*Research Article*

# A Class of Weingarten Surfaces in Euclidean 3-Space

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The class of biconservative surfaces in Euclidean 3-space $\mathbb{E}^3$ are defined in (Caddeo et al., 2012) by the equation $A(\operatorname{grad} H) = -H \operatorname{grad} H$ for the mean curvature function $H$ and the Weingarten operator $A$. In this paper, we consider the more general case that surfaces in $\mathbb{E}^3$ satisfying $A(\operatorname{grad} H) = kH \operatorname{grad} H$ for some constant $k$ are called generalized bi-conservative surfaces. We show that this class of surfaces are linear Weingarten surfaces. We also give a complete classification of generalized bi-conservative surfaces in $\mathbb{E}^3$.

## 1. Introduction

Let $x : M^n \to \mathbb{E}^m$ be an isometric immersion of a submanifold $M^n$ into Euclidean (pseudo-Euclidean) space $\mathbb{E}^m$. We denote by $x, \vec{H}$, and $\Delta$ the position vector, the mean curvature vector field, and the Laplace operator of $M^n$, respectively, with respect to the induced metric. The submanifold $M^n$ in $\mathbb{E}^m$ is said to biharmonic if it satisfies the equation $\Delta^2 x = 0$. According to the well-known Betrami's formula $\Delta x = -n\vec{H}$, the biharmonic condition in Euclidean space $\mathbb{E}^m$ is also known as the equation $\Delta \vec{H} = 0$.

There is a well-known conjecture of Chen [1].

*Chen's Conjecture.* The only biharmonic submanifolds of Euclidean spaces are the minimal ones.

This conjecture has been proved by some geometers for some special cases. For example, Chen proved that every biharmonic surface in the Euclidean 3-space $\mathbb{E}^3$ is minimal. Hasanis and Vlachos [2] proved that every biharmonic hypersurface in $\mathbb{E}^4$ is minimal, also see [3]. For the general case, the conjecture is still open so far.

The study of biharmonic submanifolds is nowadays a very active research subject. Many interesting results on biharmonic maps and submanifolds have been obtained in the last decade, see [1–13].

Very recently, Caddeo et al. introduced the notion of biconservative submanifolds in [14], which is a natural

generalization of biharmonic submanifolds. It is interesting that biconservative submanifolds form a much bigger family of submanifolds including biharmonic submanifolds.

Recall a well-known result; see, for instance, [3].

**Theorem A.** *Let $\phi : M^n \to \mathbb{E}^{n+1}$ be a hypersurface with mean curvature vector $\vec{H} = H\xi$. Then, $\phi$ is biharmonic if and only if the following equations hold:*

$$\Delta H - H|A|^2 = 0,$$
$$2A\left(\operatorname{grad} H\right) + nH \operatorname{grad} H = 0, \tag{1}$$

*where $A$ is the shape operator of the hypersurface with respect to the unit normal vector $\xi$.*

Following the definition of Caddeo et. al. in [14], a hypersurface $M^n$ in an $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ is called biconservative if

$$2A\left(\operatorname{grad} H\right) + nH \operatorname{grad} H = 0. \tag{2}$$

In general, a submanifold is biconservative if the divergence of the stress bienergy tensor vanishes.

In 1995, Hasanis and Vlachos, in [2], firstly studied biconservative hypersurfaces, which are also called *H-hypersurfaces*. The authors gave a classification of biconservative hypersurfaces in Euclidean 3-spaces and 4-spaces.

Recently, Caddeo et al. [14] investigated biconservative surfaces in the three-dimensional Riemannian space forms. Moreover, they proved that a biconservative surface in Euclidean 3-space is either a CMC (constant mean curvature) surface or a surface of revolution. This class of surfaces carry some interesting geometry. It was proved in [14] that the mean curvature function $H$ of a non-CMC biconservative surface in a three-dimensional space form $M^3(c)$ satisfies the following relation:

$$K = -3H^2 + c, \qquad (3)$$

where $K$ and $H$ denote the Gaussian curvature and mean curvature of the surfaces, respectively. Clearly, the forementioned relation implies that all the biconservative surfaces in the Euclidean 3-space are linear Weingarten surfaces.

