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Research Article

Pullback Attractors for Nonautonomous 2D-Navier-Stokes Models with Variable Delays

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Using a method based on the concept of the Kuratowski measure of the noncompactness of a bounded set as well as some new estimates of the equicontinuity of the solutions, we prove the existence of a unique pullback attractor in higher regularity space for the multivalued process associated with the nonautonomous 2D-Navier-Stokes model with delays and without the uniqueness of solutions.

1. Introduction

It is well known that the Navier-Stokes equations are very important in the understanding of fluids motion and turbulence. These equations have been studied extensively over the last decades (see [1–3], and the references cited therein). Recently, Caraballo and Real [4] considered global attractors for functional Navier-Stokes models with the uniqueness of solutions and for the delay, so that a wide range of hereditary characteristics (constant or variable delay, distributed delay, etc.) can be treated in a unified way. Very recently, Marín-Rubio and Real [5] used the theory of multivalued dynamical system to establish the existence of attractors for the 2D-Navier-Stokes model with delays, when the forcing term containing the delay is sublinear and only continuous.

For the study of asymptotic behavior for functional partial differential equations without the uniqueness of solutions, as far as we know, not many papers have been published. However, some results in the finite dimensional context can be found in [6, 7] (see also [8–10] for some preliminary and interesting results on the structure of the attractors for ordinary differential delay systems).

The pullback attractor is a possible approach to define an "attractor" for the nonautonomous dynamical systems, the long time behavior of nonautonomous dynamical systems is an interesting and challenging problem; see, for example, [11–19], and so forth. The purpose of our current paper is to study existence of pullback attractors for the following functional Navier-Stokes problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \nu\Delta u + \sum_{i=1}^{2} u_i \frac{\partial u}{\partial x_i} \\
&\quad = f\left(t, u\left(t-\rho\left(t\right)\right)\right) - \nabla p + g\left(t\right) \quad \text{in } (\tau, +\infty)\times\Omega, \\
&\qquad \operatorname{div} u = 0 \quad \text{in } (\tau, +\infty)\times\Omega, \\
&\qquad u = 0 \quad \text{on } (\tau, +\infty)\times\Gamma, \\
&\qquad u\left(\tau + t, x\right) = \phi\left(t, x\right), \quad t \in [-h, 0], \ x \in \Omega,
\end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded set with regular boundary Γ, $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p is the pressure, $\tau \in \mathbb{R}$ is the initial time, g is a nondelayed external force field, f is another external force term and contains some memory effects during a fixed interval of time of length $h > 0$, ρ is an adequate given delay function, and ϕ the initial datum on the interval $[-h, 0]$.

Using the technique of measure of noncompactness, noting that all norms on finite dimensional spaces are equivalent, we apply the new method to check the pullback ω-limit compactness given in [20] and then get the existence of the pullback attractors in C_V.

We consider the following usual abstract spaces:

$$\mathscr{V} = \left\{u \in \left(C_0^\infty(\Omega)\right)^2 : \operatorname{div} u = 0\right\}, \tag{2}$$

where H = the closure of $\mathscr{V}$ in $(L^2(\Omega))^2$ with norm $|\cdot|$ and inner product $(\cdot,\cdot)$, where for $u, v \in (L^2(\Omega))^2$,

$$(u, v) = \sum_{j=1}^{2} \int_\Omega u_j(x)\, v_j(x)\, dx, \tag{3}$$

where V = the closure of $\mathscr{V}$ in $(H_0^1(\Omega))^2$ with norm $\|\cdot\|$ and associated scalar product $((\cdot,\cdot))$, where for $u, v \in (H_0^1(\Omega))^2$,

$$((u, v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx. \tag{4}$$

Note that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact. We will use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle$ for the duality pairing between V and V'.

Define the trilinear form b on $V \times V \times V$ by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \forall u, v, w \in V. \tag{5}$$

Now, let us establish some assumptions for (1).

We assume that the given delay function satisfies $\rho \in C^1(\mathbb{R}; [0, h])$, and there exists a constant ρ_* satisfying

$$\rho'(t) \leqslant \rho_* < 1, \quad \forall t \in \mathbb{R}. \tag{6}$$

Furthermore, we suppose that f and g satisfy the following assumptions:

(H1) $f(\cdot, v) : \mathbb{R} \to H$ is measurable for all $v \in H$,

(H2) $f(t, \cdot) : H \to H$ is continuous for all $t \in \mathbb{R}$,

(H3) there exist positive constants k_1, k_2 such that for any $v \in H$,

$$\left|f(t, v)\right|^2 \leqslant k_1^2 + k_2^2 |v|^2, \quad \forall t \in \mathbb{R}, \tag{7}$$

(H4) there exists a fixed $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, the external force $g \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ satisfies

$$\int_{-\infty}^{t} \left|g(r)\right|^2 e^{\delta r} dr < \infty, \quad \forall t \in \mathbb{R}. \tag{8}$$

Set $A : V \to V'$ as $\langle Au, v\rangle = ((u, v))$, $B : V \times V \to V'$ by $\langle B(u, v), w\rangle = b(u, v, w)$, for all $u, v, w \in V$. Denote by P the corresponding orthogonal projection $P : (L^2(\Omega))^2 \to H$. We further set $A := -P\Delta$. The Stokes operator A is self-adjoint and positive from $D(A) = V \cap (H^2(\Omega))^2$ to H. The inverse operator is compact. Excluding the pressure, the system (1) can be written in the form

$$\begin{aligned} &\frac{d}{dt} u(t) + \nu A u(t) + B(u(t), u(t)) \\ &\quad = f(t, u(t - \rho(t))) + g(t) \quad \text{in } \mathscr{D}'\left(\tau, +\infty; V'\right), \\ &\quad u(\tau + t) = \phi(t), \quad t \in [-h, 0], \ x \in \Omega. \end{aligned} \tag{9}$$

2. Preliminaries

Let X be a complete metric space with metric $d_X(\cdot,\cdot)$, and denote by $\mathscr{P}(X)$ the class of nonempty subsets of X. As usual, let us denote by $H_X^*(\cdot,\cdot)$ the Hausdorff semidistance between A and B, which are defined by

$$H_X^*(A, B) = \sup_{a \in A} \operatorname{dist}_X(a, B), \tag{10}$$

where $\operatorname{dist}_X(a, B) = \inf_{b \in B} d_X(a, b)$. Finally, denote by $\mathscr{N}(A, r)$ the open neighborhood $\{y \in X \mid \operatorname{dist}_X(y, A) < r\}$ of radius $r > 0$ of a subset A of a Banach space X.

Definition 1. A family of mappings $U(t, \tau) : X \to \mathscr{P}(X)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$ is called to be a multivalued process (MVP in short) if it satisfies

(1) $U(\tau, \tau)x = \{x\}$, for all $\tau \in \mathbb{R}$, $x \in X$;

(2) $U(t, s)U(s, \tau)x = U(t, \tau)x$, for all $t \geqslant s \geqslant \tau$, $\tau \in \mathbb{R}$, $x \in X$.

Let $\mathscr{D}$ be a nonempty class of parameterized sets $\mathscr{D} = \{D(t)\}_{t \in \mathbb{R}} \subset \mathscr{P}(X)$.

