtain gap functions for generalized vector variational inequalities.

2. Generalized Vector Variational Inequalities

Throughout this paper, let the set of nonnegative real numbers be denoted by R_+, the origins of all finite-dimensional spaces denoted by 0, the norms of all finite-dimensional spaces denoted by $\|\cdot\|$, and the inner products of all finite-dimensional spaces denoted by $\langle,\rangle$. Furthermore, let $K \subseteq R^n$ be nonempty closed convex set. Let $F_i : R^n \to R^n$ $(i = 1, 2, \ldots, m)$ be vector-valued functions, and let $g_i : R^n \to R$ $(i = 1, 2, \ldots, m)$ be real-valued functions. For abbreviation, we put

$$F := (F_1, F_2, \ldots, F_m), \qquad g := (g_1, g_2, \ldots, g_m), \tag{1}$$

and for any $x, v \in R^n$,

$$\langle F(x), v\rangle := (\langle F_1(x), v\rangle, \langle F_2(x), v\rangle, \ldots, \langle F_m(x), v\rangle). \tag{2}$$

In this paper, we consider the following generalized vector variational inequality (GVVI):

$$(\mathrm{GVVI}) \begin{cases} \text{Find } x_0 \in K \text{ such that} \\ \langle F(x_0), x - x_0\rangle + g(x) - g(x_0) \notin -\operatorname{int} R^m_+, \\ \qquad\qquad\qquad \text{for any } x \in K. \end{cases} \tag{3}$$

The solution set of (GVVI) is denoted by sol (GVVI).

If $g = 0$, then (GVVI) collapses to the following vector variational inequality (VVI), introduced and studied by [2, 3]:

$$(\mathrm{VVI}) \begin{cases} \text{Find } x_0 \in K \text{ such that} \\ \langle F(x_0), x - x_0\rangle \notin -\operatorname{int} R^m_+, \\ \qquad \text{for any } x \in K. \end{cases} \tag{4}$$

The solution set of (VVI) is denoted by sol (VVI).

Clearly, for $m = 1$, (GVVI) and (VVI) collapse to the generalized variational inequality (GVI)

$$(\mathrm{GVI}) \begin{cases} \text{Find } x_0 \in K \text{ such that} \\ \langle F_1(x_0), x - x_0\rangle + g_1(x) - g_1(x_0) \geq 0, \\ \qquad\qquad\qquad \text{for any } x \in K, \end{cases} \tag{5}$$

and the variational inequality (VI)

$$(\mathrm{VI}) \begin{cases} \text{Find } x_0 \in K \text{ such that} \\ \langle F_1(x_0), x - x_0\rangle \geq 0, \\ \qquad \text{for any } x \in K, \end{cases} \tag{6}$$

respectively.

Now, by using the scalarization scheme of Lee et al. [9], we introduce scalar gap functions for (GVVI) and (VVI). So, for any $\xi \in R^m_+ \setminus \{0\}$, we consider the following scalar variational inequalities:

$$(\mathrm{GVI})_\xi \begin{cases} \text{Find } x_0 \in K \text{ such that} \\ \left\langle \sum_{i=1}^m \xi_i F_i(x_0), x - x_0 \right\rangle + \sum_{i=1}^m \xi_i g_i(x) \\ \quad -\sum_{i=1}^m \xi_i g_i(x_0) \geq 0 \quad \text{for all } x \in K, \end{cases} \tag{7}$$

$$(\mathrm{VI})_\xi \begin{cases} \text{Find } x_0 \in K \text{ such that} \\ \left\langle \sum_{i=1}^m \xi_i F_i(x_0), x - x_0 \right\rangle \geq 0 \quad \text{for all } x \in K. \end{cases}$$

The solution sets of $(\mathrm{GVI})_\xi$ and $(\mathrm{VI})_\xi$ are denoted by sol $(\mathrm{GVI})_\xi$ and sol $(\mathrm{VI})_\xi$, respectively.

