

## Research Article

# A New Fixed Point Theorem and Applications

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A new fixed point theorem is established under the setting of a generalized finitely continuous topological space (GFC-space) without the convexity structure. As applications, a weak KKM theorem and a minimax inequalities of Ky Fan type are also obtained under suitable conditions. Our results are different from known results in the literature.

## 1. Introduction

In the last decade, the theory of fixed points has been investigated by many authors; see, for example, [1–11] and references therein, which has been exploited in the existence study for almost all areas of mathematics, including optimization and applications in economics. Now, there have been a lot of generalizations of the fixed points theorem under different assumptions and different underlying space, and various applications have been given in different fields.

On the other hand, the weak KKM-type theorem introduced by Balaj [12] has attracted an increasing amount of attention and has been applied in many optimization problems so far; see [12–14] and references therein.

Inspired by the research works mentioned above, we establish a collectively fixed points theorem and a fixed point theorem. As applications, a weak KKM theorem and a minimax inequalities of Ky Fan type are also obtained under suitable conditions. Our results are new and different from known results in the literature.

The rest of the paper is organized as follows. In Section 2, we first recall some definitions and theorems. Section 3 is devoted to a new collectively fixed points theorem under noncompact situation on GFC-space and a new fixed point theorem. In Section 4, we show a new weak KKM theorem in underlying GFC-space, and, by using the weak KKM

theorem, a new minimax inequality of Ky Fan type is developed.

## 2. Preliminaries

Let  $X$  be a topological space and  $C, D \subseteq X$ . Let  $\text{int } C$  and  $\text{int}_D C$  denote the interior of  $C$  in  $X$  and in  $D$ , respectively. Let  $\langle A \rangle$  denote the set of all nonempty finite subsets of a set  $A$ , and let  $\Delta_n$  denote the standard  $n$ -dimensional simplex with vertices  $\{e_0, e_1, \dots, e_n\}$ . Let  $X$  and  $Y$  be two topological spaces. A mapping  $T : X \rightarrow 2^Y$  is said to be upper semicontinuous (u.s.c.) (resp., lower semicontinuous (l.s.c.)) if for every closed subset  $B$  of  $Y$ , the set  $\{x \in X : T(x) \cap B \neq \emptyset\}$  (resp.,  $\{x \in X : T(x) \subseteq B\}$ ) is closed.

A subset  $A$  of  $X$  is said to be compactly open (resp., compactly closed) if for each nonempty compact subset  $K$  of  $X$ ,  $A \cap K$  is open (resp., closed) in  $K$ .

These following notions were introduced by Hai et al. [15].

*Definition 1.* Let  $X$  be a topological space,  $Y$  a nonempty set, and  $\Phi$  a family of continuous mappings  $\varphi : \Delta_n \rightarrow X, n \in \mathbb{N}$ . A triple  $(X, Y, \Phi)$  is said to be a generalized finitely continuous topological space (GFC-space) if and only if for each finite subset  $N = \{y_0, y_1, \dots, y_n\}$  of  $Y$ , there is  $\varphi_N : \Delta_n \rightarrow X$  of the family  $\Phi$ .

In the sequel, we also use  $(X, Y, \{\varphi_N\})$  to denote  $(X, Y, \Phi)$ .

**Definition 2.** Let  $S : Y \rightarrow 2^X$  be a multivalued mapping. A subset  $D$  of  $Y$  is called an  $S$ -subset of  $Y$  if and only if for each  $N = \{y_0, y_1, \dots, y_n\} \subseteq Y$  and each  $\{y_{i_0}, y_{i_1}, \dots, y_{i_k}\} \subseteq N \cap D$ , one has  $\varphi_N(\Delta_k) \subset S(D)$ , where  $\Delta_k$  is the face of  $\Delta_n$  corresponding to  $\{y_{i_0}, y_{i_1}, \dots, y_{i_k}\}$ , that is, the simplex with vertices  $\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$ . Roughly speaking, if  $D$  is an  $S$ -subset of  $Y$ , then  $(S(D), D, \Phi)$  is a GFC-space.

The class of GFC-space contains a large number of spaces with various kinds of generalized convexity structures such as FC-space and  $G$ -convex space (see [15–17]).

**Definition 3** (see [8]). Let  $(X, Y, \{\varphi_N\})$  be a GFC-space and  $Z$  a nonempty set. Let  $T : X \rightarrow 2^Z$  and  $F : Y \rightarrow 2^Z$  be two set-valued mappings;  $F$  is called a weak KKM mapping with respect to  $T$ , shortly, weak  $T$ -KKM mapping if and only if for each  $N = \{y_0, y_1, \dots, y_n\} \subseteq Y$ ,  $\{y_{i_0}, y_{i_1}, \dots, y_{i_k}\} \subseteq N$  and  $x \in \varphi_N(\Delta_k)$ ,  $T(x) \cap \bigcup_{j=0}^k F(y_{i_j}) \neq \emptyset$ .