Definition 2. Let $\{U(t, \tau)\}$ be a multivalued process on X. One says that $\{U(t, \tau)\}$ is

(1) pullback $\mathscr{D}$-dissipative, if there exists a family $\mathscr{Q} = \{Q(t)\}_{t \in \mathbb{R}} \in \mathscr{D}$, so that for any $\mathscr{B} = \{B(t)\}_{t \in \mathbb{R}} \in \mathscr{D}$ and each $t \in \mathbb{R}$, there exists a $t_0 = t_0(\mathscr{B}, t) \in \mathbb{R}^+$ such that

$$U(t, t - s) B(t - s) \subset Q(t), \quad \forall s \geqslant t_0; \tag{11}$$

(2) pullback $\mathscr{D}$-limit-set compact with respect to each $t \in \mathbb{R}$, if for any $\mathscr{B} = \{B(t)\}_{t \in \mathbb{R}} \in \mathscr{D}$ and $\varepsilon > 0$, there exists a $t_1 = t_1(\mathscr{B}, t, \varepsilon) \in \mathbb{R}^+$ such that

$$k\left(\bigcup_{s \geqslant t_1} U(t, t - s) B(t - s)\right) \leqslant \varepsilon, \tag{12}$$

where k is the Kuratowski measure of noncompactness.

Definition 3. A family of nonempty compact subsets $\mathscr{A} = \{A(t)\}_{t \in \mathbb{R}} \subset \mathscr{P}(X)$ is called to be a pullback $\mathscr{D}$-attractor for the multivalued process $\{U(t, \tau)\}$, if it satisfies

(1) $\mathscr{A} = \{A(t)\}_{t \in \mathbb{R}}$ is invariant; that is,

$$U(t, \tau) A(\tau) = A(t), \quad \forall t \geqslant \tau, \ \tau \in \mathbb{R}, \tag{13}$$

(2) $\mathscr{A}$ is pullback $\mathscr{D}$-attracting; that is, for every $\mathscr{B} \in \mathscr{D}$ and any fixed $t \in \mathbb{R}$,

$$\lim_{s \to +\infty} H_X^*(U(t, t - s) B(t - s), A(t)) = 0. \tag{14}$$

Let X, Y be two Banach spaces, and let X^*, Y^* be their dual spaces, respectively. We also assume that X is a dense subspace of Y, the injection $i : X \hookrightarrow Y$ is continuous, and its adjoint $i^* : Y^* \hookrightarrow X^*$ is densely injective.

Theorem 4 (see [21, 22]). *Let X, Y be two Banach spaces satisfy the previous assumptions, and let $\{U(t,\tau)\}$ be a multivalued process on X and Y, respectively. Assume that $\{U(t,\tau)\}$ is upper semicontinuous or weak upper semicontinuous on Y. If for fixed $t \geqslant \tau$, $\tau \in \mathbb{R}$, $U(t,\tau)$ maps compact subsets of X into bounded subsets of $\mathscr{P}(X)$, then $U(t,\tau)$ is norm-to-weak upper semicontinuous on X.*

By slightly modifying the arguments of Theorem 3.4 and Remark 3.9 in [21], we have the following.

Theorem 5. *Let X be a Banach space, and let $\{U(t,\tau)\}$ be a multivalued process on X. Also let $U(t,\tau)x$ be norm-to-weak upper semicontinuous in x for fixed $t \geqslant \tau$, $\tau \in \mathbb{R}$; that is, if $x_n \to x$, then for any $y_n \in U(t,\tau)x_n$, there exist a subsequence $y_{n_k} \in U(t,\tau)x_{n_k}$ and a $y \in U(t,\tau)x$ such that $y_{n_k} \rightharpoonup y$ (weak convergence). Then the multivalued process $\{U(t,\tau)\}$ possesses a pullback $\mathscr{D}$-attractor $\mathscr{A} = \{A(t)\}_{t\in\mathbb{R}}$ in X given by*

$$A(t) = \omega_t(Q) = \bigcap_{T\in\mathbb{R}^+} \overline{\bigcup_{s\geqslant T} U(t,t-s)Q(t-s)} \subset Q(t) \tag{15}$$

if and only if $\{U(t,\tau)\}$ is pullback $\mathscr{D}$-dissipative and pullback $\mathscr{D}$-limit-set compact with respect to each $t \in \mathbb{R}$, where $\mathcal{Q} = \{Q(t)\}_{t\in\mathbb{R}} \in \mathscr{D}$ is pullback $\mathscr{D}$-absorbing for the multivalued process $\{U(t,\tau)\}$.

A multivalued process $\{U(t,\tau)\}$ is said to be pullback $\mathscr{D}$-asymptotically upper-semicompact in X if for each fixed $t \in \mathbb{R}$, any $\mathscr{B} = \{B(t)\}_{t\in\mathbb{R}} \in \mathscr{D}$, any sequence $\{T_n\}$ with $T_n \to +\infty$, $\{x_n\}$ with $x_n \in B(t-T_n)$, and any $\{y_n\}$ with $y_n \in U(t,t-T_n)x_n$; this last sequence $\{y_n\}$ is relatively compact in X.

Remark 6. Let $\{U(t,\tau)\}$ be a multivalued process on X. Then $\{U(t,\tau)\}$ is pullback $\mathscr{D}$-asymptotically upper-semicompact if and only if $\{U(t,\tau)\}$ is pullback $\mathscr{D}$-limit-set compact; see [21].

Let X be a Banach space, and let $h > 0$ be a given positive number (the delay time). Denote by C_X the Banach space $C([-h,0];X)$ endowed with the norm

$$\|\phi\|_{C_X} = \sup_{\theta\in[-h,0]} \|\phi(\theta)\|_X. \tag{16}$$

Let us consider $\mathscr{D}_{C_X}$ a class of sets parameterized in time, $\mathscr{D} = \{D(t)\}_{t\in\mathbb{R}} \subset \mathscr{P}(C_X)$. To study the pullback $\mathscr{D}$-limit-set compactness of the multivalued process on C_X, we need the following result from [20].

Theorem 7. *Let $\{U(t,\tau)\}$ be a multivalued process on C_X. Suppose that for each $t \in \mathbb{R}$, any $\mathscr{B} \in \mathscr{D}_{C_X}$ and $\varepsilon > 0$, there exist $\tau_0 = \tau_0(t,\mathscr{B},\varepsilon) > 0$, a finite dimensional subspace X_1 of X, and a $\delta > 0$ such that*

(1) *for each fixed $\theta \in [-h,0]$,*

$$\left\| \bigcup_{s\geqslant\tau_0} \bigcup_{u_t(\cdot)\in U(t,t-s)B(t-s)} Pu(t+\theta) \right\|_X \text{ is bounded;} \tag{17}$$

(2) *for all $s \geqslant \tau_0$, $u_t(\cdot) \in U(t,t-s)B(t-s)$, $\theta_1, \theta_2 \in [-h,0]$ with $|\theta_2 - \theta_1| < \delta$,*

$$\|P(u(t+\theta_1) - u(t+\theta_2))\|_X < \varepsilon; \tag{18}$$

(3) *for all $s \geqslant \tau_0$, $u_t(\cdot) \in U(t,t-s)B(t-s)$,*

$$\sup_{\theta\in[-h,0]} \|(I-P)u(t+\theta)\|_X < \varepsilon, \tag{19}$$

where $P : X \to X_1$ is the canonical projector. Then $\{U(t,\tau)\}$ is pullback $\mathscr{D}$-limit-set compact in C_X with respect to each $t \in \mathbb{R}$.

3. Existence of an Absorbing Family of Sets in C_V

By the classical Faedo-Galerkin scheme and compactness method, analogous to the arguments in [5], we have the following.