A surface is called a Weingarten surface if there exists the Jacobi equation $\Psi(H, K) = 0$ between the Gaussian curvature $K$ and the mean curvature $H$ on the surface. Weingarten surfaces were introduced by Weingarten in 1861 in the context of the problem of finding all surfaces isometric to a given surface of revolution. Along the history, they have been of interest for geometers. There is a great amount of literature on Weingarten surfaces, beginning with works of Chern, Hartman, Winter, and Hopf in the fifties of the last century. For a long time, many geometers tried to look for examples of linear Weingarten surfaces, for example, see [10].

For surfaces in $\mathbb{E}^3$, the biconservative condition is equivalent to the equation

$$A(\operatorname{grad} H) = -H \operatorname{grad} H. \qquad (4)$$

Observe from [14] that the forementioned equation corresponds to a class of linear Weingarten surfaces, which include CMC surfaces and a family of surfaces of revolution. From the view of equation, a natural idea is to extend this class of surfaces in order to search for more examples of Weingarten surfaces of revolution. Hence, from the view of geometry, we propose to study the surfaces in $\mathbb{E}^3$ satisfying a more general equation:

$$A(\operatorname{grad} H) = kH \operatorname{grad} H, \quad k \in \mathbb{R}. \qquad (5)$$

We would like to call this new class of surfaces is *generalized biconservative surfaces*.

In this note, we focus on the equation and study this class of surfaces in $\mathbb{E}^3$. Precisely, we will prove that any generalized biconservative surface in Euclidean 3-space is a linear Weingarten surface satisfying a more general relation $K = cH^2$ for some constant $c$. A local classification of generalized biconservative surfaces in $\mathbb{E}^3$ is also obtained. Note that our method is slightly different from the method developed by Caddeo et al. in [14].

## 2. Preliminaries

Let $x : M \to \mathbb{E}^3$ be an isometric immersion of a surface $M$ into $\mathbb{E}^3$. Denote the Levi-Civita connections of $M$ and $\mathbb{E}^3$ by $\nabla$ and $\widetilde{\nabla}$, respectively. Let $X$ and $Y$ denote vector fields

tangent to $M$, and let $\xi$ be a normal vector field. The Gauss and Weingarten formulas are given, respectively, by (cf. [8, 15])

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad (6)$$

$$\widetilde{\nabla}_X \xi = -A_\xi X, \qquad (7)$$

where $h$, $A$ are the second fundamental form and the shape operator. It is well known that the second fundamental form $h$ and the shape operator $A$ are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle. \qquad (8)$$

The Gauss and Codazzi equations are given respectively by

$$\langle R(X, Y) Z, W \rangle = \langle h(Y, Z), h(X, W) \rangle$$
$$- \langle h(X, Z), h(Y, W) \rangle, \qquad (9)$$
$$(\nabla_X A) Y = (\nabla_Y A) X,$$

where $R$ is the curvature tensor of the Levi-Civita connection on $M$. The mean curvature vector field $\overrightarrow{H}$ and the Gauss curvature of $M$ are given respectively, by

$$\overrightarrow{H} = \frac{1}{2} \operatorname{trace} h, \qquad K = \det A. \qquad (10)$$

As known from the Introduction, a surface $M$ in $\mathbb{E}^3$ is biconservative if the mean curvature function $H$ satisfies

$$A(\operatorname{grad} H) + H \operatorname{grad} H = 0. \qquad (11)$$

Motivated by the above equation for biconservative surfaces in $\mathbb{E}^3$, we propose to the notion of generalized biconservative surfaces in $\mathbb{E}^3$.

*Definition 1.* A surface $M$ in Euclidean 3-space $\mathbb{E}^3$ is called generalized biconservative surface if the mean curvature function $H$ and the Weingarten operator satisfy a equation

$$A(\operatorname{grad} H) = kH \operatorname{grad} H \qquad (12)$$

for some $k \in \mathbb{R}$.

Note that this class of surfaces include all the biconservative surfaces as a subclass when $k = -1$.

Clearly, all of the CMC surfaces in $\mathbb{E}^3$ are trivially generalized biconservative surfaces. This is also the case of biconservative surfaces. We are interested in the case of non-CMC surfaces in $\mathbb{E}^3$.

## 3. The Characterizations of Generalized Biconservative Surfaces

In this section, let us focus on the situation of non-CMC generalized biconservative surfaces in $\mathbb{E}^3$.