**Definition 4** (see [8]). Let  $X$  be a Hausdorff space,  $(X, Y, \{\varphi_N\})$  a GFC-space,  $Z$  a topological space,  $T : X \rightarrow 2^Z$ ,  $f : Y \times Z \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ , and  $g : X \times Z \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ . Let  $\lambda \in \mathbf{R}$ .  $f$  is called  $(\lambda, T, g)$ -GFC quasiconvex if and only if for each  $x \in X$ ,  $z \in T(x)$ ,  $N = \{y_0, y_1, \dots, y_n\} \in \langle Y \rangle$ , and  $N_k = \{y_{i_0}, y_{i_1}, \dots, y_{i_k}\} \subseteq N$ , one has the implication  $f(y_{i_j}, z) < \lambda$ , for all  $j = 0, 1, \dots, k$  implies that  $g(x', z) < \lambda$  for all  $x' \in \varphi_N(\Delta_k)$ .

For  $\lambda \in \mathbf{R}$ , define  $\beta \in \mathbf{R}$  and  $H_\lambda : Y \rightarrow 2^Z$  by  $\beta = \inf_{x \in X} \sup_{z \in T(x)} g(x, z)$  and  $H_\lambda(y) = \{z \in Z : f(y, z) \geq \lambda\}$ , respectively.

**Lemma 5** (see [8]). For  $\lambda < \beta$ , if  $f$  is  $(\lambda, T, g)$ -GFC quasiconvex, then  $H_\lambda$  is a weak  $T$ -KKM mapping.

The following result is the obvious corollary of Theorem 3.1 of Khanh et al. [8].

**Lemma 6.** Let  $\{(X_i, Y_i, \{\varphi_{N_i}\})\}_{i \in I}$  be a family of GFC-spaces and  $X = \prod_{i \in I} X_i$  a compact Hausdorff space. For each  $i \in I$ , let  $G_i : X \rightarrow 2^{X_i}$  and  $F_i : X \rightarrow 2^{Y_i}$  be such that the conditions hold as follows:

(i) for each  $x \in X$ , each  $N_i = \{y_{i_0}^j, y_{i_1}^j, \dots, y_{i_{n_i}}^j\} \subseteq Y_i$  and each  $\{y_{i_0}^j, y_{i_1}^j, \dots, y_{i_{k_i}}^j\} \subseteq N_i \cap F_i(x)$ , one has  $\varphi_{N_i}(\Delta_k) \subseteq G_i(x)$  for all  $i \in I$ ,

(ii)  $X = \bigcup_{y^j \in Y_i} \text{int } F_i^{-1}(y^j)$  for all  $i \in I$ .

Then, there exists  $\bar{x} = (\bar{x}_i)_{i \in I} \in X$  such that  $\bar{x}_i \in G_i(\bar{x})$  for all  $i \in I$ .

### 3. Fixed Points Theorems

Let  $I$  be an index set,  $X_i$  topological spaces,  $X = \prod_{i \in I} X_i$ , and  $G_i : X \rightarrow 2^{X_i}$ . The collectively fixed points problem is to find  $\bar{x} = (\bar{x}_i)_{i \in I} \in X$  such that  $\bar{x}_i \in G_i(\bar{x})$ , for all  $i \in I$ .

**Theorem 7.** Let  $\{(X_i, Y_i, \{\varphi_{N_i}\})\}_{i \in I}$  be a family of GFC-spaces and  $X = \prod_{i \in I} X_i$  a Hausdorff space. For each  $i \in I$ , let  $G_i : X \rightarrow 2^{X_i}$ ,  $F_i : X \rightarrow 2^{Y_i}$ , and  $S_i : Y_i \rightarrow 2^{X_i}$  with the following properties:

(i) for each  $x \in X$ ,  $N_i = \{y_{i_0}^j, y_{i_1}^j, \dots, y_{i_{n_i}}^j\} \subseteq Y_i$ , and  $\{y_{i_0}^j, y_{i_1}^j, \dots, y_{i_{k_i}}^j\} \subseteq N_i \cap F_i(x)$ , one has  $\varphi_{N_i}(\Delta_k) \subseteq G_i(x)$  for all  $i \in I$ ,

(ii) for each compact subset  $K$  of  $X$  and each  $i \in I$ ,  $K \subseteq \bigcup_{y^j \in Y_i} \text{int } F_i^{-1}(y^j)$ ;