Theorem 8. *Let one consider $\phi \in C_H$, $g \in L^2_{\mathrm{loc}}(\mathbb{R};H)$, and assume that $f : \mathbb{R}\times H \to H$ satisfies the hypotheses $(H1)$–$(H3)$. Then, for each $\tau \in \mathbb{R}$,*

(a) *there exists a weak solution u to problem (9) satisfying*

$$u \in C([\tau-h,T];H) \cap L^\infty(\tau,T;H) \cap L^2(\tau,T;V) \quad \forall T \geqslant \tau; \tag{20}$$

(b) *if $\phi \in C_V$, then there exists a strong solution u to problem (9); that is,*

$$u \in C([\tau-h,T];V) \cap L^\infty(\tau,T;V) \cap L^2(\tau,T;D(A)), \quad \forall T \geqslant \tau. \tag{21}$$

Given $T > \tau$ and $u : [\tau-h,T) \to H$, for each $t \in [\tau,T)$, we denote by u_t the function defined on $[-h,0]$ by the relation $u_t(s) = u(t+s)$, $s \in [-h,0]$. We also denote $C_H = C([-h,0];H)$ and $C_V = C([-h,0];V)$. Let C be the arbitrary positive constants, which may be different from line to line and even in the same line.

Thanks to Theorem 8, we can define a multivalued process $(C_V, \{U(\cdot,\cdot)\})$ as

$$U(t,\tau)(\phi) = \{u_t(\cdot;\tau,\phi) \mid u(\cdot) \text{ is a strong solution of (9) with initial datum } \phi \in C_V\}. \tag{22}$$

We first need a priori estimates for the solution u of (9) in the space C_H and a necessary bound on the term $\int_{t-1}^t e^{\alpha r}\|u(r)\|^2 dr$, which will be very useful in our analysis; it relates the absorption property for the multivalued process $\{U(t,\tau)\}$ on C_V.

Lemma 9. *In addition to the assumptions $(H1)$–$(H4)$, assume that*

$$k_2^2 < \left(\frac{\nu\lambda_1}{2}\right)^2 (1-\rho_*) \tag{23}$$

holds true. Then

$$\begin{aligned}\|u_t\|^2_{C_H} &\leqslant \left(1+\frac{2k_2^2e^{\alpha h}}{\nu\lambda_1(1-\rho_*)\alpha}\right)e^{\alpha(\tau-t+h)}\|\phi\|^2_{C_H}\\ &\quad+\frac{2k_1^2}{\nu\lambda_1\alpha}+\frac{e^{-\alpha(t-h)}}{2\varepsilon_2}\int_{-\infty}^{t}e^{\alpha s}|g(s)|^2ds,\\ &\qquad\qquad \forall t\geqslant\tau+h,\end{aligned}\tag{24}$$

$$\begin{aligned}&\nu\int_{t-1}^{t}e^{\alpha r}\|u(r)\|^2dr\\ &\quad\leqslant Ce^{\alpha\tau}\|\phi\|^2_{C_H}+Ce^{\alpha t}\\ &\qquad+C\int_{-\infty}^{t}e^{\alpha r}|g(r)|^2dr,\quad \forall t\geqslant\tau+h+1,\end{aligned}\tag{25}$$

provided that $\alpha>0$ is small enough.

Proof. By the energy inequality and the Poincaré inequality, we have

$$\begin{aligned}&\frac{d}{dt}|u(t)|^2+\nu\lambda_1|u(t)|^2+\nu\|u(t)\|^2\\ &\quad\leqslant 2\left(f\left(t,u\left(t-\rho(t)\right)\right),u(t)\right)+2\left(g(t),u(t)\right).\end{aligned}\tag{26}$$

We fixed two positive parameters ε_1 and ε_2 to be chosen later on. Then by (H3) and Young's inequality, we can deduce that

$$\begin{aligned}\left|\left(f\left(t,u\left(t-\rho(t)\right)\right),u(t)\right)\right| &\leqslant \left|f\left(t,u\left(t-\rho(t)\right)\right)\right||u(t)|\\ &\leqslant \varepsilon_1|u(t)|^2\\ &\quad+\frac{k_1^2+k_2^2\left|u\left(t-\rho(t)\right)\right|^2}{4\varepsilon_1},\\ \left|g(t),u(t)\right| &\leqslant \varepsilon_2|u(t)|^2+\frac{1}{4\varepsilon_2}|g(t)|^2.\end{aligned}\tag{27}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}|u(t)|^2+\nu\|u(t)\|^2 &\leqslant \left(2\varepsilon_1+2\varepsilon_2-\nu\lambda_1\right)|u(t)|^2\\ &\quad+\frac{k_1^2+k_2^2\left|u\left(t-\rho(t)\right)\right|^2}{2\varepsilon_1}\\ &\quad+\frac{|g(t)|^2}{2\varepsilon_2}.\end{aligned}\tag{28}$$

Let $\alpha>0$ to be determined later on. Then it follows that

$$\begin{aligned}&\frac{d}{dt}\left(e^{\alpha t}|u(t)|^2\right)\\ &\quad=\alpha e^{\alpha t}|u(t)|^2+e^{\alpha t}\frac{d}{dt}|u(t)|^2\\ &\quad\leqslant\left(\alpha+2\varepsilon_1+2\varepsilon_2-\nu\lambda_1\right)e^{\alpha t}|u(t)|^2+\frac{e^{\alpha t}k_1^2}{2\varepsilon_1}\\ &\qquad+\frac{e^{\alpha t}k_2^2\left|u\left(t-\rho(t)\right)\right|^2}{2\varepsilon_1}+\frac{e^{\alpha t}|g(t)|^2}{2\varepsilon_2}.\end{aligned}\tag{29}$$

Integrating between τ and t $(\geqslant\tau)$, we have

$$\begin{aligned}e^{\alpha t}|u(t)|^2 &\leqslant e^{\alpha\tau}|u(\tau)|^2\\ &\quad+\left(\alpha+2\varepsilon_1+2\varepsilon_2-\nu\lambda_1\right)\int_{\tau}^{t}e^{\alpha s}|u(s)|^2ds\\ &\quad+\frac{k_1^2}{2\varepsilon_1}\int_{\tau}^{t}e^{\alpha s}ds\\ &\quad+\frac{k_2^2}{2\varepsilon_1}\int_{\tau}^{t}e^{\alpha s}\left|u\left(s-\rho(s)\right)\right|^2ds\\ &\quad+\frac{1}{2\varepsilon_2}\int_{\tau}^{t}e^{\alpha s}|g(s)|^2ds.\end{aligned}\tag{30}$$

Let $r=s-\rho(s)$; note that $\rho(s)\in[0,h]$ and $1/(1-\rho'(s))\leqslant 1/(1-\rho_*)$ for all $s\in\mathbb{R}$. Hence,

$$\begin{aligned}&\frac{k_2^2}{2\varepsilon_1}\int_{\tau}^{t}e^{\alpha s}\left|u\left(s-\rho(s)\right)\right|^2ds\\ &\quad\leqslant\frac{k_2^2}{2\varepsilon_1}\frac{1}{1-\rho_*}\int_{\tau-h}^{t}e^{\alpha(r+h)}|u(r)|^2dr\\ &\quad\leqslant\frac{k_2^2e^{\alpha h}}{2\varepsilon_1(1-\rho_*)}\\ &\qquad\times\left(\int_{\tau-h}^{\tau}e^{\alpha r}|u(r)|^2dr+\int_{\tau}^{t}e^{\alpha r}|u(r)|^2dr\right)\\ &\quad\leqslant\frac{k_2^2e^{\alpha(h+\tau)}\|\phi\|^2_{C_H}}{2\varepsilon_1(1-\rho_*)\alpha}\\ &\qquad+\frac{k_2^2e^{\alpha h}}{2\varepsilon_1(1-\rho_*)}\int_{\tau}^{t}e^{\alpha r}|u(r)|^2dr.\end{aligned}\tag{31}$$

