Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 421328, 10 pages
http://dx.doi.org/10.1155/2013/421328

Research Article

Positive Interpolation Operators with Exponential-Type Weights

Hee Sun Jung[1] and Ryozi Sakai[2]

[1] Department of Mathematics Education, Sungkyunkwan University, Seoul 110-745, Republic of Korea
[2] Department of Mathematics, Meijo University, Nagoya 468-8502, Japan

Correspondence should be addressed to Hee Sun Jung; hsun90@skku.edu

Received 26 December 2012; Accepted 7 March 2013

Academic Editor: Roberto Barrio

We consider positive operators on the real line $\mathbb{R}$ with property of interpolation, and we show the weighted L_p-convergence of the operators. We will construct an analogical operator of one which is studied by Knopfmacher (1986). Furthermore, we treat the Shepard-type interpolatory operator (cf. Xie et al. (1998)).

1. Introduction

In this paper, we consider two interpolatory positive operators. For $\gamma > 1$ and $-\infty < x_{n,n} < \cdots < x_{1,n} < \infty$, we construct an operator

$$\mathscr{F}_{n,\gamma}[f](x) = \frac{\sum_{k=1}^{n} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} f(x_{k,n}) |K_n(x, x_{k,n})|^{\gamma}}{\sum_{k=1}^{n} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} |K_n(x, x_{k,n})|^{\gamma}}. \tag{1}$$

The details will be stated later, and the result is written in Section 2. Knopfmacher [1] studied the positive operator

$$F_{n,\gamma}[f](x) = \frac{\sum_{k=1}^{n} \lambda_{kn} f(x_{k,n}) |K_n(x, x_{k,n})|^{\gamma}}{\sum_{k=1}^{n} \lambda_{kn} |K_n(x, x_{k,n})|^{\gamma}}, \tag{2}$$

and for $1 < \gamma \leqslant 2$, he obtained a certain weighted-convergence theorem on the compact interval $I \subset \mathbb{R} = (-\infty, \infty)$. The operators (1) and (2) have the property of Hermite-Fejér interpolation, that is,

$$H_n[f](x_{k,n}) = f(x_{k,n}), \qquad H_n[f]'(x_{k,n}) = 0, \quad k = 1, 2, \ldots, n. \tag{3}$$

We also treat the interpolatory positive operator of Shepard-type. Let us define $S_{n,\lambda}(f;x)$ for $f \in C(\mathbb{R})$ by

$$S_{n,\lambda}(f;x) := \frac{\sum_{j=1}^{n} f(x_{j,n}) \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-\lambda}}{\sum_{j=1}^{n} \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-\lambda}}, \quad \lambda \geqslant 1, \; x \in \mathbb{R}. \tag{4}$$

The operator $S_{n,\lambda}(f;x)$ is linear and positive, furthermore it interpolates $f(x)$ at the zeros $\{x_{i,n}\}_{i=1}^{n}$. In fact, we see that

$$S_{n,\lambda}(f;x_{k,n}) = \lim_{\substack{x \neq x_{k,n},\\ x \to x_{k,n}}} \Bigg(f(x_{k,n}) \Phi_n^{(\lambda-1)/2}(x_{k,n}) + \sum_{j \neq k} f(x_{j,n}) \Phi_n^{(\lambda-1)/2}(x_{j,n}) \times |x - x_{j,n}|^{-\lambda} |x - x_{k,n}|^{\lambda} \Bigg)$$

$$\times\left(\Phi_n^{(\lambda-1)/2}(x_{k,n})+\sum_{j\neq k}\Phi_n^{(\lambda-1)/2}(x_{j,n})\times\left|x-x_{j,n}\right|^{-\lambda}\left|x-x_{k,n}\right|^{\lambda}\right)^{-1}$$
$$= f(x_{k,n}),\quad k=1,2,\ldots,n. \tag{5}$$

The related theorem is written in Section 4.

First we need the following definition from [2]. We say that $f:\mathbb{R}\to[0,\infty)$ is quasi-increasing (quasi-decreasing) if there exists $C>0$ such that $f(x)\leqslant Cf(y)(f(x)\geqslant Cf(y))$, $0<x<y$.

Definition 1. Let $Q:\mathbb{R}\to[0,\infty)$ be an even function and satisfying the following properties.

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0)=0$.

(b) $Q''(x)$ exists and is positive in $\mathbb{R}\setminus\{0\}$.

(c) $\lim_{x\to\infty}Q(x)=\infty$.

(d) The function

$$T(x):=\frac{xQ'(x)}{Q(x)},\quad x\neq 0 \tag{6}$$

is quasi-increasing in $(0,\infty)$, with

$$T(x)\geqslant\Lambda>1,\quad x\in\mathbb{R}^{+}\setminus\{0\}. \tag{7}$$

(e) There exists $C_1>0$ such that

$$\frac{Q''(x)}{\left|Q'(x)\right|}\leqslant C_1\frac{\left|Q'(x)\right|}{Q(x)},\quad \text{a.e. } x\in\mathbb{R}\setminus\{0\}. \tag{8}$$

Then, we write $w(x)=\exp(-Q(x))\in\mathcal{F}(C^2)$. If there also exist a compact subinterval $J(\ni 0)$ of $\mathbb{R}$ and $C_2>0$ such that

$$\frac{Q''(x)}{\left|Q'(x)\right|}\geqslant C_2\frac{\left|Q'(x)\right|}{Q(x)},\quad \text{a.e. } x\in\mathbb{R}\setminus J, \tag{9}$$

then we write $w(x)=\exp(-Q(x))\in\mathcal{F}(C^2+)$.

Example 2. There are some typical examples of $Q(x)$ satisfying $w=\exp(-Q)\in\mathcal{F}(C^2+)$.

(1) If $T(x)$ is bounded, then the weight $w=\exp(-Q)$ is the so-called the Freud-type weight. Then the typical Freud-type example would be

$$Q(x)=|x|^{\alpha},\quad \alpha>1. \tag{10}$$

(2) If $T(x)$ is unbounded, then the weight $w=\exp(-Q)$ is called the Erdös-type weight. Erdös-type examples $w=\exp(-Q)\in\mathcal{F}(C^2+)$ are as follows.