Lemma 1. *The following properties hold.*

(i) *If g_i is an affine function for every i, then,*

$$\bigcup_{\xi \in \operatorname{int} R^m_+} sol\ (GVI)_\xi \subset sol\ (GVVI) = \bigcup_{\xi \in R^m_+ \setminus \{0\}} sol\ (GVI)_\xi. \tag{8}$$

(ii) *If F_i and g_i are continuous functions for every i, then sol (GVVI) is a closed set.*

Proof. (i) We first prove the inclusion. In fact, take any $x_0 \in$ sol $(\mathrm{GVI})_\xi$, where $\xi \in \operatorname{int} R^m_+$. Then, for any $x \in K$,

$$\begin{aligned} &\left\langle \sum_{i=1}^m \xi_i F_i(x_0), x - x_0 \right\rangle + \sum_{i=1}^m \xi_i g_i(x) - \sum_{i=1}^m \xi_i g_i(x_0) \\ &\quad = \sum_{i=1}^m \xi_i \left(\langle F_i(x_0), x - x_0\rangle + g_i(x) - g_i(x_0)\right) \\ &\quad = \langle \xi, \langle F(x_0), x - x_0\rangle + g(x) - g(x_0)\rangle \\ &\quad \geq 0. \end{aligned} \tag{9}$$

Thus, there cannot exist $x \in K$ such that

$$\langle F(x_0), x - x_0\rangle + g(x) - g(x_0) \in -\operatorname{int} R^m_+, \tag{10}$$

which means that $x_0 \in$ sol (GVVI).

Now, we prove the equality in (8). If $x_0 \in$ sol (GVVI), then,

$$\begin{aligned} &\{\langle F(x_0), x - x_0\rangle + g(x) - g(x_0) : x \in K\} \\ &\quad \cap (-\operatorname{int} R^m_+) = \emptyset. \end{aligned} \tag{11}$$

Moreover, since each g_i is an affine function

$$\{\langle F(x_0), x - x_0\rangle + g(x) - g(x_0) : x \in K\} \tag{12}$$

is a convex set. Thus, by using the separation theorem (see [16, Theorem 11.3]), there exists $\xi \in R^m \setminus \{0\}$ such that

$$\begin{aligned} &\inf_{x \in K} \left\{\xi^T \left(\langle F(x_0), x - x_0\rangle + g(x) - g(x_0)\right)\right\} \\ &\quad \geq \sup_{v \in -\operatorname{int} R^m_+} \xi^T v. \end{aligned} \tag{13}$$

This means that $\xi \in R^m \setminus \{0\}$, and for any $x \in K$,

$$\xi^T \left(\langle F(x_0), x - x_0\rangle + g(x) - g(x_0)\right) \geq 0. \tag{14}$$

Then, $x_0 \in$ sol $(\mathrm{GVI})_\xi$. Conversely, for any $x_0 \in$ sol $(\mathrm{GVI})_\xi$, where $\xi \in R^m_+ \setminus \{0\}$, it is easy to see that $x_0 \in$ sol (GVVI).

(ii) Set

$$\Delta := R^m \setminus (-\operatorname{int} R^m_+). \tag{15}$$

Then, Δ is a closed set. For any $x \in K$, let

$$K(x) := \{y \in K : \langle F(y), x - y\rangle + g(x) - g(y) \in \Delta\}. \tag{16}$$

Since F and g are continuous, $K(x)$ is a closed set. Moreover, since

$$\mathrm{sol}\,(\mathrm{GVVI}) = \bigcap_{x\in K} K(x), \tag{17}$$

we get that sol (GVVI) is a closed set. The proof is complete. □

Taking $g = 0$ in Lemma 1, we can easily get the following result.

Corollary 2 (see [9]). *The following properties hold.*

(i)

$$\bigcup_{\xi\in \operatorname{int} R^m_+} sol\,(VI)_\xi \subset sol\,(VVI) = \bigcup_{\xi\in R^m_+\setminus\{0\}} sol\,(VI)_\xi. \tag{18}$$

(ii) *If F_i is a continuous function for every i, then sol (VVI) is a closed set.*

3. Gap Functions for (GVVI) and (VVI)

In this section, we propose some new gap functions for (GVVI). Now, we first introduce the definitions of gap functions for (GVVI) and (VVI).

Definition 3. A real-valued function $\psi : X \to R$ is said to be a scalar-valued gap function of (GVVI) if it satisfies the following conditions:

(i) $\psi(x) \ge 0$, for any $x \in K$;

(ii) $\psi(x_0) = 0$ if and only if $x_0 \in K$ is a solution of (GVVI).