Combining (30) and (31) together, we get

$$\begin{aligned}e^{\alpha t}|u(t)|^2 &\leqslant \left(1+\frac{k_2^2e^{\alpha h}}{2\varepsilon_1(1-\rho_*)\alpha}\right)e^{\alpha\tau}\|\phi\|^2_{C_H}\\ &\quad+\frac{k_1^2e^{\alpha t}}{2\varepsilon_1\alpha}+\frac{1}{2\varepsilon_2}\int_{-\infty}^{t}e^{\alpha s}|g(s)|^2ds\\ &\quad+\left(\alpha+2\varepsilon_1+2\varepsilon_2-\nu\lambda_1+\frac{k_2^2e^{\alpha h}}{2\varepsilon_1(1-\rho_*)}\right)\\ &\quad\times\int_{\tau}^{t}e^{\alpha s}|u(s)|^2ds.\end{aligned}\tag{32}$$

Let $\varepsilon_1 = \nu\lambda_1/4$ and using (23), so we can choose positive constants α and ε_2 small enough such that $\alpha + 2\varepsilon_1 + 2\varepsilon_2 - \nu\lambda_1 + (k_2^2 e^{\alpha h}/2\varepsilon_1(1-\rho_*)) < 0$ and $\alpha < \delta_0$ (where δ_0 is given in the assumption (H4)). Then, it follows that

$$e^{\alpha t}|u(t)|^2 \leqslant \left(1 + \frac{2k_2^2 e^{\alpha h}}{\nu\lambda_1(1-\rho_*)\alpha}\right) e^{\alpha\tau}\|\phi\|_{C_H}^2 + \frac{2k_1^2 e^{\alpha t}}{\nu\lambda_1\alpha} + \frac{1}{2\varepsilon_2}\int_{-\infty}^{t} e^{\alpha s}|g(s)|^2 ds. \tag{33}$$

Setting now $t+\theta$ instead of t (where $\theta \in [-h, 0]$), multiplying by $e^{-\alpha(t+\theta)}$, it holds

$$\begin{aligned}|u(t+\theta)|^2 &\leqslant \left(1 + \frac{2k_2^2 e^{\alpha h}}{\nu\lambda_1(1-\rho_*)\alpha}\right) e^{\alpha(\tau-t-\theta)}\|\phi\|_{C_H}^2 + \frac{2k_1^2}{\nu\lambda_1\alpha} + \frac{e^{-\alpha(t+\theta)}}{2\varepsilon_2}\int_{-\infty}^{t+\theta} e^{\alpha s}|g(s)|^2 ds \\ &\leqslant \left(1 + \frac{2k_2^2 e^{\alpha h}}{\nu\lambda_1(1-\rho_*)\alpha}\right) e^{\alpha(\tau-t+h)}\|\phi\|_{C_H}^2 + \frac{2k_1^2}{\nu\lambda_1\alpha} + \frac{e^{-\alpha(t-h)}}{2\varepsilon_2}\int_{-\infty}^{t} e^{\alpha s}|g(s)|^2 ds.\end{aligned} \tag{34}$$

Note that $\| u_t \|_{C_H}^2 = \sup_{\theta\in[-h,0]}|u(t+\theta)|^2$, thus the conclusion (24) follows immediately from (34).

Finally, we will obtain the bound on the term $\nu\int_{t-1}^{t} e^{\alpha r}\| u(r) \|^2 dr$. It follows from (28) that

$$\begin{aligned}\nu e^{\alpha t}\|u(t)\|^2 \leqslant{}& (2\varepsilon_1 + 2\varepsilon_2 - \nu\lambda_1) e^{\alpha t}|u(t)|^2 + \frac{k_1^2 e^{\alpha t}}{2\varepsilon_1} + \frac{k_2^2 e^{\alpha t}|u(t-\rho(t))|^2}{2\varepsilon_1} \\ &+ \frac{e^{\alpha t}|g(t)|^2}{2\varepsilon_2} + \alpha e^{\alpha t}|u(t)|^2 - \frac{d}{dt}\left(e^{\alpha t}|u(t)|^2\right).\end{aligned} \tag{35}$$

Integrating from $t-1$ to t, we have

$$\begin{aligned}\nu\int_{t-1}^{t} e^{\alpha r}\|u(r)\|^2 dr \leqslant{}& (\alpha + 2\varepsilon_1 + 2\varepsilon_2 - \nu\lambda_1)\int_{t-1}^{t} e^{\alpha r}|u(r)|^2 dr + \frac{k_1^2}{2\varepsilon_1}\int_{t-1}^{t} e^{\alpha r} dr \\ &+ \frac{k_2^2}{2\varepsilon_1}\int_{t-1}^{t} e^{\alpha r}|u(r-\rho(r))|^2 dr + \frac{1}{2\varepsilon_2}\int_{t-1}^{t} e^{\alpha r}|g(r)|^2 dr \\ &+ e^{\alpha(t-1)}|u(t-1)|^2.\end{aligned} \tag{36}$$

Similar to the arguments of (31), we can deduce that

$$\begin{aligned}&\frac{k_2^2}{2\varepsilon_1}\int_{t-1}^{t} e^{\alpha r}|u(r-\rho(r))|^2 dr \\ &\leqslant \frac{k_2^2}{2\varepsilon_1}\frac{1}{1-\rho_*}\int_{t-1-h}^{t} e^{\alpha(r+h)}|u(r)|^2 dr \\ &\leqslant \frac{k_2^2 e^{\alpha h}}{2\varepsilon_1(1-\rho_*)} \times \left(\int_{t-1-h}^{t-1} e^{\alpha r}|u(r)|^2 dr + \int_{t-1}^{t} e^{\alpha r}|u(r)|^2 dr\right) \\ &\leqslant \frac{k_2^2 e^{\alpha(h+t-1)}\|u_{t-1}\|_{C_H}^2}{2\varepsilon_1(1-\rho_*)\alpha} + \frac{k_2^2 e^{\alpha h}}{2\varepsilon_1(1-\rho_*)}\int_{t-1}^{t} e^{\alpha r}|u(r)|^2 dr.\end{aligned} \tag{37}$$

Recall that $\varepsilon_1 = \nu\lambda_1/4$ and $\alpha + 2\varepsilon_1 + 2\varepsilon_2 - \nu\lambda_1 + (k_2^2 e^{\alpha h}/2\varepsilon_1(1-\rho_*)) < 0$. By (24) and (36)-(37), we have (25) as desired, and thus the proof of this lemma is completed. □

By slightly modifying the proof of Lemma 1.1 in [23], we have the following result.