(a) (see [2, Example 1.2], [3, Theorem 3.1]) For $\alpha>1$, $l=1,2,3,\ldots$.

$$Q(x)=Q_{l,\alpha}(x)=\exp_l\left(|x|^{\alpha}\right)-\exp_l(0), \tag{11}$$

where

$$\exp_l(x)=\exp\left(\exp\left(\exp\cdots\exp x\right)\cdots\right)\quad (l\text{-times}). \tag{12}$$

More precisely, we define for $\alpha+m>1$, $m\geqslant 0$, $l\geqslant 1$ and $\alpha\geqslant 0$,

$$Q_{l,\alpha,m}(x):=|x|^{m}\left(\exp_l\left(|x|^{\alpha}\right)-\alpha^{*}\exp_l(0)\right), \tag{13}$$

where $\alpha^*=0$ if $\alpha=0$, otherwise $\alpha^*=1$ (but, note that $Q_{l,0,m}$ gives a Freud-type weight).

(b) (see [3, Theorem 3.5]) For $\alpha>1$, put $Q_\alpha(x):=(1+|x|)^{|x|^{\alpha}}-1$, $\alpha>1$.

We construct the orthonormal polynomials $p_n(x)=p_n(w^2,x)$ of degree n for $w^2(x)$, that is,

$$\int_{-\infty}^{\infty}p_n\left(w^2,x\right)p_m\left(w^2,x\right)w^2(x)\,dx = \delta_{mn}\quad (\text{Kronecker delta}). \tag{14}$$

Let $fw\in L_p(\mathbb{R})$. The Fourier-type series of f is defined by

$$\tilde{f}(x):=\sum_{k=0}^{\infty}a_k\left(w^2,f\right)p_k\left(w^2,x\right),\qquad a_k\left(w^2,f\right):=\int_{-\infty}^{\infty}f(t)\,p_k\left(w^2,t\right)w^2(t)\,dt. \tag{15}$$

We denote the partial sum of $\tilde{f}(x)$ by

$$s_n(f,x):=s_n\left(w^2,f,x\right):=\sum_{k=0}^{n-1}a_k\left(w^2,f\right)p_k\left(w^2,x\right). \tag{16}$$

If we use the Christoffel-Darboux formula, then we obtain

$$s_n(f,x)=\int_{-\infty}^{\infty}K_n(x,t)\,f(t)\,w^2(t)\,dt. \tag{17}$$

Here,

$$K_n(x,t):=\sum_{k=0}^{n-1}p_k(x)\,p_k(t) = \frac{\gamma_{n-1}}{\gamma_n}\frac{p_n(x)\,p_{n-1}(t)-p_n(t)\,p_{n-1}(x)}{x-t}, \tag{18}$$

where $p_n(x) = \gamma_n x^n + \cdots$. The polynomials of degree $\leqslant n$ are denoted by $\mathscr{P}_n$. We define the Christoffel numbers $\lambda_n(w;x)$ by

$$\lambda_n(w;x) := \inf_{P\in\mathscr{P}_{n-1}} \int_{-\infty}^{\infty} \frac{|Pw|^2(t)\,dt}{|P(x)|^2}, \quad (19)$$

then we have

$$\lambda_n(w;x) = \frac{1}{K_n(x,x)} = \frac{1}{\sum_{k=0}^{n-1} p_k^2(w^2,x)}. \quad (20)$$

We denote the zeros of the orthonormal polynomial $p_n(w^2,x)$ by $x_{n,n} < x_{n-1,n} < \cdots < x_{1,n}$. Then we define the Christoffel numbers $\lambda_{k,n}$, $k = 1,2,\ldots,n$ such as $\lambda_{k,n} := \lambda_n(w, x_{k,n})$.

2. Preliminaries and Theorems

We need the Mhaskar-Rakhmanov-Saff number a_x;

$$x = \frac{2}{\pi}\int_0^1 \frac{a_x t Q'(a_x t)}{(1-t^2)^{1/2}}\,dt, \quad x > 0. \quad (21)$$

We define

$$\varphi_u(x) = \begin{cases} \dfrac{a_u}{u}\dfrac{1-(|x|/a_{2u})}{\sqrt{1-(|x|/a_u)+\delta_u}}, & |x| \leqslant a_u; \\ \varphi_u(a_u), & a_u < |x|, \end{cases} \quad (22)$$

$$\Phi_n(x) = \begin{cases} 1 - \dfrac{|x|}{a_n} + \delta_n, & |x| \leqslant a_n; \\ \delta_n, & a_n < |x|, \end{cases} \quad (23)$$

where

$$\delta_u = \{uT(a_u)\}^{-2/3} \quad u > 0. \quad (24)$$

Moreover, we define a function $\psi_n(x)$ for $\gamma > 1$ and $x \in \mathbb{R}$

$$\psi_n(x) = \begin{cases} a_n^{2-\gamma}\varphi_n^{\gamma-1}(x), & 1 < \gamma < 2; \\ \varphi_n(x)\log a_n, & \gamma = 2; \\ \varphi_n(x), & \gamma > 2, \end{cases} \quad (25)$$

$$\psi_n^* := \begin{cases} a_n^{2-\gamma}\phi_n^{\gamma-1}, & 1 < \gamma < 2; \\ \phi_n \log n, & \gamma = 2; \\ \phi_n, & 2 < \gamma, \end{cases} \quad (26)$$

where $\phi_n := \max\{a_n/n, a_n\delta_n\}$. For the Freud-type weight w we suppose to hold $\psi_n^* \to 0$ as $n \to \infty$. If $w \in \mathscr{F}(C^2+)$ is the Erdös-type weight, then it always holds. So for the Freud-type weight we need to limit slightly the weights.

To state our main result, we assume some conditions for $h(x)$ as follows.

(1) $h(x)$ is even, positive, and quasi-decreasing on $[0,\infty)$.

(2) $h(x_{k,n}) \sim h(x_{k+1,n})$ for $k = 1,2,\ldots,n$ and $n = 1,2,\ldots$

(3) $h(x)\Phi_n^{-\gamma/4}(x)$ is bounded on $\mathbb{R}$ for $n = 1,2,\ldots$.

Let $\{x_{j,n}\}_{j=1}^n$ be the zeros of the orthonormal polynomial $p_n(w^2,x)$. Then we define the operator $\mathscr{F}_{n,\gamma}[f](x)$ by (1) with $\gamma > 1$, $h(x)$, and for each $f \in C(\mathbb{R})$ we define a pointwise modulus of continuity $\omega_x(f;t) = \sup_{\{y;|x-y|\leqslant t, y\in\mathbb{R}\}}|f(x) - f(y)|$. When $f \in C(\mathbb{R})$ is uniformly continuous on $\mathbb{R}$, we set

$$\Omega(f;t) = \sup_{x\in\mathbb{R}} \omega_x(f;t). \quad (27)$$

Then our first theorem is as follows.