Definition 4. A real-valued function $\varphi : X \to R$ is said to be a scalar-valued gap function of (VVI) if it satisfies the following conditions:

(i) $\varphi(x) \ge 0$, for any $x \in K$;

(ii) $\varphi(x_0) = 0$ if and only if $x_0 \in K$ is a solution of (VVI).

Now, by using Lemma 1 and Corollary 2, we generalize the gap function introduced by Auslender [17] for scalar variational inequalities to the case of vector variational inequalities. The gap functions for (GVVI) and (VVI) are defined by

$$\psi(x) = \inf_{\xi\in S^m}\left\{\sup_{y\in K}\left\{\left\langle\sum_{i=1}^m \xi_i F_i(x), x-y\right\rangle - \sum_{i=1}^m \xi_i g_i(y)\right\} + \sum_{i=1}^m \xi_i g_i(x)\right\}, \tag{19}$$

$$\varphi(x) = \inf_{\xi\in S^m}\sup_{y\in K}\left\langle\sum_{i=1}^m \xi_i F_i(x), x-y\right\rangle, \tag{20}$$

respectively. The symbol S^m in the above expression denotes the unit simplex in R^m_+; that is, it is given as

$$S^m = \left\{x \in \mathbb{R}^m_+ : \sum_{i=1}^m x_i = 1\right\}. \tag{21}$$

The use of S^m in the above expression is to stress the fact that the vector $\xi \neq 0$, and we just express the normalized version. Further, use of S^m has an advantage since if additionally K is compact and each g_i is convex for any $i = 1, 2, \dots, m$, then, the functions ψ and φ are finite.

Theorem 5. *If g_i is an affine function for every i, then, the function ψ defined by (19) is a gap function for (GVVI).*

Proof. (i) It is easy to prove that $\psi(x) \ge 0$ for all $x \in K$.

(ii) If there exists $\overline{x} \in K$ such that $\psi(\overline{x}) = 0$, set

$$\theta(\overline{x},\xi) = \sup_{y\in K}\left\{\left\langle\sum_{i=1}^m \xi_i F_i(\overline{x}), \overline{x}-y\right\rangle - \sum_{i=1}^m \xi_i g_i(y)\right\}. \tag{22}$$

Then,

$$\psi(\overline{x}) = \inf_{\xi\in S^m}\left\{\theta(\overline{x},\xi) + \sum_{i=1}^m \xi_i g_i(\overline{x})\right\}. \tag{23}$$

It is easy to observe that for $\overline{x} \in K$, the function $\theta(\overline{x},\cdot)$ is a convex function. Moreover, since $\psi(\overline{x}) = 0$, $\theta(\overline{x},\cdot)$ is a proper lower semicontinuous convex function. Then, there exists $\xi^* \in S^m$ such that

$$\theta(\overline{x},\xi^*) + \sum_{i=1}^m \xi_i g_i(\overline{x}) = \psi(\overline{x}) = 0, \tag{24}$$

which follows that for all $y \in K$,

$$\left\langle\sum_{i=1}^m \xi_i F_i(\overline{x}), \overline{x}-y\right\rangle + \sum_{i=1}^m \xi_i g_i(\overline{x}) - \sum_{i=1}^m \xi_i g_i(y) \le 0. \tag{25}$$

Then,

$$\left\langle\sum_{i=1}^m \xi_i F_i(\overline{x}), y-\overline{x}\right\rangle + \sum_{i=1}^m \xi_i g_i(y) - \sum_{i=1}^m \xi_i g_i(\overline{x}) \ge 0. \tag{26}$$

This means that $\overline{x}$ solves $(\mathrm{GVI})_\xi$. Thus, using Lemma 1, we get that $\overline{x}$ is a solution of (GVVI).

Conversely, let $\overline{x} \in$ sol (GVVI). By Lemma 1, there exists $\xi' \in S^m$ such that $\overline{x} \in \mathrm{sol}\,(\mathrm{GVI})_{\xi'}$. Then, for all $y \in K$,

$$\left\langle\sum_{i=1}^m \xi'_i F_i(\overline{x}), y-\overline{x}\right\rangle + \sum_{i=1}^m \xi'_i g_i(y) - \sum_{i=1}^m \xi'_i g_i(\overline{x}) \ge 0; \tag{27}$$

that is,

$$\left\langle\sum_{i=1}^m \xi'_i F_i(\overline{x}), \overline{x}-y\right\rangle + \sum_{i=1}^m \xi'_i g_i(\overline{x}) - \sum_{i=1}^m \xi'_i g_i(y) \le 0. \tag{28}$$

Then,

$$\theta\left(\overline{x}, \xi'\right) + \sum_{i=1}^{m} \xi_i' g_i\left(\overline{x}\right) \leq 0. \tag{29}$$

So, $\psi(\overline{x}) \leq 0$. Moreover, as $\psi(x) \geq 0$ for all $x \in K$, then, $\psi(\overline{x}) = 0$ and the proof is complete. □

By Theorem 5, it is easy to see that the following result holds.