Suppose that $\operatorname{grad} H \neq 0$ on any point $p \in M$. It follows from (12) that $\operatorname{grad} H$ is a principal direction and $kH$ is the corresponding principal curvature. We can choose a

local orthonormal frame field $\{e_1, e_2\}$ such that $e_1$ is parallel to grad $H$. Therefore, we have $e_2(H) = 0$. Since (12) gives $Ae_1 = kHe_1$, it follows that $Ae_2 = (2-k)He_2$. According to the Gauss equation, the Gaussian curvature $K$ is given by

$$K = k(2-k)H^2, \qquad (13)$$

which implies the following.

**Theorem 2.** *The generalized biconservative surfaces in $\mathbb{E}^3$ are linear Weingarten surfaces.*

If we put $\nabla_X e_1 = \omega(X)e_2$, then $\nabla_X e_2 = -\omega(X)e_1$. Using the remark above, the Codazzi equation reduces to

$$2(k-1)H\omega(e_1) = 0,$$
$$(2-k)e_1(H) = 2(k-1)H\omega(e_2). \qquad (14)$$

Since $H$ is nonconstant, from $e_2(H) = 0$, one has $e_1(H) \neq 0$. So, the second equation of (14) yields $k \neq 1$. Moreover, the first equation of (14) implies that $\omega(e_1) = 0$. Without the loss of generality, one assumes that $H > 0$.

According to the second equation of (14), one divides it into the following two cases.

*Case A ($k = 2$).* In this case, the surface is flat. Then, the second equation of (14) yields $\omega(e_2) = 0$ as well. Choose the local coordinates on $M$ as $\partial/\partial s = e_1$ and $\partial/\partial t = e_2$. By applying the Gauss and Weingarten formulas (6) and (7) respectively, the immersion satisfies that

$$x_{ss} = 2H\xi,$$
$$x_{st} = 0, \qquad (15)$$
$$x_{tt} = 0,$$

$$(\xi)_s = -2H\frac{\partial}{\partial s}, \qquad (\xi)_t = 0. \qquad (16)$$

By solving the second and third equations of (15), we obtain that

$$x = c_1 t + \alpha(s), \qquad (17)$$

for a constant vector $c_1$ and a curve $\alpha(s)$ in $\mathbb{E}^3$. Substitute (17) into the first equation of (15) and the first equation of (16), respectively. Combining these equations, we obtain a three-order differential equation as follows:

$$\alpha''' = \frac{H'}{H}\alpha'' - 4H^2\alpha'. \qquad (18)$$

In order to solve the above equation, we introduce two functions, $\beta(s)$ (vector-valued function) and $u(s)$, by putting $\beta = \alpha'$ and $u' = 2H$. Note that $u'$ is the nonzero principal curvature and $u'$ is not constant.

Denote by "·" the derivative with respect to the new variable $u$. With these symbols, (18) becomes

$$\ddot{\beta} + \beta = 0, \qquad (19)$$

whose solution is given by

$$\beta(s) = c_2 \cos u + c_3 \sin u, \qquad (20)$$

where $c_2$ and $c_3$ are constant vectors in $\mathbb{E}^3$. Consequently, by a suitable translation, the immersion $x$ is given by

$$x = c_1 t + c_2 \int^s \cos u\, ds + c_3 \int^s \sin u\, ds. \qquad (21)$$

Considering the metric of surfaces, we may choose that

$$c_1 = (1, 0, 0), \qquad c_2 = (0, 1, 0), \qquad c_3 = (0, 0, 1). \qquad (22)$$

Hence, the surface can be expressed as

$$x(s,t) = \left(t, \int^s \cos u(s)\, ds, \int^s \sin u(s)\, ds\right). \qquad (23)$$

Remark that the surface (23) is a cylinder, but not a circular cylinder, since the curvature of the curve $\alpha$ is not constant.