Theorem 3. *Let $w \in \mathscr{F}(C^2+)$, and let $\psi_n^* \to 0$ as $n \to \infty$. Let $\gamma > 1$. Then we have the following.*

(a) *For $x_{n,n} \leqslant x \leqslant x_{1,n}$,*

$$\left|\mathscr{F}_{n,\gamma}[f](x) - f(x)\right| \leqslant C\omega_x(f;\psi_n(x))h^{-1}(x)\Phi_n^{-\gamma/4}(x), \quad (28)$$

and for $|x| > x_{1,n}$

$$\left|\mathscr{F}_{n,\gamma}[f](x) - f(x)\right| \leqslant C\omega_x(f;\psi_n(x))|x|\psi_n^{-1}(x). \quad (29)$$

(b) *Let $0 < p \leqslant \infty$ and w^* be an integrable function satisfying the following condition:*

$$\left\|w^*(x)h^{-1}(x)\Phi_n^{-\gamma/4}(x)\right\|_{L_p([x_{n,n},x_{1,n}])} + \left\|x\psi_n^{-1}(x)w^*(x)\right\|_{L_p(|x|\geqslant x_{1,n})} < \infty. \quad (30)$$

Then one has for $f(x)$ being uniformly continuous and bounded on $\mathbb{R}$

$$\left\|w^*\{\mathscr{F}_{n,\gamma}[f] - f\}\right\|_{L_p(\mathbb{R})} = O(1)\Omega(f;\psi_n^*), \quad (31)$$

where ψ_n^* are defined in (26).

We prepare some lemmas for the proof of the theorem.

Lemma 4. *Let $w = \exp(-Q) \in \mathscr{F}(C^2)$.*

(1) *(see [2, Lemma 3.5 (3.27)–(3.29)]) For fixed $L > 0$ and uniformly for $t > 0$,*

$$a_{Lt} \sim a_t, \qquad T(a_{Lt}) \sim T(a_t), \\ Q^{(j)}(a_{Lt}) \sim Q^{(j)}(a_t), \quad j = 0,1. \quad (32)$$

Moreover,

$$T(a_{Lt}) \sim T(a_t). \quad (33)$$

(2) *(see [2, Lemma 3.4 (3.18),(3.17), Lemma 3.8 (3.42)])*

$$Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}}, \qquad Q'(a_t) \sim \frac{t\sqrt{T(a_t)}}{a_t}, \quad (34)$$

and for $x \in [0, a_n/2]$,

$$Q'(x) \sim \frac{n}{a_n}\left(\frac{x}{a_n}\right)^{\Lambda-1}, \tag{35}$$

where $\Lambda > 1$ *is defined in Definition 1(d).*

(3) *(see [2, Lemma 3.11 (a), (b)]) Given fixed* $0 < \alpha$, *one has uniformly for* $t > 0$,

$$\left|1 - \frac{a_{\alpha t}}{a_t}\right| \sim \frac{1}{T(a_t)}. \tag{36}$$

(4) *(see [2, Lemma 3.7 (3.38)]) For some* $0 < \varepsilon \leqslant 2$, *and for large enough* t,

$$T(a_t) \leqslant t^{2-\varepsilon}. \tag{37}$$

(5) *(see [2, Lemma 3.8 (a)]) For* $x \in [0, a_t)$,

$$Q'(x) \leqslant C\frac{t}{a_t}\frac{1}{\sqrt{1-(x/a_t)}}. \tag{38}$$

Lemma 5 ([4, Theorem 2.7]). *There exists* $C > 0$ *such that*

$$\sup_{x\in\mathbb{R}}\left|p_n(x)\,w(x)\,\Phi_n^{1/4}(x)\right| \leqslant Ca_n^{-1/2}. \tag{39}$$

Lemma 6. *Let* $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$.

(1) *Let* $x_{j,n}$ *be the zero of* $p_n(x)$. *Then for* $n \geqslant 1$ *and* $1 \leqslant j \leqslant n-1$,

$$x_{j,n} - x_{j+1,n} \sim \varphi_n(x_{j,n}), \tag{40}$$

$$\varphi_n(x_{j,n}) \sim \varphi_n(x_{j+1,n}). \tag{41}$$

(2) *For* $n \geqslant 1$ *and* $1 \leqslant j \leqslant n-1$,

$$\Phi_n(x_{j,n}) \sim \Phi_n(x_{j+1,n}). \tag{42}$$

Proof. (1) This follows from [2, Corollary 13.4, Theorem 5.7 (b)].

(2) Recall the definition of $\Phi_n(x)$ in (21). We have

$$\begin{aligned}\Phi_n(x_{j,n}) &= 1 - \frac{|x_{j,n}|}{a_n} + \delta_n \\ &= 1 - \frac{|x_{j+1,n}|}{a_n} \\ &\quad + \delta_n - \frac{x_{j,n} - x_{j+1,n}}{a_n} \\ &\quad \sim 1 - \frac{|x_{j+1,n}|}{a_n} + \delta_n - \frac{\varphi_n(x_{j,n})}{a_n} \\ &= 1 - \frac{|x_{j+1,n}|}{a_n} + \delta_n \\ &\quad - \frac{1}{n}\frac{1 - |x_{j,n}/a_{2n}|}{\sqrt{1 - |x_{j,n}|/a_n}}.\end{aligned} \tag{43}$$

Hence, if $|x_{k,n}|, |x_{k+1,n}| \leqslant a_{n/2}$, then we see

$$\Phi_n(x_{j,n}) \sim 1 - \frac{|x_{j+1,n}|}{a_n} + \delta_n - \frac{1}{n}\sqrt{\frac{1-|x_{j,n}|}{a_n} + \delta_n}. \tag{44}$$

Here we see

$$\begin{aligned}&\Phi_n(x_{j,n}) + \frac{C}{n}\sqrt{\Phi_n(x_{j,n})} \\ &\quad = \sqrt{\Phi_n(x_{j,n})}\left\{\sqrt{\Phi_n(x_{j,n})} + \frac{C}{n}\right\} \sim \Phi_n(x_{j,n}),\end{aligned} \tag{45}$$

because of $\sqrt{\Phi_n(x_{j,n})} \geqslant C/\sqrt{T(a_n)} > 1/n^{1-\varepsilon} > (1/n)(\varepsilon > 0)$. (see Lemma 4 (3), (4)). Therefore, we have

$$\Phi_n(x_{j,n}) \sim \Phi_n(x_{j+1,n}). \tag{46}$$

Let $a_{n/2} < |x_{j,n}|$. Then we see

$$\frac{1}{n}\frac{1 - |x_{j,n}/a_{2n}|}{\sqrt{1 - |x_{j,n}|/a_n + \delta_n}} \leqslant C\frac{1}{n}\frac{(nT(a_n))^{1/3}}{T(a_n)} \sim \delta_n. \tag{47}$$