Lemma 10. *Let $t \in \mathbb{R}$ be given arbitrarily. Let g, h, and y be three positive locally integrable functions on $(-\infty, t]$ such that y' is locally integrable on $(-\infty, t]$, which satisfy that*

$$\begin{aligned}&\frac{dy}{ds} \leqslant gy + h \quad \text{for } s \leqslant t, \\ &\int_{t-1}^{t} g(s)\,ds \leqslant a_1, \qquad \int_{t-1}^{t} h(s)\,ds \leqslant a_2, \\ &\int_{t-1}^{t} y(s)\,ds \leqslant a_3,\end{aligned} \tag{38}$$

where a_1, a_2, and a_3 are positive constants. Then

$$y(t) \leqslant \exp(a_1)(a_3 + a_2). \tag{39}$$

Now we state and prove the main result in this section.

Theorem 11. *Suppose in addition to the hypotheses in Lemma 9, assume that*

$$\lim_{t\to-\infty}\int_{-\infty}^{t} e^{-\gamma(t-r)}|g(r)|^2 dr < \infty \quad \forall\gamma > 0 \tag{40}$$

holds true. Then the multivalued process $\{U(t,\tau)\}$ on C_V is pullback $\mathcal{D}$-dissipative.

Proof. We take the inner product of (9) with $Au(t)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu|Au(t)|^2 + (B(u(t), u(t)), Au(t)) \\ &\quad = (f(t, u(t-\rho(t))), Au(t)) + (g(t), Au(t)).\end{aligned} \tag{41}$$

Now we evaluate the terms, using (H3) and Young's inequality, and we arrive to

$$\begin{aligned}&\left|\left(f\left(t,u\left(t-\rho\left(t\right)\right)\right),Au\left(t\right)\right)\right|+\left|\left(g\left(t\right),Au\left(t\right)\right)\right|\\&\leqslant\frac{\nu}{2}\left|Au\left(t\right)\right|^{2}+\frac{\left|f\left(t,u\left(t-\rho\left(t\right)\right)\right)\right|^{2}}{\nu}+\frac{\left|g\left(t\right)\right|^{2}}{\nu}\\&\leqslant\frac{\nu}{2}\left|Au\left(t\right)\right|^{2}+\frac{k_{1}^{2}+k_{2}^{2}\left\|u_{t}\right\|_{C_{H}}^{2}}{\nu}+\frac{\left|g\left(t\right)\right|^{2}}{\nu}.\end{aligned}\tag{42}$$

Next,

$$\begin{aligned}&\left|\left(B\left(u\left(t\right),u\left(t\right)\right),Au\left(t\right)\right)\right|\\&\leqslant C_{1}\left|u\left(t\right)\right|^{1/2}\left\|u\left(t\right)\right\|\left|Au\left(t\right)\right|^{3/2}\\&\leqslant\frac{\nu}{4}\left|Au\left(t\right)\right|^{2}+\frac{C_{2}}{\nu^{3}}\left|u\left(t\right)\right|^{2}\left\|u(t)\right\|^{4}.\end{aligned}\tag{43}$$

Thanks to (41)–(43) and the fact that $\|\varphi\|^{2}\leqslant\lambda_{1}^{-1}|A\varphi|^{2}$ for $\varphi\in D(A)$, we can deduce that

$$\begin{aligned}&\frac{d}{dt}\left\|u(t)\right\|^{2}+\frac{\nu\lambda_{1}}{2}\left\|u(t)\right\|^{2}\\&\leqslant\frac{2k_{1}^{2}+2k_{2}^{2}\left\|u_{t}\right\|_{C_{H}}^{2}}{\nu}+\frac{2\left|g\left(t\right)\right|^{2}}{\nu}\\&\quad+\frac{2C_{2}}{\nu^{3}}\left|u\left(t\right)\right|^{2}\left\|u(t)\right\|^{4},\end{aligned}\tag{44}$$

and consequently,

$$\begin{aligned}&\frac{d}{dt}\left(e^{\alpha t}\left\|u(t)\right\|^{2}\right)+\left(\frac{\nu\lambda_{1}}{2}-\alpha\right)e^{\alpha t}\left\|u(t)\right\|^{2}\\&\leqslant\frac{2k_{1}^{2}+2k_{2}^{2}\left\|u_{t}\right\|_{C_{H}}^{2}}{\nu}e^{\alpha t}+\frac{2\left|g\left(t\right)\right|^{2}}{\nu}e^{\alpha t}\\&\quad+\frac{2C_{2}e^{\alpha t}}{\nu^{3}}\left|u\left(t\right)\right|^{2}\left\|u(t)\right\|^{4}.\end{aligned}\tag{45}$$

Since $\varepsilon_{1}=\nu\lambda_{1}/4$ and $\alpha+2\varepsilon_{1}+2\varepsilon_{2}-\nu\lambda_{1}+(k_{2}^{2}e^{\alpha h}/2\varepsilon_{1}(1-\rho_{*}))<0$, it is easy to see that $(\nu\lambda_{1}/2)-\alpha>0$. Then

$$\begin{aligned}&\frac{d}{dt}\left(e^{\alpha t}\left\|u\left(t\right)\right\|^{2}\right)\\&\leqslant\frac{2k_{1}^{2}+2k_{2}^{2}\left\|u_{t}\right\|_{C_{H}}^{2}}{\nu}e^{\alpha t}+\frac{2\left|g\left(t\right)\right|^{2}}{\nu}e^{\alpha t}\\&\quad+\frac{2C_{2}}{\nu^{3}}\left|u\left(t\right)\right|^{2}\left\|u(t)\right\|^{2}\left\|u(t)\right\|^{2}e^{\alpha t}.\end{aligned}\tag{46}$$

Let $t\in\mathbb{R}$ be given arbitrarily and taking τ such that $t\geqslant\tau+h+1$. In order to apply Lemma 10, in view of (24), now we firstly obtain

$$\begin{aligned}&\int_{t-1}^{t}\left(\frac{2k_{1}^{2}+2k_{2}^{2}\left\|u_{r}\right\|_{C_{H}}^{2}}{\nu}e^{\alpha r}+\frac{2\left|g\left(r\right)\right|^{2}}{\nu}e^{\alpha r}\right)dr\\&\leqslant\int_{t-1}^{t}\frac{2k_{1}^{2}+2k_{2}^{2}\left\|u_{r}\right\|_{C_{H}}^{2}}{\nu}e^{\alpha r}dr\\&\quad+\int_{t-1}^{t}\frac{2\left|g\left(r\right)\right|^{2}}{\nu}e^{\alpha r}dr\\&\leqslant Ce^{\alpha t}+Ce^{\alpha\tau}\left\|\phi\right\|_{C_{H}}^{2}+C\int_{-\infty}^{t}e^{\alpha r}\left|g\left(r\right)\right|^{2}dr.\end{aligned}\tag{47}$$