*Case B ($k \neq 2, 1$).* Let $u = H^{(2-k)/(2k-2)}$. Since $\omega(e_1) = 0$, it follows from the second equation of (14) that $[e_1, ue_2] = 0$. Therefore, there exist local coordinates $(s, t)$ on $M$ such that $\partial/\partial s = e_1$ and $\partial/\partial t = ue_2$. Then, the metric tensor of $M$ is given by

$$g = ds^2 + u^2 dt^2. \qquad (24)$$

Since $e_2(H) = 0$, we have that $u = H^{(2-k)/(2k-2)}$ is a function depending only on the variable $s$. Consequently, the Levi-Civita connection $\nabla$ is given by the expressions

$$\nabla_{\partial/\partial s}\frac{\partial}{\partial s} = 0, \qquad \nabla_{\partial/\partial t}\frac{\partial}{\partial s} = \frac{u'}{u}\frac{\partial}{\partial t}, \qquad \nabla_{\partial/\partial t}\frac{\partial}{\partial t} = -uu'\frac{\partial}{\partial s}, \qquad (25)$$

and the second fundamental form is given by

$$h\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = ku^{(2k-2)/(2-k)}\xi, \qquad h\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = 0,$$
$$h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = (2-k)u^{2/(2-k)}\xi. \qquad (26)$$

Moreover, it follows from (25) and second fund form that the Gauss and Weingarten formulas (6) and (7) yield, respectively

$$x_{ss} = ku^{(2k-2)/(2-k)}\xi,$$
$$x_{st} = \frac{u'}{u}x_t, \qquad (27)$$
$$x_{tt} = -uu'x_s + (2-k)u^{2/(2-k)}\xi,$$
$$(\xi)_s = -ku^{(2k-2)/(2-k)}\frac{\partial}{\partial s},$$
$$(\xi)_t = -(2-k)u^{(2k-2)/(2-k)}\frac{\partial}{\partial t}. \qquad (28)$$

By (28), the compatibility condition of PDE system (27) is given by

$$u'' = -k(2-k)u^{(3k-2)/(2-k)}. \qquad (29)$$

Integrating on (29), we obtain

$$u'^2 + (2-k)^2 u^{2k/(2-k)} = C \qquad (30)$$

for some integral constant $C$. Clearly, we have $C > 0$ for nonconstant function $u$. Solving the second equation of (27), the immersion $x$ is given by

$$x = u(s)a(t) + b(s) \qquad (31)$$

for two vector-valued functions $a(t)$ and $b(s)$ in $\mathbb{E}^3$.

*Case B.1 ($k = 0$).* In this case, (27) and (28) become

$$x_{ss} = 0,$$

$$x_{st} = \frac{u'}{u}x_t, \qquad (32)$$

$$x_{tt} = -uu'x_s + 2u\xi,$$

$$(\xi)_s = 0, \qquad (\xi)_t = -2u^{-1}\frac{\partial}{\partial t}. \qquad (33)$$

It follows from (30) that

$$u = \sqrt{C-4}s + C_1 \qquad (34)$$

for some constant $C_1$, and $C > 4$. Substituting (31) into the first equation of (32), after a suitable translation, we obtain the immersion

$$x(s,t) = u(s)a(t) + c_1 u(s) = u(s)\bar{a}(t), \qquad (35)$$

where $\bar{a}(t)$ is another curve in $\mathbb{E}^3$. Substituting (35) into the third equation of (32) and applying (33), we have the following three-order differential equation:

$$\bar{a}'''(t) + (u'^2 + 4)\bar{a}'(t) = 0. \qquad (36)$$

By (30), the solution of (36) is given by

$$\bar{a}(t) = c_2 \cos\left(\sqrt{C}t\right) + c_3 \sin\left(\sqrt{C}t\right) + c_4 \qquad (37)$$

for constant vector $c_i$, $i = 2, 3, 4$ in $\mathbb{E}^3$. Hence, the immersion becomes

$$x(s,t) = u(s)a(t) + c_1 u(s)$$
$$= u(s)\left(c_2 \cos\left(\sqrt{C}t\right) + c_3 \sin\left(\sqrt{C}t\right) + c_4\right). \qquad (38)$$

In view of the metric (24), one can obtain that $c_2$, $c_3$, $c_4$ are mutual orthonormal and

$$\langle c_2, c_2 \rangle = \langle c_3, c_3 \rangle = \frac{1}{C}, \qquad \langle c_4, c_4 \rangle = \frac{4}{C(C-4)}. \qquad (39)$$

After choosing $c_2$, $c_3$, and $c_4$ as

$$c_2 = \frac{1}{\sqrt{C}}(1,0,0), \qquad c_3 = \frac{1}{\sqrt{C}}(0,1,0),$$
$$c_4 = \frac{2}{\sqrt{C(C-4)}}(0,0,1), \qquad (40)$$

the immersion can be expressed as

$$x(u,t) = \frac{u}{\sqrt{C}}\left(\cos\sqrt{C}t, \sin\sqrt{C}t, \frac{2}{\sqrt{C-4}}\right). \qquad (41)$$

Note that this surface is a cone.