Therefore we see

$$\Phi_n(x_{j,n}) \sim 1 - \frac{|x_{j+1,n}|}{a_n} + \delta_n = \Phi_n(x_{j+1,n}). \tag{48}$$

□

Lemma 7 ([2, Theorem 13.3 (13.9)]). *If* $x \in [x_{k+1,n}, x_{k,n}]$, *then*

$$(l_{k,n}w)(x)\,w^{-1}(x_{k,n}) + (l_{k+1,n}w)(x)\,w^{-1}(x_{k+,n}) \sim 1. \tag{49}$$

Lemma 8. *Let* $w = \exp(-Q) \in \mathcal{F}(C^2+)$. *Then the following results hold.*

(*a*) *For* $|x| \leqslant a_n(1+\delta_n)$,

$$K_n(x,x) = \sum_{k=0}^{n-1} p_k^2(w^2,x) = \lambda_{n,2}^{-1}(w;x) \sim \varphi_n^{-1}(x)\,w^{-2}(x). \quad (50)$$

(*b*) *For* $x \in \mathbb{R}$

$$K_n(x,x) \leqslant C\varphi_n^{-1}(x)\,w^{-2}(x). \quad (51)$$

Proof. From [2, Theorem 9.3], we have the following.

(1) Uniformly for $n \geqslant 1$ and $|x| \leqslant a_n(1+\eta_n)$, we have

$$\lambda_n(w;x) \sim \varphi_n(x)\,w^2(x). \quad (52)$$

(2) Moreover, uniformly for $n \geqslant 1$ and $x \in \mathbb{R}$,

$$\lambda_n(w;x) \geqslant C\varphi_n(x)\,w^2(x). \quad (53)$$

Since $K_n(x,x) = 1/\lambda_n(w;x)$, we have the following results. □

3. Proof of Theorem 3

To estimate the difference $|\mathcal{F}_{n,\gamma}[f](x) - f(x)|$, we split $\sum_{k=1}^{n}$ into two parts.

To prove the theorem we start the estimation of the denominator for the operator $\mathcal{F}_{n,\gamma}$. We will need it in Step 4.

Step 1. Let

$$H_{n,\gamma}(x) := \sum_{k=1}^{n} h(x_{k,n})\,w^{\gamma-2}(x_{k,n})\,\lambda_{kn}\left|K_n(x,x_{k,n})\right|^{\gamma}. \quad (54)$$

Then we have the following.

Lemma 9. *There exists $C > 0$ such that uniformly, for $x \in [x_{n,n}, x_{1,n}]$,*

$$H_{n,\gamma}(x) \geqslant Ch(x)\,\varphi_n^{1-\gamma}(x)\,w^{-\gamma}(x). \quad (55)$$

Proof. By Lemma 7, if $x \in [x_{k+1,n}, x_{k,n}]$, $x_{k+1,n} \geqslant 0$, $k \geqslant 1$, then

$$(l_{k,n}w)(x)\,w^{-1}(x_{k,n}) + (l_{k+1,n}w)(x)\,w^{-1}(x_{k+1,n}) \sim 1. \quad (56)$$

Since $l_{k,n}(x) = \lambda_{k,n}K_n(x,x_{k,n})$, we see

$$\lambda_{k,n}\frac{w(x)}{w(x_{k,n})}K_n(x,x_{k,n}) + \lambda_{k+1,n}\frac{w(x)}{w(x_{k+1,n})}K_n(x,x_{k+1,n}) \sim 1, \quad (57)$$

and this implies that

$$\begin{aligned} 0 &< C \\ &\leqslant \left(\lambda_{kn}\frac{w(x)}{w(x_{k,n})}\left|K_n(x,x_{k,n})\right| + \lambda_{k+1,n}\frac{w(x)}{w(x_{k+1,n})}\left|K_n(x,x_{k+1,n})\right|\right) \\ &= \left(\frac{\lambda_{kn}w(x)}{w^2(x_{k,n})}w(x_{k,n})\left|K_n(x,x_{k,n})\right| + \frac{\lambda_{k+1,n}w(x)}{w^2(x_{k+1,n})}w(x_{k+1,n})\left|K_n(x,x_{k+1,n})\right|\right) \\ &\sim \frac{\lambda_{kn}w(x)}{w^2(x_{k,n})}\left(w(x_{k,n})\left|K_n(x,x_{k,n})\right| + w(x_{k+1,n})\left|K_n(x,x_{k+1,n})\right|\right) \\ &\leqslant C\frac{\lambda_{kn}w(x)}{w^2(x_{k,n})}\left(\left|K_n(x,x_{k,n})\,w(x_{k,n})\right|^{\gamma} + \left|K_n(x,x_{k+1,n})\,w(x_{k+1,n})\right|^{\gamma}\right)^{1/\gamma}. \end{aligned} \quad (58)$$

Therefore, from (41) and (52) we can obtain

$$\begin{aligned} &\varphi_n^{1-\gamma}(x)\,w^{-\gamma}(x) \\ &\sim \left(\frac{\lambda_{kn}}{w^2(x_{k,n})}\right)^{1-\gamma} w^{-\gamma}(x) \\ &\leqslant C\frac{\lambda_{kn}}{w^2(x_{k,n})}\left(\left|K_n(x,x_{k,n})\,w(x_{k,n})\right|^{\gamma} + \left|K_n(x,x_{k+1,n})\,w(x_{k+1,n})\right|^{\gamma}\right). \end{aligned} \quad (59)$$

Using the fact $h(x_{k,n}) \sim h(x) \sim h(x_{k+1,n})$ (see the definition of $h(x)$), we have by (41) and (52)

$$\begin{aligned} &H_{n,\gamma}(x) \\ &\geqslant h(x_{k,n})\,\lambda_{k,n}w^{\gamma-2}(x_{k,n})\left|K_n(x,x_{k,n})\right|^{\gamma} + h(x_{k+1,n})\,\lambda_{k+1,n}w^{\gamma-2}(x_{k+1,n})\left|K_n(x,x_{k+1,n})\right|^{\gamma} \\ &\geqslant Ch(x)\frac{\lambda_{k,n}}{w^2(x_{k,n})}\left(\left|K_n(x,x_{k,n})\,w(x_{k,n})\right|^{\gamma} + \left|K_n(x,x_{k+1,n})\,w(x_{k+1,n})\right|^{\gamma}\right) \\ &\geqslant Ch(x)\,\varphi_n^{1-\gamma}(x)\,w^{-\gamma}(x). \end{aligned} \quad (60)$$

In another case, that is, when $x_{k+1,n} < 0$, we also have the same result. □

Step 2. Let $|x - x_{k,n}| \leqslant \varphi_n(x)$. Let $f(x)$ be uniformly continuous and bounded on $\mathbb{R}$, and let $\gamma > 1$. Then we have

$$\left|f(x) - f(x_{k,n})\right| \leqslant \omega_x(f;\varphi_n(x)). \quad (61)$$

Now, let

$$\sum_1 := \frac{1}{H_{n,\gamma}(x)} \sum_{|x-x_{k,n}|\leqslant\varphi_n(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} \times |f(x) - f(x_{k,n})| |K_n(x, x_{k,n})|^{\gamma}. \quad (62)$$

We have the following estimation.