(iii) there exists a nonempty compact subset  $K_i$  of  $X_i$  and for each  $N_i \in \langle Y_i \rangle$ , there exists an  $S_i$ -subset  $L_{N_i}$  of  $Y_i$  containing  $N_i$  with  $S_i(L_{N_i})$  being compact such that

$$S(L_N) \setminus K \subset \bigcup_{y^j \in L_{N_i}} \text{int } F_i^{-1}(y^j), \quad (1)$$

where  $L_N = \prod_{i \in I} L_{N_i}$ ,  $K = \prod_{i \in I} K_i$ , and  $S(L_N) = \prod_{i \in I} S_i(L_{N_i})$ .

Then, there exists  $\bar{x} = (\bar{x}_i)_{i \in I} \in X$  such that  $\bar{x}_i \in G_i(\bar{x})$  for all  $i \in I$ .

*Proof.* As  $K$  is a compact subset of  $X$ , by the condition (ii), there exists a finite set  $N_i = \{y_{i_0}^j, y_{i_1}^j, \dots, y_{i_{n_i}}^j\} \subseteq Y_i$ , such that

$$K \subseteq \bigcup_{k=0}^{n_i} \text{int } F_i^{-1}(y_{i_k}^j). \quad (2)$$

By the condition (iii), there exists an  $S_i$ -subset  $L_{N_i}$  of  $Y_i$  containing  $N_i$  such that

$$S(L_N) \setminus K \subset \bigcup_{y^j \in L_{N_i}} \text{int } F_i^{-1}(y^j), \quad (3)$$

and it follows that

$$S(L_N) \subset \bigcup_{y^j \in L_{N_i}} \text{int } F_i^{-1}(y^j). \quad (4)$$

We observe that the family  $\{S_i(L_{N_i}), L_{N_i}, \{\varphi_{N_i}\}\}_{i \in I}$  is a family of GFC-space and  $S_i(L_{N_i})$  is compact for each  $i \in I$ , defining set-valued mapping  $G_i^* : S_i(L_{N_i}) \rightarrow 2^{S_i(L_{N_i})}$  and  $F_i^* : S_i(L_{N_i}) \rightarrow 2^{L_{N_i}}$  as follows:

$$\begin{aligned} G_i^*(x) &= G_i(x) \cap S_i(L_{N_i}), \\ F_i^*(x) &= F_i(x) \cap L_{N_i}. \end{aligned} \quad (5)$$

We check assumptions (i) and (ii) of Lemma 6 for replaced  $G_i$  and  $F_i$  by  $G_i^*$  and  $F_i^*$ , respectively. By (i) and the definition of  $S$ -subset, for each  $x \in S_i(L_{N_i})$ , each  $N_i = \{y_{i_0}^j, y_{i_1}^j, \dots, y_{i_{n_i}}^j\} \subseteq L_{N_i}$  and each  $\{y_{i_0}^j, y_{i_1}^j, \dots, y_{i_{k_i}}^j\} \subseteq N_i \cap F_i^*(x) = N_i \cap F_i(x) \cap L_{N_i}$ , we have

$$\varphi_{N_i}(\Delta_k) \subseteq G_i(x) \cap S_i(L_{N_i}) = G_i^*(x), \quad (6)$$

then assumption (i) of Lemma 6 is satisfied.

By (4), we have

$$\begin{aligned}
 S(L_N) &= \left( \bigcup_{y^i \in L_{N_i}} \text{int } F_i^{-1}(y^i) \right) \cap S(L_N) \\
 &= \bigcup_{y^i \in L_{N_i}} \text{int}_{S(L_N)} (F_i^{-1}(y^i) \cap S(L_N)).
 \end{aligned}
 \tag{7}$$

On the other hand, for all  $y^i \in L_{N_i}$ ,

$$\begin{aligned}
 (F_i^*)^{-1}(y^i) &= \{x \in X : y^i \in F_i(x)\} \cap S(L_N) \\
 &= F_i^{-1}(y^i) \cap S(L_N).
 \end{aligned}
 \tag{8}$$

Hence,

$$S(L_N) = \bigcup_{y^i \in L_{N_i}} \text{int}_{S(L_N)} (F_i^*)^{-1}(y^i).
 \tag{9}$$

Thus, (ii) of Lemma 6 is also satisfied. According to Lemma 6, there exists a point  $\bar{x} = (\bar{x}_i)_{i \in I} \in X$  such that  $\bar{x}_i \in G_i(\bar{x})$  for all  $i \in I$ .  $\square$

*Remark 8.* Theorem 7 generalizes Theorem 3.4 of Ding [6] from FC-space to GFC-space, and our condition (iii) is different from its condition (iii). Theorem 7 also extends Theorem 3 in [18]. Note that Theorem 7 is the variation of Theorem 3.2 in [8].