Then, it follows from (24) and (25) that

$$\begin{aligned}&\frac{2C_{2}}{\nu^{3}}\int_{t-1}^{t}\left|u\left(r\right)\right|^{2}\left\|u(r)\right\|^{2}dr\\&\leqslant\frac{2C_{2}}{\nu^{3}}\int_{t-1}^{t}\left\|u_{r}\right\|_{C_{H}}^{2}\left\|u(r)\right\|^{2}dr\\&\leqslant\frac{2C_{2}}{\nu^{3}}\int_{t-1}^{t}\left(1+\frac{2k_{2}^{2}e^{\alpha h}}{\nu\lambda_{1}\left(1-\rho_{*}\right)\alpha}\right)e^{\alpha(\tau-r+h)}\\&\quad\times\left\|\phi\right\|_{C_{H}}^{2}\left\|u(r)\right\|^{2}dr\\&\quad+\frac{4C_{2}k_{1}^{2}}{\nu^{4}\lambda_{1}\alpha}\int_{t-1}^{t}\left\|u(r)\right\|^{2}dr\\&\quad+\frac{2C_{2}}{\nu^{3}}\int_{t-1}^{t}\left(\frac{e^{-\alpha(r-h)}}{2\varepsilon_{2}}\left\|u(r)\right\|^{2}\int_{-\infty}^{r}e^{\alpha s}\left|g\left(s\right)\right|^{2}ds\right)dr\\&\leqslant Ce^{2\alpha\tau}e^{-2\alpha t}\left\|\phi\right\|_{C_{H}}^{2}+C\\&\quad+Ce^{-\alpha t}\int_{-\infty}^{t}e^{\alpha r}\left|g\left(r\right)\right|^{2}dr+Ce^{\alpha\tau}e^{-\alpha t}\left\|\phi\right\|_{C_{H}}^{2}\\&\quad+Ce^{\alpha\tau}e^{-2\alpha t}\left\|\phi\right\|_{C_{H}}^{2}\int_{-\infty}^{t}e^{\alpha r}\left|g\left(r\right)\right|^{2}dr\\&\quad+Ce^{-2\alpha t}\left(\int_{-\infty}^{t}e^{\alpha r}\left|g\left(r\right)\right|^{2}dr\right)^{2}.\end{aligned}\tag{48}$$

Combining (25) and (47)-(48) together, by Lemma 10, we can conclude that

$$\left\|u(t)\right\|^{2}\leqslant\left(a_{3}+a_{2}\right)e^{a_{1}}\quad\forall t\geqslant\tau+h+1,\tag{49}$$

where

$$\begin{aligned}
a_3 &= Ce^{\alpha\tau}\|\phi\|^2_{C_H} + Ce^{\alpha t} + C\int_{-\infty}^{t} e^{\alpha r}|g(r)|^2 dr,\\
a_2 &= Ce^{\alpha t} + Ce^{\alpha\tau}\|\phi\|^2_{C_H} + C\int_{-\infty}^{t} e^{\alpha r}|g(r)|^2 dr,\\
a_1 &= Ce^{2\alpha\tau}e^{-2\alpha t}\|\phi\|^2_{C_H} + C\\
&\quad + Ce^{-\alpha t}\int_{-\infty}^{t} e^{\alpha r}|g(r)|^2 dr + Ce^{\alpha\tau}e^{-\alpha t}\|\phi\|^2_{C_H}\\
&\quad + Ce^{\alpha\tau}e^{-2\alpha t}\|\phi\|^2_{C_H}\int_{-\infty}^{t} e^{\alpha r}|g(r)|^2 dr\\
&\quad + Ce^{-2\alpha t}\left(\int_{-\infty}^{t} e^{\alpha r}|g(r)|^2 dr\right)^2.
\end{aligned} \tag{50}$$

Therefore, if we take τ such that $t \geqslant \tau + 1 + 2h$, then similar to the above mentioned, we get

$$\|u_t\|^2_{C_V} = \sup_{\theta\in[-h,0]}\|u(t+\theta)\|^2 \leqslant (a_3 + a_2)e^{a_1}. \tag{51}$$

We denote by $\mathscr{R}$ the set of all functions $r : \mathbb{R} \to (0, +\infty)$ such that

$$\lim_{t\to-\infty} r^2(t) = 0, \tag{52}$$

and denote by $\mathscr{D}_{C_V}$ the class of all families $\mathscr{D} = \{D(t)\}_{t\in\mathbb{R}} \subset \mathscr{P}(C_V)$ such that $D(t) \subset \overline{\mathscr{N}}(0, r_{\mathscr{D}}(t))$, for some $r_{\mathscr{D}} \in \mathscr{R}$, where $\mathscr{P}(C_V)$ denotes the family of all nonempty subsets of C_V and $\overline{\mathscr{N}}(0, r_{\mathscr{D}}(t))$ denotes the closed ball in C_V centered at zero with radius $r_{\mathscr{D}}(t)$.

Denote by $R(t)$ the nonnegative number given for each $t \in \mathbb{R}$ by

$$\begin{aligned}
(R(t))^2 &= \left(Ce^{\alpha t} + C\int_{-\infty}^{t} e^{\alpha r}|g(r)|^2 dr\right)\\
&\quad \times \exp\left(Ce^{-\alpha t}\int_{-\infty}^{t} e^{\alpha r}|g(r)|^2 dr\right.\\
&\quad \left. + Ce^{-2\alpha t}\left(\int_{-\infty}^{t} e^{\alpha r}|g(r)|^2 dr\right)^2\right),
\end{aligned} \tag{53}$$

and consider the family of closed balls $\mathscr{Q} = \{Q(t)\}_{t\in\mathbb{R}}$ in C_V defined by

$$Q(t) = \left\{\psi \in C_V : \|\psi\|_{C_V} \leqslant R(t)\right\}. \tag{54}$$

It is straightforward to check that $\mathscr{Q} \in \mathscr{D}_{C_V}$, and moreover, by (51) and (52), the family of $\mathscr{Q}$ is pullback $\mathscr{D}$-absorbing for the multivalued process $\{U(t,\tau)\}$ on C_V.

The proof of Theorem 11 is completed. □

4. Existence of the Pullback Attractors in C_V

Theorem 12. *Suppose in addition to the hypotheses in Theorem 11 that $g \in C(\mathbb{R}; H)$. Then there exists a unique pullback $\mathscr{D}$-attractor $\{A_{C_V}(t)\}_{t\in R}$ for the multivalued process $\{U(t,\tau)\}$ in C_V.*

Proof. Since A^{-1} is a continuous compact operator in H, by the classical spectral theory, there exist a sequence $\{\lambda_j\}_{j=1}^{\infty}$,

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots, \quad \lambda_j \longrightarrow +\infty, \text{ as } j \longrightarrow +\infty, \tag{55}$$

and a family of elements $\{w_j\}_{j=1}^{\infty}$ of $D(A)$ which are orthonormal in H such that

$$Aw_j = \lambda_j w_j \quad \forall j \in \mathbb{N}. \tag{56}$$

Let $V_m = \operatorname{span}\{w_1, \ldots, w_m\}$ in V and $P_m : V \to V_m$ be an orthogonal projector.

Let $u = u_1 + u_2$, where $u_1 = P_m u$ and $u_2 = (I - P_m)u$. We decompose (9) as follows:

$$\begin{aligned}
&\frac{\partial u_2(t)}{\partial t} + \nu A u_2(t) + B(u(t), u(t)) - P_m B(u_1(t), u_1(t))\\
&\quad = f(t, u(t-\rho(t)))\\
&\qquad - P_m f(t, u_1(t-\rho(t))) + (I - P_m)g(t),\\
&\quad u_2(\tau + t) = (I - P_m)\phi(t), \quad t \in [-h, 0],
\end{aligned} \tag{57}$$

$$\begin{aligned}
&\frac{\partial u_1(t)}{\partial t} + \nu A u_1(t) + P_m B(u_1(t), u_1(t))\\
&\quad = P_m f(t, u_1(t-\rho(t))) + P_m g(t),\\
&\quad u_1(\tau + t) = P_m\phi(t), \quad t \in [-h, 0].
\end{aligned} \tag{58}$$

We divide the proof into three steps.