Lemma 10. *For $x \in \mathbb{R}$,*

$$\sum_1 \leqslant \omega_x(f; \varphi_n(x)). \quad (63)$$

Proof. By (61),

$$\sum_1 \leqslant \omega_x(f; \varphi_n(x)) \frac{1}{H_{n,\gamma}(x)} \times \sum_{|x-x_{k,n}|\leqslant\varphi_n(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} |K_n(x, x_{k,n})|^{\gamma} \leqslant \omega_x(f; \varphi_n(x)), \quad (64)$$

because we know from the definition of $H_{n,\gamma}(x)$ in (54) that

$$\sum_{|x-x_{k,n}|\leqslant\varphi_n(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} |K_n(x, x_{k,n})|^{\gamma} \leqslant H_{n,\gamma}(x). \quad (65)$$

□

Step 3. Next, we estimate $\sum_{|x-x_{k,n}|>\varphi_n(x)}$. Let $|x - x_{k,n}| > \varphi_n(x)$ and let $|x - x_{m,n}| = \min\{|x - x_{k,n}|, k = 1, 2, \ldots, n\}$. To do so, we prepare the following. By Lemma 6,

$$|K_n(x, x_{k,n})| \leqslant C a_n \frac{|p_n(x) p_{n-1}(x_{k,n})|}{|x - x_{k,n}|} \leqslant C w^{-1}(x) w^{-1}(x_{k,n}) \Phi_n^{-1/4}(x) \times \Phi_n^{-1/4}(x_{k,n}) \frac{1}{|x - x_{k,n}|}. \quad (66)$$

From the property of the modulus of continuity we have, for $|x - x_{k,n}| > \varphi_n(x_{k,n})$,

$$|f(x) - f(x_{k,n})| \leqslant C\left(|x - x_{k,n}| \psi_n^{-1}(x) + 1\right) \omega_x(f; \psi_n(x)), \quad (67)$$

where $\psi_n(x)$ is defined in (25) as $\psi_n(x) \to 0$ uniformly in $\mathbb{R}$ as $n \to \infty$.

We have the following estimate.

Lemma 11. *For any $x \in \mathbb{R}$,*

$$B_{n,k}(x) := \frac{1}{H_{n,\gamma}(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{k,n} |K_n(x, x_{k,n})|^{\gamma}. \quad (68)$$

Then

$$\sum_{|x-x_{k,n}|>\varphi_n(x)} B_{n,k}(x) \leqslant 1. \quad (69)$$

Proof.

$$\sum_{|x-x_{k,n}|>\varphi_n(x)} B_{n,k}(x) = \frac{1}{H_{n,\gamma}(x)} \sum_{|x-x_{k,n}|>\varphi_n(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} \times |K_n(x, x_{k,n})|^{\gamma} \leqslant 1, \quad (70)$$

because we know from the definition of $H_{n,\gamma}(x)$ in (54) that

$$\sum_{|x-x_{k,n}|>\varphi_n(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} |K_n(x, x_{k,n})|^{\gamma} \leqslant H_{n,\gamma}(x). \quad (71)$$

□

Step 4. Let $|x - x_{k,n}| > \varphi_n(x)$. Using the result of Step 1, we have the following estimate.

Lemma 12. *For any $x \in \mathbb{R}$, one sets*

$$C_{n,k}(x) := \frac{1}{H_{n,\gamma}(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{k,n} \times |x - x_{k,n}| |K_n(x, x_{k,n})|^{\gamma}. \quad (72)$$

Then for $x \in [x_{n,n}, x_{1,n}]$,

$$\sum_{|x-x_{k,n}|>\varphi_n(x)} C_{n,k}(x) \leqslant C h^{-1}(x) \Phi_n^{-\gamma/4}(x) \psi_n(x), \quad (73)$$

and for $|x| > x_{1,n}$,

$$\sum_{|x-x_{k,n}|>\varphi_n(x)} C_{n,k}(x) \leqslant 2|x|. \quad (74)$$

Proof. First, let $x \in [x_{n,n}, x_{1,n}]$. Then using (52), (66), and Lemma 9, we have

$$C_{n,k}(x) \leqslant C h(x_{k,n}) \varphi_n(x_{k,n}) \Phi_n^{-\gamma/4} \times (x) \Phi_n^{-\gamma/4}(x_{k,n}) \frac{1}{|x - x_{k,n}|^{\gamma-1}} \times h^{-1}(x) \varphi_n^{\gamma-1}(x). \quad (75)$$

From the fact that $h(x)\Phi_n^{-\gamma/4}(x)$ is bounded (recall the definition of $h(x)$), we can continue as

$$C_{n,k}(x) \leqslant C h^{-1}(x) \Phi_n^{-\gamma/4}(x) \varphi_n^{\gamma-1}(x) \frac{\varphi_n(x_{k,n})}{|x - x_{k,n}|^{\gamma-1}}. \quad (76)$$

Then by (25) and (40),

$$\begin{aligned}
&\sum_{|x-x_{k,n}|>\varphi_n(x)} C_{n,k}(x)\\
&\leqslant C \sum_{|x-x_{k,n}|>\varphi_n(x)} \frac{\varphi_n(x_{k,n})}{|x-x_{k,n}|^{\gamma-1}} h^{-1}(x)\,\Phi_n^{-\gamma/4}(x)\,\varphi_n^{\gamma-1}(x)\\
&\leqslant Ch^{-1}(x)\,\Phi_n^{-\gamma/4}(x)\,\varphi_n^{\gamma-1}(x)\\
&\quad\times \begin{cases} a_n^{2-\gamma}, & 1<\gamma<2;\\ \log a_n, & \gamma=2;\\ \varphi_n^{2-\gamma}(x), & \gamma>2, \end{cases}\\
&\leqslant Ch^{-1}(x)\,\Phi_n^{-\gamma/4}(x)\,\psi_n(x).
\end{aligned} \tag{77}$$