*Case B.2 ($k \neq 0, 1, 2$).* Substituting (31) into the first and third equations of (27), we have

$$ua''(t) = -uu'\left(u'a(t) + b'(s)\right) + \frac{2-k}{k}u^2\left(u''a(t) + b''(s)\right), \qquad (42)$$

which is equivalent to

$$a''(t) = \left(-u'^2 + \frac{2-k}{k}uu''\right)a(t) - u'b'(s) + \frac{2-k}{k}ub''(s). \qquad (43)$$

Substituting (29) and (30) into the previous equation in succession, we have

$$a''(t) + Ca(t) = -u'b'(s) + \frac{2-k}{k}ub''(s). \qquad (44)$$

In view of (44), the two sides of the equation have different variables, respectively. Hence, we have

$$a''(t) + Ca(t) = c, \qquad (45)$$

$$-u'b'(s) + \frac{2-k}{k}ub''(s) = c \qquad (46)$$

for some constant vector $c$ in $\mathbb{E}^3$. Solving (45) gives

$$a(t) = c_1 \cos\left(\sqrt{C}t\right) + c_2 \sin\left(\sqrt{C}t\right) + \frac{1}{C}c \qquad (47)$$

for two constant vectors $c_1$ and $c_2$ in $\mathbb{E}^3$. Looking at (31), we may assume that $c = 0$. In fact, the immersion can be rewritten as

$$x = u(s)\bar{a}(t) + \bar{b}(s), \qquad (48)$$

where $\bar{a}(t) = c_1 \cos(\sqrt{C}t) + c_2 \sin(\sqrt{C}t)$ and $\bar{b}(s) = (1/C)u(s)c + b(s)$. Hence, the immersion becomes

$$x = u(s)\left(c_1 \cos\left(\sqrt{C}t\right) + c_2 \sin\left(\sqrt{C}t\right)\right) + b(s). \qquad (49)$$

Solving (46) gives

$$b'(s) = u^{k/(k-2)}c_3 \qquad (50)$$

for a constant vector $c_3$.

One can compute from (49) that

$$x_s = u' \left( c_1 \cos \left( \sqrt{C} t \right) + c_2 \sin \left( \sqrt{C} t \right) \right) + u^{k/(k-2)} c_3,$$
$$x_t = \sqrt{C} u(s) \left( -c_1 \sin \left( \sqrt{C} t \right) + c_2 \cos \left( \sqrt{C} t \right) \right). \tag{51}$$

It follows from the above expressions and the metric (24) that

$$\langle c_1, c_1 \rangle = \langle c_2, c_2 \rangle = \frac{1}{C},$$
$$\langle c_1, c_2 \rangle = \langle c_1, c_3 \rangle = \langle c_2, c_3 \rangle = 0, \tag{52}$$
$$\frac{1}{C} u'^2 + u^{2k/(k-2)} \langle c_3, c_3 \rangle = 1.$$

Combining the previous expression (52) with (30) gives

$$\langle c_3, c_3 \rangle = \frac{(2-k)^2}{C}. \tag{53}$$

After a change of the variable $t$, we can assume $C = 1$. Hence, the three vectors $c_1$, $c_2$, $c_3$ in $\mathbb{E}^3$ can be chosen as

$$c_1 = (1, 0, 0), \qquad c_2 = (0, 1, 0), \qquad c_3 = (0, 0, 2-k). \tag{54}$$

Now, let us consider (50), which can be rewritten as

$$\dot{b}(u) u'(s) = u^{k/(k-2)} c_3, \tag{55}$$

where $\dot{b}$ is the derivative of $b$ with respect to $u$. By applying (30), (55) becomes

$$\dot{b}(u) = \pm \frac{1}{\sqrt{u^{2k/(k-2)} - (2-k)^2}} c_3. \tag{56}$$

By solving (56)

$$b(u) = \pm c_3 \int^u \frac{1}{\sqrt{u^{2k/(k-2)} - (2-k)^2}} du + c_4 \tag{57}$$

for some constant vector $c_4$ in $\mathbb{E}^3$.