As a special case of Theorem 7, we have the following fixed point theorem that will be used to prove a weak KKM theorem in Section 4.

**Corollary 9.** *Let  $X$  be the Hausdorff space,  $(X, Y, \{\varphi_N\})$  a GFC-space,  $G : X \rightarrow 2^X$ ,  $F : X \rightarrow 2^Y$ , and  $S : Y \rightarrow 2^X$  with the following properties:*

- (i) *for each  $x \in X$ ,  $N = \{y_0, y_1, \dots, y_n\} \subseteq Y$ , and  $\{y_{i_0}, y_{i_1}, \dots, y_{i_k}\} \subseteq N \cap F(x)$ , one has  $\varphi_N(\Delta_k) \subseteq G(x)$ ,*
- (ii) *for each compact subset  $K$  of  $X$ ,  $K \subseteq \bigcup_{y \in Y} \text{int } F^{-1}(y)$ ,*
- (iii) *for each  $N \in \langle Y \rangle$ , there exists an  $S$ -subset  $L_N$  of  $Y$  containing  $N$  with  $S(L_N)$  being compact such that*

$$S(L_N) \setminus K \subset \bigcup_{y \in L_N} \text{int } F^{-1}(y).
 \tag{10}$$

*Then, there exists  $\bar{x} \in X$  such that  $\bar{x} \in G(\bar{x})$ .*

#### 4. Applications

**Theorem 10.** *Let  $X$  be a Hausdorff space,  $(X, Y, \{\varphi_N\})$  a GFC-space,  $Z$  a nonempty set,  $T : X \rightarrow 2^Z$ ,  $H : Y \rightarrow 2^Z$ , and  $S : Y \rightarrow 2^X$ ; assume that*

- (i)  *$H$  is a weak  $T$ -KKM mapping,*
- (ii) *for each  $y \in Y$ , the set  $\{x \in X : T(x) \cap H(y) \neq \emptyset\}$  is compactly closed,*

- (iii) *there exists a compact  $K$  of  $X$ , and, for any  $N \in \langle Y \rangle$ , there exists an  $S$ -subset  $L_N$  of  $Y$  containing  $N$  with  $S(L_N)$  being compact such that*

$$S(L_N) \setminus K \subset \bigcup_{y \in L_N} \text{int } \{x \in X : T(x) \cap H(y) = \emptyset\}.
 \tag{11}$$

*Then, there exists a point  $\bar{x} \in X$  such that  $T(\bar{x}) \cap H(y) \neq \emptyset$  for each  $y \in Y$ .*

*Proof.* Define  $F : X \rightarrow 2^Y$  and  $G : X \rightarrow 2^X$  by

$$\begin{aligned}
 F(x) &= \{y \in Y : T(x) \cap H(y) = \emptyset\}, \\
 G(x) &= \{x' \in X : \exists y \in F(x), T(x') \cap H(y) \neq \emptyset\}.
 \end{aligned}
 \tag{12}$$

Suppose the conclusion does not hold. Then, for each  $x \in X$ , there exists a  $y \in Y$  such that

$$T(x) \cap H(y) = \emptyset.
 \tag{13}$$

It is easy to see that  $F$  has nonempty values. By (ii), for each  $y \in Y$ ,

$$F^{-1}(y) = \{x \in X : T(x) \cap H(y) = \emptyset\}
 \tag{14}$$

is compactly open. Then,

$$X = \bigcup_{y \in Y} \text{int } F^{-1}(y).
 \tag{15}$$

Since  $K$  is a compact subset of  $X$ , then there exists  $N \in \langle Y \rangle$  such that

$$K \subseteq \bigcup_{y \in N} \text{int } F^{-1}(y).
 \tag{16}$$

Then, assumption (ii) of Corollary 9 is satisfied.

It follows from (iii) that there exists a compact  $K$  of  $X$  and for any  $N \in \langle Y \rangle$ , there exists a  $S$ -subset  $L_N$  of  $Y$  containing  $N$  with  $S(L_N)$  being compact such that

$$S(L_N) \setminus K \subset \bigcup_{y \in L_N} \text{int } F^{-1}(y).
 \tag{17}$$

Therefore, assumption (iii) of Corollary 9 is also satisfied.