Next, suppose $x > x_{1,n}$. Then since

$$\begin{aligned}
C_{n,k}(x) \leqslant 2|x| \frac{1}{H_{n,\gamma}(x)} h(x_{k,n})\,w^{\gamma-2}(x_{k,n})\,\lambda_{k,n}\\
\times |K_n(x,x_{k,n})|^{\gamma},
\end{aligned} \tag{78}$$

we have from Lemma 11,

$$\sum_{|x-x_{k,n}|>\varphi_n(x)} C_{n,k}(x) \leqslant 2|x|. \tag{79}$$

□

Step 5. Using (67) and Lemmas 11 and 12, we can estimate the part $\sum_{|x-x_{k,n}|>\varphi_n(x)}$ as follows:

$$\begin{aligned}
\sum_2 := &\Bigg(\sum_{|x-x_{k,n}|>\varphi_n(x)} h(x_{k,n})\,w^{\gamma-2}(x_{k,n})\,\lambda_{kn}\\
&\quad \times |f(x)-f(x_{k,n})|\,|K_n(x,x_{k,n})|^{\gamma} \Bigg)\\
&\times \left(\sum_{k=1}^{n} h(x_{k,n})\,w^{\gamma-2}(x_{k,n})\,\lambda_{kn}|K_n(x,x_{k,n})|^{\gamma} \right)^{-1}.
\end{aligned} \tag{80}$$

Then for $x \in [x_{n,n}, x_{1,n}]$

$$\begin{aligned}
\sum_2 &\leqslant C\omega_x(f;\psi_n(x))\\
&\quad\times \Bigg(\sum_{|x-x_{k,n}|>\varphi_n(x)} B_{n,k}(x) + \psi_n^{-1}(x)\\
&\quad\times \sum_{|x-x_{k,n}|>\varphi_n(x)} C_{n,k}(x) \Bigg)\\
&\leqslant C\omega_x(f;\psi_n(x))\left(1+h^{-1}(x)\,\Phi_n^{-\gamma/4}(x)\right)\\
&\leqslant C\omega_x(f;\psi_n(x))\,h^{-1}(x)\,\Phi_n^{-\gamma/4}(x),
\end{aligned} \tag{81}$$

and for $|x| \geqslant x_{1,n}$,

$$\begin{aligned}
\sum_2 &\leqslant C\omega_x(f;\psi_n(x))\\
&\quad\times \Bigg(\sum_{|x-x_{k,n}|>\varphi_n(x)} B_{n,k}(x) + \psi_n^{-1}(x)\\
&\quad\times \sum_{|x-x_{k,n}|>\varphi_n(x)} C_{n,k}(x) \Bigg)\\
&\leqslant C\omega_x(f;\psi_n(x))\left(1+|x|\,\psi_n^{-1}(x)\right)\\
&\leqslant C\omega_x(f;\psi_n(x))\,|x|\,\psi_n^{-1}(x).
\end{aligned} \tag{82}$$

Therefore, with Lemma 10 we have the following result.

Lemma 13. *For $x_{n,n} \leqslant x \leqslant x_{1,n}$,*

$$\left|\mathcal{F}_{n,\gamma}[f](x) - f(x)\right| \leqslant C\omega_x(f;\psi_n(x))\,h^{-1}(x)\,\Phi_n^{-\gamma/4}(x) \tag{83}$$

and for $|x| > x_{1,n}$

$$\left|\mathcal{F}_{n,\gamma}[f](x) - f(x)\right| \leqslant C\omega_x(f;\psi_n(x))\,|x|\,\psi_n^{-1}(x). \tag{84}$$

Proof of Theorem 3. (a) follows from Lemma 13. We will show (b). Let $0 < p \leqslant \infty$. Then since we know that $\varphi(x) \leqslant C\phi_n$ and so $\psi_n(x) \leqslant C\psi_n^*$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
\left\| w^*\left(\mathcal{F}_{n,\gamma}[f]-f\right)\right\|_{L_p(\mathbb{R})} &= O(1)\left\| w^*\left(\sum_1 + \sum_2\right)\right\|_{L_p(\mathbb{R})}\\
&= O(1)\,\Omega(f;\psi_n^*).
\end{aligned} \tag{85}$$

□

Example 14. Let $h(x) = \Phi^{\gamma/4}(x)$ and

$$w^*(x) = \frac{\Phi^{\gamma/2}(x)}{(1+|x|)^{\beta+1}}, \quad \beta p > 1, \tag{86}$$

where

$$\Phi(x) := \frac{1}{(1+Q(x))^{2/3}T(x)}. \tag{87}$$

Then the condition (30) is satisfied.

4. Shepard-Type Operator

Let us define the positive interpolatory operator (4) for $f \in C(\mathbb{R})$ and the zeros $\{x_{j,n}\}_{j=1}^{n}$ of the orthonormal polynomial $p_n(w^2,x)$.

Let

$$\Phi(x) := \frac{1}{(1+Q(x))^{2/3}T(x)}. \tag{88}$$

Lemma 15 ([5, Lemma 3.3]). *For $x \in \mathbb{R}$, one has*

$$\Phi(x) \leqslant C\Phi_n(x), \quad n \geqslant 1. \tag{89}$$

Assumption 1. We suppose that, for each $\varepsilon > 0$,

$$T(a_n) \leqslant C(\varepsilon) n^{\varepsilon}, \quad n = 1, 2, 3, \ldots, \tag{90}$$

where $C(\varepsilon)$ is a constant depending only on ε.

Remark 16. Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$, and let us define

$$\nu := \limsup_{x\to\infty} \frac{Q''(x)/Q'(x)}{Q'(x)/Q(x)}, \qquad \mu := \liminf_{x\to\infty} \frac{Q''(x)/Q'(x)}{Q'(x)/Q(x)}. \tag{91}$$

If $\nu = \mu$, then we say that the weight w is regular. The regular weights satisfy the condition (90) (see [6, Corollry 5.5]). All weights in Example 2 are regular weights.