(1) For every fixed $t \in \mathbb{R}$, any $\mathscr{B} = \{B(t)\}_{t\in\mathbb{R}} \in \mathscr{D}_{C_V}$ and $\varepsilon > 0$, we observe that for any $T \geqslant t - s$ with $s \geqslant 0$,

$$\begin{aligned}
&U(T, t-s)(\phi)\\
&\quad = \{u_T(\cdot; t-s, \phi) \mid u(\cdot) \text{ is a strong solution}\\
&\qquad \text{of (9) with } \phi \in B(t-s)\}.
\end{aligned} \tag{59}$$

Taking the inner product in H of (57) with $Au_2 = A(I - P_m)u$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dT}\|u_2(T)\|^2 + \nu|Au_2(T)|^2\\
&\quad \leqslant |(f(T, u(T-\rho(T))), Au_2(T))|\\
&\qquad + |(P_m f(T, u_1(T-\rho(T))), Au_2(T))|\\
&\qquad + |(B(u(T), u(T)), Au_2(T))|\\
&\qquad + |(P_m B(u_1(T), u_1(T)), Au_2(T))|\\
&\qquad + |((I - P_m)g(T), Au_2(T))|.
\end{aligned} \tag{60}$$

By (H3) and Young's inequality, we have

$$\begin{aligned}
&|(f(T, u(T-\rho(T))), Au_2(T))| + |(g(T), Au_2(T))|\\
&\quad \leqslant \frac{\nu}{8}|Au_2(T)|^2 + C + C\|u_T\|^2_{C_H} + C|g(T)|^2.
\end{aligned} \tag{61}$$

To estimate $(B(u(T),u(T)),Au_2(T))$, we recall some inequalities [19]:

$$|\varphi|_{(L^\infty(\Omega))^2} \leqslant C_3 \|\varphi\| \left(1+\log\frac{|A\varphi|^2}{\lambda_1\|\varphi\|^2}\right)^{1/2} \quad \forall \varphi \in D(A), \tag{62}$$

and thus

$$\begin{aligned} |B(u,v)| &\leqslant C_4 |(u\cdot\nabla)v| \leqslant C_4|u|_{L^\infty(\Omega)}\|v\| \\ &\leqslant C_4C_3\|u\|\|v\|\left(1+\log\frac{|Au|^2}{\lambda_1\|u\|^2}\right)^{1/2}. \end{aligned} \tag{63}$$

Note that $|Au_1|^2 \leqslant \lambda_m\|u_1\|^2$, and set $L = 1+\log(\lambda_{m+1}/\lambda_1)$. Then by Young's inequality, we can deduce that

$$\begin{aligned} &\left|\left(B(u(T),u(T)),Au_2(T)\right)\right| \\ &\quad\leqslant \left|\left(B(u_2(T),u_1(T)+u_2(T)\right),Au_2(T)\right)\right| \\ &\qquad+\left|\left(B(u_1(T),u_1(T)+u_2(T)\right),Au_2(T)\right)\right| \\ &\quad\leqslant C_1|u_2(T)|^{1/2}|Au_2(T)|^{3/2} \\ &\qquad\times\left(\|u_1(T)\|+\|u_2(T)\|\right) \\ &\qquad+C_3C_4L^{1/2}\|u_1(T)\|\,|Au_2(T)| \\ &\qquad\times\left(\|u_1(T)\|+\|u_2(T)\|\right) \\ &\quad\leqslant \frac{\nu}{8}|Au_2(T)|^2+C|u(T)|^2\|u(T)\|^4+C\|u(T)\|^4. \end{aligned} \tag{64}$$

By (60)–(64) and Poincaré inequality, we obtain

$$\begin{aligned} &\frac{d}{dT}\|u_2(T)\|^2+\nu\lambda_{m+1}\|u_2(T)\|^2 \\ &\qquad\leqslant C+C\|u_T\|^2_{C_H}+C|g(T)|^2 \\ &\qquad\quad+C|u(T)|^2\|u(T)\|^4+C\|u(T)\|^4. \end{aligned} \tag{65}$$

Applying the Gronwall's lemma in the interval $[t-s,t+\theta]$, it yields

$$\begin{aligned} &\|u_2(t+\theta)\|^2 \\ &\leqslant \|u_2(t-s)\|^2e^{-\nu\lambda_{m+1}(\theta+s)} \\ &\quad+C\int_{t-s}^{t+\theta}e^{-\nu\lambda_{m+1}(t+\theta-r)} \\ &\qquad\times\left(1+\|u_r\|^2_{C_H}+|g(r)|^2+|u(r)|^2\|u(r)\|^4+\|u(r)\|^4\right)dr. \end{aligned} \tag{66}$$

Let $\varepsilon>0$ be given arbitrarily. Note that $g\in C(\mathbb{R};H)$, then we can take $m+1$ large enough such that for any fixed $\eta>0$,

$$C\int_{t-h-\eta}^{t+\theta}e^{-\nu\lambda_{m+1}(t+\theta-r)}|g(r)|^2dr \leqslant \frac{C}{\nu\lambda_{m+1}} < \frac{\varepsilon}{4}, \tag{67}$$

$$\begin{aligned} &\sup_{\theta\in[-h,0]}C\int_{-\infty}^{t-h-\eta}e^{-\nu\lambda_{m+1}(t+\theta-r)}|g(r)|^2dr \\ &\quad\leqslant C\int_{-\infty}^{t-h-\eta}e^{-\nu\lambda_{m+1}(t-h-r)}|g(r)|^2dr \\ &\quad\leqslant Ce^{-\nu\lambda_{m+1}(t-h)} \\ &\qquad\times\left(\int_{t-h-\eta-1}^{t-h-\eta}e^{\nu\lambda_{m+1}r}|g(r)|^2dr\right. \\ &\qquad\qquad\left.+\int_{t-h-\eta-2}^{t-h-\eta-1}e^{\nu\lambda_{m+1}r}|g(r)|^2dr+\cdots\right) \\ &\quad\leqslant Ce^{-\nu\lambda_{m+1}(t-h)} \\ &\qquad\times\left(e^{(\nu\lambda_{m+1}-\alpha)(t-h-\eta)}+e^{(\nu\lambda_{m+1}-\alpha)(t-h-\eta-1)}+\cdots\right) \\ &\qquad\times\int_{-\infty}^{t}e^{\alpha r}|g(r)|^2dr \\ &\quad\leqslant \frac{Ce^{-\nu\lambda_{m+1}\eta}e^{-\alpha(t-h-\eta)}}{1-e^{-(\nu\lambda_{m+1}-\alpha)}}\int_{-\infty}^{t}e^{\alpha r}|g(r)|^2dr \\ &\quad< \frac{\varepsilon}{4}. \end{aligned} \tag{68}$$

Combining (67) and (68) together, we can get for $m+1$ large enough,

$$\sup_{\theta\in[-h,0]}C\int_{-\infty}^{t+\theta}e^{-\nu\lambda_{m+1}(t+\theta-r)}|g(r)|^2dr < \frac{\varepsilon}{2}. \tag{69}$$

On the other hand, thanks to Lemma 9 and Theorem 11, we can deduce that when $m+1$ and s are large enough,

$$\begin{aligned} &\sup_{\theta\in[-h,0]}\|u_2(t-s)\|^2e^{-\nu\lambda_{m+1}(\theta+s)} \\ &\qquad\leqslant \|u_2(t-s)\|^2e^{-\nu\lambda_{m+1}(s-h)} < \frac{\varepsilon}{4}, \\ &\sup_{\theta\in[-h,0]}C\int_{t-s}^{t+\theta}e^{-\nu\lambda_{m+1}(t+\theta-r)} \\ &\qquad\qquad\times\left(C+\|u_r\|^2_{C_H}+|u(r)|^2\|u(r)\|^4+\|u(r)\|^4\right)dr \\ &\qquad< \frac{\varepsilon}{4}. \end{aligned} \tag{70}$$