Furthermore,  $G$  has no fixed point. Indeed, if  $x \in G(x)$ , then there exists  $y \in F(x)$  such that

$$T(x) \cap H(y) \neq \emptyset,
 \tag{18}$$

which contracts the definition of  $F$ . Thus, assumption (i) of Corollary 9 must be violated; that is, there exist an  $\bar{x} \in X$ ,  $\bar{N} = \{\bar{y}_0, \bar{y}_1, \dots, \bar{y}_n\} \subseteq Y$ , and

$$\bar{N}_k = \{\bar{y}_{i_0}, \bar{y}_{i_1}, \dots, \bar{y}_{i_k}\} \subseteq \bar{N} \cap F(\bar{x})
 \tag{19}$$

such that

$$\varphi_{\bar{N}}(\Delta_k) \not\subseteq G(\bar{x}).
 \tag{20}$$

That is, for each  $y \in F(\bar{x})$ ,

$$T(\bar{x}) \cap H(y) = \emptyset. \quad (21)$$

Hence,

$$T(\bar{x}) \cap H(\bar{N}_k) = \emptyset. \quad (22)$$

On the other hand, since  $H$  is a weak T-KKM mapping and  $\bar{x} \in \varphi_{\bar{N}}(\Delta_k)$ , we have

$$T(\bar{x}) \cap H(\bar{N}_k) \neq \emptyset, \quad (23)$$

which is contradict. This completes the proof.  $\square$

*Remark 11.* (1) Theorem 10 extends Theorem 1 in [13] from the G-convex space to GFC-space, and our proof techniques are different. Theorem 10 also generalizes Theorem 4.1 of [8] from the compactness assumption to noncompact situation.

(2) If  $Z$  is a topological space, condition (ii) in Theorem 10 is fulfilled in any of the following cases (see [13]):

- (i)  $H$  has closed values, and  $T$  is u.s.c, on each compact subset of  $X$ .
- (ii)  $H$  has compactly closed values, and  $T$  is u.s.c, on each compact of subset of  $X$  and its values are compact.

**Theorem 12.** Let  $X$  be a Hausdorff space,  $(X, Y, \{\varphi_N\})$  a GFC-space,  $Z$  a topological space,  $T : X \rightarrow 2^Z$  u.s.c.,  $f : Y \times Z \rightarrow \mathbf{R} \cup \{-\infty, +\infty\}$ . and  $S : Y \rightarrow 2^X$ ; assume that

- (i) for each  $y \in Y$ ,  $f(y, \cdot)$  is u.s.c. on each compact subset of  $Z$ ,
- (ii)  $f$  is  $(\lambda, T, g)$ -GFC quasiconvex for all  $\lambda < \beta$  sufficiently close to  $\beta$ ,
- (iii) there exists a compact  $K$  of  $X$ , and, for any  $N \in \langle Y \rangle$ , there exists an  $S$ -subset  $L_N$  of  $Y$  containing  $N$  with  $S(L_N)$  being compact such that

$$S(L_N) \setminus K \subset \bigcup_{y \in L_N} \text{int} \{x \in X : T(x) \cap H_\lambda(y) = \emptyset\}. \quad (24)$$

Then,

$$\inf_{x \in X} \sup_{z \in T(x)} g(x, z) \leq \sup_{x \in X} \inf_{y \in Y} \sup_{z \in T(x)} f(y, z). \quad (25)$$

*Proof.* Let  $\lambda < \beta$  be arbitrary. By Lemma 5 and condition (ii),  $H_\lambda$  is a weak T-KKM mapping. It follows from condition (i) that  $H_\lambda$  has closed values. Hence, the set  $\{x \in X : T(x) \cap H_\lambda(y) \neq \emptyset\}$  is compactly closed for all  $y \in Y$  (see Remark 11 (2)). Thus, all the conditions of Theorem 10 are satisfied, and so there exists an  $\bar{x} \in X$  such that

$$T(\bar{x}) \cap H_\lambda(y) \neq \emptyset, \quad \forall y \in Y. \quad (26)$$

This implies that  $\lambda \leq \inf_{y \in Y} \sup_{z \in T(\bar{x})} f(y, z)$  and so

$$\lambda \leq \sup_{x \in X} \inf_{y \in Y} \sup_{z \in T(\bar{x})} f(y, z). \quad (27)$$

Since  $\lambda < \beta$  is arbitrary, we get the conclusion. This completes the proof.  $\square$

*Remark 13.* Theorem 12 improves Theorem 4.2 of [8] from the compactness assumption to noncompact situation. Theorem 12 also extends Theorem 4 of [12] from compact G-convex space to noncompact GFC-space. Our result includes corresponding earlier Fan-type minimax inequalities due to Tan [19], Park [20], Liu [21], and Kim [22].

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