Lemma 17 ([3, Theorem 1.6]). *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$, and let a_n be defined by* (21). *Then there exists $C > 0$ such that for every $n > 0$*

$$a_n \leqslant C n^{1/\Lambda}, \tag{92}$$

where $\Lambda > 1$ is defined in Definition 1 (d). In particular, for the weight w_α one has $\Lambda = \alpha$. Furthermore, if w is an Erdös-type, then for any $\eta > 0$, there exists $C(\eta) > 0$ such that, for every $n > 0$,

$$a_n \leqslant C(\eta) n^{\eta}. \tag{93}$$

For each $3/2 < \lambda < 3$ let us set

$$\mu_n = \begin{cases} \dfrac{a_n T^{\lambda/3}(a_n)}{n^{1-\lambda/3}}, & 2 < \lambda < 3; \\[2ex] \dfrac{a_n T^{\lambda/3}(a_n)\log n}{n^{1-\lambda/3}}, & \lambda = 2; \\[2ex] \dfrac{a_n T^{\lambda/3}(a_n)}{n^{(2\lambda-3)/3}}, & \dfrac{3}{2} < \lambda < 2. \end{cases} \tag{94}$$

Our second theorem is as follows.

Theorem 18. *Let $f \in C(\mathbb{R})$ be uniformly continuous on $\mathbb{R}$ and let $3/2 < \lambda < 3$. Assume $U(x)$ is a nonnegative and decreasing function with $U(x) \leqslant C\Phi^{(\lambda-1)/2}(x)$. Then one has for the Erdös-type weights,*

$$\left\| U(x)\left(S_{n,\lambda}(f;x) - f(x)\right)\right\|_{L_\infty(\mathbb{R})} \leqslant C\Omega(f;\mu_n), \tag{95}$$

where μ_n is defined in (94).

For the Freud weights we have the following. For $\Lambda > 3$, let us set $(3/2)(1 + (1/\Lambda)) < \lambda < 3(1 - (1/\Lambda))$ and

$$\mu_{n,\Lambda} = \begin{cases} \dfrac{1}{n^{1-\lambda/3-1/\Lambda}}, & 2 < \lambda < 3\left(1 - \dfrac{1}{\Lambda}\right); \\[2ex] \dfrac{1}{n^{1/3-1/\Lambda}}, & \lambda = 2; \\[2ex] \dfrac{1}{n^{(2\lambda-3)/3-1/\Lambda}}, & \dfrac{3}{2}\left(1 + \dfrac{1}{\Lambda}\right) < \lambda < 2 \end{cases} \tag{96}$$

(note (92) and (94)).

Corollary 19. *Let $\Lambda > 3$, where Λ is defined in Definition 1 (d), and let $(3/2)(1 + (1/\Lambda)) < \lambda < 3(1 - (1/\Lambda))$. Then, for the Freud-type weights,* (95) *holds with $\mu_{n,\Lambda}$. In particular, when $w(x) = \exp(-|x|^\alpha)$, one can take $\Lambda = \alpha$.*

Remark 20. For the Freud-type weights we see $\lim_{n\to\infty}\mu_{n,\Lambda} = 0$. If we assume (90), then for the Erdös-type weights, from Lemma 17 (93), we also have $\lim_{n\to\infty}\mu_n = 0$.

Proof of Theorem 18. Let $3/2 < \lambda < 3$. We see that

$$S_{n,\lambda}(f;x) - f(x) = \frac{\sum_{j=1}^n \left\{ f(x_{j,n}) - f(x)\right\} \Phi_n^{(\lambda-1)/2}(x_{j,n}) \left| x - x_{j,n}\right|^{-\lambda}}{\sum_{j=1}^n \Phi_n^{(\lambda-1)/2}(x_{j,n}) \left| x - x_{j,n}\right|^{-\lambda}}. \tag{97}$$

Let $(x_{m+1,n} + x_{m,n})/2 < x \leqslant x_{m,n}$ or $(x_{m,n} + x_{m-1,n})/2 < x \leqslant x_{m,n}$. Then, we see

$$\begin{aligned} \left| f(x_{m,n}) - f(x)\right| &\leqslant \omega_x\left(f; \left|x_{m,n} - x\right|\right) \\ &\leqslant C\omega_x\left(f;\varphi_n(x)\right) \leqslant C\omega(f;\mu_n), \end{aligned} \tag{98}$$

where μ_n is defined in (94). If $j \neq m$, then we have

$$\begin{aligned} \left| f(x_{j,n}) - f(x)\right| &\leqslant \omega_x\left(f; \left|x - x_{j,n}\right|\right) \\ &\leqslant \left(\left|x - x_{j,n}\right| \mu_n^{-1} + 1\right)\Omega(f;\mu_n). \end{aligned} \tag{99}$$

Let

$$\sum_1 := \frac{\sum_{j\neq m} \Phi_n^{(\lambda-1)/2}(x_{j,n}) \left|x - x_{j,n}\right|^{-\lambda}}{\sum_{j=1}^n \Phi_n^{\lambda-1/2}(x_{j,n}) \left|x - x_{j,n}\right|^{-\lambda}}, \qquad \sum_2 := \frac{\sum_{j\neq m} \Phi_n^{(\lambda-1)/2}(x_{j,n}) \left|x - x_{j,n}\right|^{-(\lambda-1)}}{\sum_{j=1}^n \Phi_n^{\lambda-1/2}(x_{j,n}) \left|x - x_{j,n}\right|^{-\lambda}}. \tag{100}$$

Then we see that $0 < \sum_1 \leqslant 1$. Now, we will estimate $\sum_2$. We see that

$$\frac{1}{\left|x-x_{j,n}\right|} \sim \left(\sum_{\substack{j\leqslant s\leqslant m+1 \text{ or}\\ m-1\leqslant s\leqslant j}} \varphi_n\left(x_{s,n}\right)\right)^{-1}$$
$$\sim \frac{n}{a_n}\left(\sum_{\substack{j\leqslant s\leqslant m+1 \text{ or}\\ m-1\leqslant s\leqslant j}} \frac{1-\left|x_{s,n}\right|/a_{2n}}{\left(1-\left|x_{s,n}\right|/a_n+\delta_n\right)^{1/2}}\right)^{-1}$$
$$\geqslant \frac{n}{a_n}\left(nT\left(a_n\right)\right)^{-1/3}\left(\sum_{\substack{j\leqslant s\leqslant m+1 \text{ or}\\ m-1\leqslant s\leqslant j}} \left(1-\left|x_{s,n}\right|/a_{2n}\right)\right)^{-1}$$
$$\geqslant \frac{n^{2/3}}{a_nT^{1/3}\left(a_n\right)}\frac{1}{\left|m-j\right|}. \tag{101}$$