Thanks to (69) and (70), it follows from (66) that when $m+1$ and s are large enough,

$$\begin{aligned}
&\|u_{2t}\|^2_{C_V} \\
&= \sup_{\theta\in[-h,0]} \|u_2(t+\theta)\|^2 \\
&\leqslant \sup_{\theta\in[-h,0]} \|u_2(t-s)\|^2 e^{-\nu\lambda_{m+1}(\theta+s)} \\
&\quad + \sup_{\theta\in[-h,0]} C\int_{t-s}^{t+\theta} e^{-\nu\lambda_{m+1}(t+\theta-r)} \\
&\qquad \times \left(C + \|u_r\|^2_{C_H} + |g(r)|^2 + |u(r)|^2\|u(r)\|^4 \right. \\
&\qquad \left. + \|u(r)\|^4\right) dr \\
&< \varepsilon.
\end{aligned} \tag{71}$$

(2) Now we consider the ordinary functional differential system (58) and check the condition (2) in Theorem 7. Note that $|Au_1|^2 \leqslant \lambda_m \| u_1 \|^2 \leqslant \lambda_m^2 |u_1|^2$. Without generality, we assume that $\theta_1, \theta_2 \in [-h, 0]$ with $0 < \theta_1 - \theta_2 < 1$. Hence

$$\begin{aligned}
&\|u_1(t+\theta_1) - u_1(t+\theta_2)\| \\
&\leqslant \sqrt{\lambda_m}\,|u_1(t+\theta_1) - u_1(t+\theta_2)| \\
&\leqslant \sqrt{\lambda_m}\int_{t+\theta_2}^{t+\theta_1} \left|\frac{du_1(T)}{dT}\right| dT \\
&\leqslant \sqrt{\lambda_m}\int_{t+\theta_2}^{t+\theta_1} \left(\nu\,|Au_1(T)| + |B(u_1(T), u_1(T))| \right. \\
&\qquad \left. + |f(T, u_1(T-\rho(T)))| + |P_m g(T)|\right) dT.
\end{aligned} \tag{72}$$

Notice that

$$|B(u_1,u_1)| \leqslant C\,|Au_1|\,\|u_1\| \leqslant C\sqrt{\lambda_m}\|u_1\|^2 \leqslant C\lambda_m^{3/2}|u_1|^2. \tag{73}$$

Then, it follows from (H3), (H4), and (24) that

$$\begin{aligned}
&\int_{t+\theta_2}^{t+\theta_1} \left(|Au_1(T)| + |B(u_1(T), u_1(T))|\right) dT \\
&\quad \leqslant C\int_{t+\theta_2}^{t+\theta_1} |u_1(T)|\,dT + C\int_{t+\theta_2}^{t+\theta_1} |u_1(T)|^2 dT \\
&\quad \leqslant C\int_{t+\theta_2}^{t+\theta_1} |u_1(T)|^2 dT + C(\theta_1 - \theta_2) \\
&\quad \leqslant C\left(e^{-\alpha\theta_2} - e^{-\alpha\theta_1}\right) + C(\theta_1 - \theta_2) \\
&\qquad + C\left(e^{-\alpha\theta_2} - e^{-\alpha\theta_1}\right) e^{-\alpha t}\int_{-\infty}^{t} e^{\alpha r}|g(r)|^2 dr, \\
&\int_{t+\theta_2}^{t+\theta_1} |f(T, u_1(T-\rho(T)))|\,dT \\
&\quad \leqslant \int_{t+\theta_2}^{t+\theta_1} \left(|f(T, u_1(T-\rho(T)))|^2 + C\right) dT \\
&\quad \leqslant \int_{t+\theta_2}^{t+\theta_1} \left(k_2^2\|u_{1T}\|^2_{C_H} + C\right) dT \\
&\quad \leqslant C\left(e^{-\alpha\theta_2} - e^{-\alpha\theta_1}\right) + C(\theta_1 - \theta_2) \\
&\qquad + C\left(e^{-\alpha\theta_2} - e^{-\alpha\theta_1}\right) e^{-\alpha t}\int_{-\infty}^{t} e^{\alpha r}|g(r)|^2 dr.
\end{aligned} \tag{74}$$

Since $g \in C(\mathbb{R}; H)$ and t is fixed,

$$\int_{t+\theta_2}^{t+\theta_1} |P_m g(T)|\,dT \leqslant C(\theta_1 - \theta_2). \tag{75}$$

Equations (74)–(75) imply that the condition (2) in Theorem 7 is proved.

(3) Invoking Theorem 7, in view of the previous arguments and Theorem 11, we can see that the multivalued process $\{U(t,\tau)\}$ is pullback $\mathscr{D}$-limit-set compact and pullback $\mathscr{D}$-dissipative in C_V.

In order to get the existence of pullback $\mathscr{D}$-attractors, by the proof of Theorem 3.2 in [21], now we only need to show the negative invariance of $\{A_{C_V}(t)\}_{t\in\mathbb{R}}$, where

$$\begin{aligned}
A_{C_V}(t) &= \omega_t(\mathcal{Q}) \\
&= \bigcap_{T\in\mathbb{R}^+} \overline{\bigcup_{s\geqslant T} U(t, t-s)Q(t-s)}, \quad \forall t \in \mathbb{R},
\end{aligned} \tag{76}$$

and $\mathcal{Q} = \{Q(t)\}_{t\in\mathbb{R}} \in \mathscr{D}_{C_V}$ is a pullback $\mathscr{D}$-absorbing set of $\{U(t,\tau)\}$ in C_V.

Let $y \in A_{C_V}(t)$. Then there exist sequences $s_n \in \mathbb{R}^+$, $s_n \to +\infty$ $(n \to \infty)$, $x_n \in Q(t-s_n)$, and $y_n \in U(t, t-s_n)x_n$ such that

$$y_n \longrightarrow y \quad \text{in } C_V \text{ as } n \longrightarrow \infty. \tag{77}$$

On the other hand, for n sufficiently large,

$$y_n \in U(t, t-s_n)x_n = U(t,\tau)U(\tau, t-s_n)x_n. \tag{78}$$

Then by the pullback $\mathscr{D}$-limit-set compactness of the multivalued process $\{U(t,\tau)\}$, there is a subsequence of $\widetilde{x}_n \in U(\tau, t-s_n)x_n = U(\tau, \tau-(\tau+s_n-t))x_n$, which we still relabel as $\widetilde{x}_n$ such that $y_n \in U(t,\tau)\widetilde{x}_n$ and

$$\widetilde{x}_n \longrightarrow x \quad \text{in } C_V \text{ as } n \longrightarrow \infty. \tag{79}$$

Clearly, $x \in A_{C_V}(\tau)$.

We observe that y_n is bounded in C_V for n sufficiently large. Then by slightly modifying the proof of the existence of solutions (see [16] for details), in view of Theorem 2.11 in [21], we can see that

$$y_n(\cdot) \rightharpoonup u(\cdot + t, \tau, x) \quad \text{in } L^2([-h, 0]; V). \tag{80}$$

This together with (77)–(79), we can deduce that $y \in U(t,\tau)x \subset U(t,\tau)A_{C_V}(\tau)$, and thus the proof of Theorem 12 is finished. □

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