Corollary 6. *The function* φ *defined by* (20) *is a gap function for* (VVI).

4. Conclusions

In this paper, by using the scalarization approach of [9], we transform a generalized vector variational inequality into an equivalent scalar variational inequality. Then, we establish some relationships between the solutions of vector variational inequalities and variational inequalities. By using these relationships and some mild conditions, we obtain gap functions for the generalized vector variational inequalities and vector variational inequalities.

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References

[1] F. Giannessi, "Theorems of alternative, quadratic programs and complementarity problems," in *Variational Inequalities and Complementarity Problems*, pp. 151–186, John Wiley & Sons, New York, NY, USA, 1980.

[2] F. Giannessi, *Vector Variational Inequalities and Vector Equilibria*, vol. 38 of *Mathematical Theories*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[3] G.-y. Chen, X. Huang, and X. Yang, *Vector Optimization: Set-Valued and Variational Analysis*, vol. 541 of *Lecture Notes in Economics and Mathematical Systems*, Springer, Berlin, Germany, 2005.

[4] J. C. Yao, "Variational inequalities with generalized monotone operators," *Mathematics of Operations Research*, vol. 19, no. 3, pp. 691–705, 1994.

[5] J. C. Yao, "Multi-valued variational inequalities with K-pseudomonotone operators," *Journal of Optimization Theory and Applications*, vol. 83, no. 2, pp. 391–403, 1994.

[6] S. J. Yu and J. C. Yao, "On vector variational inequalities," *Journal of Optimization Theory and Applications*, vol. 89, no. 3, pp. 749–769, 1996.

[7] L.-C. Zeng and J.-C. Yao, "Existence of solutions of generalized vector variational inequalities in reflexive Banach spaces," *Journal of Global Optimization*, vol. 36, no. 4, pp. 483–497, 2006.

[8] G. Y. Chen and S. J. Li, "Existence of solutions for a generalized vector quasivariational inequality," *Journal of Optimization Theory and Applications*, vol. 90, no. 2, pp. 321–334, 1996.

[9] G. M. Lee, D. S. Kim, B. S. Lee, and N. D. Yen, "Vector variational inequality as a tool for studying vector optimization problems," *Nonlinear Analysis*, vol. 34, no. 5, pp. 745–765, 1998.

[10] I. V. Konnov, "A scalarization approach for vector variational inequalities with applications," *Journal of Global Optimization*, vol. 32, no. 4, pp. 517–527, 2005.

[11] X. Q. Yang and J. C. Yao, "Gap functions and existence of solutions to set-valued vector variational inequalities," *Journal of Optimization Theory and Applications*, vol. 115, no. 2, pp. 407–417, 2002.

[12] G.-y. Chen, C.-J. Goh, and X. Q. Yang, "On gap functions for vector variational inequalities," in *Vector Variational Inequalities and Vector Equilibria*, vol. 38 of *Mathematical Theories*, pp. 55–70, Kluwer Academic, Boston, Mass, USA, 2000.

[13] S. J. Li and G. Y. Chen, "Properties of gap function for vector variational inequality," in *Variational Analysis and Applications*, F. Giannessi and A. Maugeri, Eds., vol. 79, pp. 605–631, Springer, Berlin, Germany, 2005.

[14] S. J. Li, H. Yan, and G. Y. Chen, "Differential and sensitivity properties of gap functions for vector variational inequalities," *Mathematical Methods of Operations Research*, vol. 57, no. 3, pp. 377–391, 2003.

[15] K. W. Meng and S. J. Li, "Differential and sensitivity properties of gap functions for Minty vector variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 1, pp. 386–398, 2008.

[16] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, USA, 1970.

[17] A. Auslender, *Optimisation: Methodes Numeriques*, Masson, Paris, France, 1976.