Combining (49) with (54) and (57), and by a suitable translation, we obtain the immersion

$$x(u, t) = (u \cos t, u \sin t, f(u)), \tag{58}$$

where

$$f(u) = \int^u \frac{1}{\sqrt{\left( 1/(2-k)^2 \right) u^{2k/(k-2)} - 1}} du. \tag{59}$$

In this case, the immersion is a surface of revolution with non-constant mean curvature.

In summary, we have the following classification result.

**Theorem 3.** *Let $x : M \rightarrow \mathbb{E}^3$ be a nondegenerate generalized biconservative surface immersed in the 3-dimensional Euclidean space $\mathbb{E}^3$. Then, the immersion $x(M)$ is either a CMC*
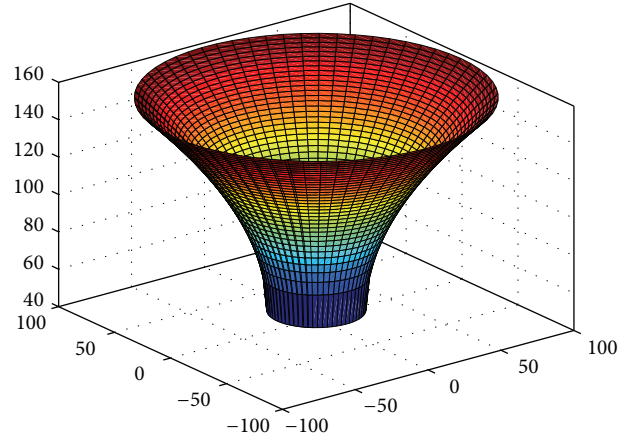


Figure 1: The non-CMC biconservative surfaces.

*surface or locally given by one of the following three surfaces:*

(1) *a cylinder given by*

$$x(s, t) = \left( t, \int^s \cos u(s) \, ds, \int^s \sin u(s) \, ds \right), \tag{60}$$

*where the function $u$ satisfies $u''(s) \neq 0$;*

(2) *a cone given by*

$$x(u, t) = \frac{u}{\sqrt{C}} \left( \cos \sqrt{C} t, \sin \sqrt{C} t, \frac{2}{\sqrt{C-4}} \right), \tag{61}$$

*where $C \in (4, +\infty)$;*

(3) *a surface of revolution given by*

$$x(u, t) = (u \cos t, u \sin t, f(u)), \tag{62}$$

*where $f$ is defined as*

$$f(u) = \int^u \frac{1}{\sqrt{\left( 1/(2-k)^2 \right) u^{2k/(k-2)} - 1}} du \tag{63}$$

*for $k \in (-\infty, 0) \cup (0, 1) \cup (1, 2) \cup (2, +\infty)$.*

## 4. Some Examples of Generalized Biconservative Surfaces

In this section, we give some examples of generalized biconservative surfaces (3) in Theorem 3, depending on different values for $k$.

*Example 1.* In the case $k = -1$, the function $f(u)$ can be integrated as

$$f(u) = \int^u \frac{1}{\sqrt{(1/9) u^{2/3} - 1}} du$$
$$= \frac{9}{2} \left( u^{1/3} \sqrt{u^{2/3} - 9} + 9 \ln \left( u^{1/3} + \sqrt{u^{2/3} - 9} \right) \right) + C_0 \tag{64}$$

for some integral constant $C_0$. Hence, by a suitable translation, the non-CMC biconservative surface in $\mathbb{E}^3$ (see also [14]) is given by

$$x(u,t) = \left(u\cos t, u\sin t, f(u)\right), \tag{65}$$

where $f$ is defined as

$$f(u) = \frac{9}{2}\left(u^{1/3}\sqrt{u^{2/3}-9} + 9\ln\left(u^{1/3} + \sqrt{u^{2/3}-9}\right)\right). \tag{66}$$

See Figure 1.

*Example 2.* For $k = -2$, the function $f(u)$ can be integrated as

$$f(u) = \int^u \frac{1}{\sqrt{(1/16)\,u - 1}}\,du = 8\sqrt{u-16} + C_1 \tag{67}$$

for some integral constant $C_1$ and $u > 16$. We have a non-CMC generalized biconservative surface (after a suitable translation) in $\mathbb{E}^3$, given by

$$x(u,t) = \left(u\cos t, u\sin t, 8\sqrt{u-16}\right). \tag{68}$$

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