Hence we have

$$\sum_{j=1}^{n}\Phi_n^{(\lambda-1)/2}\left(x_{j,n}\right)\left|x-x_{j,n}\right|^{-\lambda}$$
$$\geqslant \left(\frac{n^{2/3}}{a_nT^{1/3}\left(a_n\right)}\right)^{\lambda}\sum_{j\neq m}\Phi_n^{(\lambda-1)/2}\left(x_{j,n}\right)\frac{1}{\left|m-j\right|^{\lambda}}$$
$$\geqslant \left(\frac{n^{2/3}}{a_nT^{1/3}\left(a_n\right)}\right)^{\lambda}\sum_{\substack{|x_{j,n}|\leqslant a_n/2\\ j\neq m}}\Phi_n^{(\lambda-1)/2}\left(x_{j,n}\right)\frac{1}{\left|m-j\right|^{\lambda}} \tag{102}$$
$$\geqslant C\left(\frac{n^{2/3}}{a_nT^{1/3}\left(a_n\right)}\right)^{\lambda}\sum_{\substack{|x_{j,n}|\leqslant a_n/2\\ j\neq m}}\frac{1}{\left|m-j\right|^{\lambda}}$$
$$\geqslant C\left(\frac{n^{2/3}}{a_nT^{1/3}\left(a_n\right)}\right)^{\lambda}\begin{cases}1, & \lambda>1;\\ \log n, & \lambda=1.\end{cases}$$

Using for $1 \leqslant j \leqslant n$

$$1-\frac{\left|x_{j,n}\right|}{a_{2n}} \geqslant C\left(1-\frac{\left|x_{j,n}\right|}{a_n}+\delta_n\right), \tag{103}$$

we see that

$$\frac{1}{\left|x-x_{j,n}\right|} \sim \left(\sum_{\substack{j\leqslant s\leqslant m+1 \text{ or}\\ m-1\leqslant s\leqslant j}} \varphi_n\left(x_{s,n}\right)\right)^{-1}$$
$$\sim \frac{n}{a_n}\left(\sum_{\substack{j\leqslant s\leqslant m+1 \text{ or}\\ m-1\leqslant s\leqslant j}} \frac{1-\left|x_{s,n}\right|/a_{2n}}{\left(1-\left|x_{s,n}\right|/a_n+\delta_n\right)^{1/2}}\right)^{-1}$$
$$\leqslant C\frac{n}{a_n}\left(\sum_{\substack{j\leqslant s\leqslant m+1 \text{ or}\\ m-1\leqslant s\leqslant j}} \left(1-\left|x_{s,n}\right|/a_n+\delta_n\right)^{1/2}\right)^{-1}$$
$$\leqslant C\frac{n}{a_n}\left(\Phi_n^{-1/2}(x)+\Phi_n^{-1/2}\left(x_{j,n}\right)\right)\frac{1}{\left|m-j\right|}. \tag{104}$$

Therefore, we have

$$\sum_{j\neq m}U(x)\,\Phi_n^{(\lambda-1)/2}\left(x_{j,n}\right)\left|x-x_{j,n}\right|^{-(\lambda-1)}$$
$$\leqslant C\left(\frac{n}{a_n}\right)^{\lambda-1}\sum_{j\neq m}\left(U(x)\,\Phi_n^{-(\lambda-1)/2}(x)\times\Phi_n^{(\lambda-1)/2}\left(x_{j,n}\right)+U(x)\right)\times\frac{1}{\left|m-j\right|^{\lambda-1}} \tag{105}$$
$$\leqslant C\left(\frac{n}{a_n}\right)^{\lambda-1}\sum_{j\neq m}\frac{1}{\left|m-j\right|^{\lambda-1}}$$
$$\leqslant C\left(\frac{n}{a_n}\right)^{\lambda-1}\begin{cases}1, & 2<\lambda;\\ \log n, & \lambda=2;\\ n^{2-\lambda}, & 1\leqslant\lambda<2.\end{cases}$$

Then, with (102) we see

$$\left|U(x)\sum_2(x)\right|$$
$$=\left|\frac{\sum_{j\neq m}U(x)\,\Phi_n^{(\lambda-1)/2}\left(x_{j,n}\right)\left|x-x_{j,n}\right|^{-(\lambda-1)}}{\sum_{j=1}^{n}\Phi_n^{(\lambda-1)/2}\left(x_{j,n}\right)\left|x-x_{j,n}\right|^{-\lambda}}\right|$$
$$\leqslant C\left(\frac{a_nT^{1/3}\left(a_n\right)}{n^{2/3}}\right)^{\lambda}\left(\frac{n}{a_n}\right)^{\lambda-1}$$

$$\times \begin{cases} 1, & 2 < \lambda; \\ \log n, & \lambda = 2; \\ n^{2-\lambda}, & 1 < \lambda < 2; \\ \dfrac{n^{2-\lambda}}{\log n}, & \lambda = 1, \end{cases}$$

$$\leqslant C \frac{a_n T^{\lambda/3}(a_n)}{n^{1-\lambda/3}} \begin{cases} 1, & 2 < \lambda; \\ \log n, & \lambda = 2; \\ n^{2-\lambda}, & 1 < \lambda < 2; \\ \dfrac{n^{2-\lambda}}{\log n}, & \lambda = 1. \end{cases} \tag{106}$$

Hence, using μ_n in (94), we have that, for $3/2 < \lambda < 3$,

$$\left| U(x) \sum_2 (x) \right| \leqslant C\mu_n. \tag{107}$$

Consequently, with $0 < \sum_1 \leqslant 1$ we have

$$U(x) \left| S_{n,\lambda}(f; x) - f(x) \right| \leqslant C\Omega(f; \mu_n). \tag{108}$$

□

References

[1] A. Knopfmacher, "Positive convergent approximation operators associated with orthogonal polynomials for weights on the whole real line," *Journal of Approximation Theory*, vol. 46, no. 2, pp. 182–203, 1986.

[2] E. Levin and D. S. Lubinsky, *Orthogonal Polynomials for Exponential Weights*, Springer, New York, NY, USA, 2001.

[3] H. Jung and R. Sakai, "Specific examples of exponential weights," *Communications of the Korean Mathematical Society*, vol. 24, no. 2, pp. 303–319, 2009.

[4] H. S. Jung and R. Sakai, "Orthonormal polynomials with exponential-type weights," *Journal of Approximation Theory*, vol. 152, no. 2, pp. 215–238, 2008.

[5] H. S. Jung, G. Nakamura, R. Sakai, and N. Suzuki, "Convergence and divergence of higher-order Hermite or Hermite-Fejér interpolation polynomials with exponential-type weights," *ISRN Mathematical Analysis*, vol. 2012, Article ID 904169, 31 pages, 2012.

[6] R. Sakai and N. Suzuki, "Mollification of exponential-type weights and its application to Markov-Bernstein inequality," *Pioneer Journal of Mathematics and Mathematical Siences*, vol. 7, no. 1, pp. 83–101, 2013.