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Research Article

Inertial Iteration for Split Common Fixed-Point Problem for Quasi-Nonexpansive Operators

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Inspired by the note on split common fixed-point problem for quasi-nonexpansive operators presented by Moudafi (2011), based on the very recent work by Dang et al. (2012), in this paper, we propose an inertial iterative algorithm for solving the split common fixed-point problem for quasi-nonexpansive operators in the Hilbert space. We also prove the asymptotical convergence of the algorithm under some suitable conditions. The results improve and develop previously discussed feasibility problems and related algorithms.

1. Introduction

The convex feasibility problem (CFP), as an important optimization problem [1], is to find a common point in the intersection of finitely many convex sets. It has been applied to many areas, for instance, approximation theory [2], image reconstruction from projections [3, 4], control [5], and so on. When there are only two sets and constraints are imposed on the solutions in the domain of a linear operator as well as in this operator's ranges, the problem is said to be a split feasibility problem (SFP) which has the following formula: finding a point x satisfying

$$x \in C, \quad Ax \in Q, \tag{1}$$

where C is a closed convex subset of a Hilbert space H_1, Q is a closed convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator. The SFP was originally introduced in [6], and it has also broad applications in many fields, such as image reconstruction problem, signal processing, and radiation therapy. Many projection methods have also been developed for solving the SFP; see [7–9]. Denote by P_C the orthogonal projection onto C; that is, $P_C(x) = \arg\min_{y\in C}\|x - y\|$, over all $x \in C$. Assuming that the SFP is consistent (i.e., (1) has a solution), it is not hard to see that $x \in C$ solves (1) if and only if it solves the fixed-point equation:

$$x = P_C\left[\left(I - \gamma A^*\left(I - P_Q\right)A\right)(x)\right], \tag{2}$$

where $0 < \gamma$ is any positive constant and A^* denotes the adjoint of A.

To solve (2), in [10], Byrne introduced the so-called CQ algorithm, which generates a sequence $\{x^k\}$ by

$$x^{k+1} = P_C\left[\left(I - \gamma A^*\left(I - P_Q\right)A\right)\left(x^k\right)\right], \tag{3}$$

where $0 < \gamma < 2/\rho(A^T A)$ and $\rho(A^T A)$ is the spectral radius of $A^* A$.

The split common fixed-point problem (SCFP) is a generalization of the split feasibility problem (SFP) and the convex feasibility problem (CFP); see [11]. Our main purpose here is to give an extension of the results developed in [12] to the split common fixed-point problem for quasi-nonexpansive operators, and we will introduce weak symposium convergence result of the algorithm under some suitable conditions. This will be done in the context of general Hilbert spaces.

The paper is organized as follows. In Section 2, we recall some preliminaries. In Section 3, we present an inertial CQ algorithm and show its convergence.

2. Preliminaries

Throughout the rest of the paper, I denotes the identity operator and $\mathrm{Fix}(T)$ denotes the set of the fixed points of an operator T, that is, $\mathrm{Fix}(T) := \{x \mid x = T(x)\}$.

Recall that a mapping T is said to be quasi-nonexpansive (ε_Q) if

$$\|Tx - q\| \le \|x - q\|, \quad \forall (x,q) \in H \times \mathrm{Fix}(T). \tag{4}$$

A mapping T is called nonexpansive (ε_N) if

$$\|T(x) - T(y)\| \le \|x - y\|, \quad \forall (x,y) \in H \times H. \tag{5}$$

A mapping T is called firmly nonexpansive (ε_{FN}) if

$$\|T(x) - T(y)\|^2 \le \|x - y\|^2 - \|(x - y) - (T(x) - T(y))\|^2, \quad \forall (x,y) \in H \times H. \tag{6}$$

A mapping T is called firmly quasi-nonexpansive (ε_{FQ}) if

$$\|T(x) - q\|^2 \le \|x - q\|^2 - \|x - T(x)\|^2, \quad \forall (x,q) \in H \times \mathrm{Fix}(H). \tag{7}$$

It is easily observed that $\varepsilon_{FN} \subset \varepsilon_N \subset \varepsilon_Q$ and that $\varepsilon_{FN} \subset \varepsilon_{FQ} \subset \varepsilon_Q$. Furthermore, ε_{FN} is well known to include resolvents and projection operators, while ε_{FQ} contains subgradient projection operators (see, e.g., [13], and the references therein).

Recently, Bauschke and Combettes [14] have considered a class of mappings satisfying the condition

$$\langle q - Tx, x - Tx \rangle \le 0, \quad \forall (x,q) \in H \times \mathrm{Fix}(T). \tag{8}$$

It can easily be seen that the class of mappings satisfying the latter condition coincides with that of firmly quasi-nonexpansive mappings.

Usually, the convergence of fixed-point algorithms requires some additional smoothness properties of the mapping T such as demiclosedness.

Definition 1. A mapping T is said to be demiclosed if for any sequence $\{x^k\}$ which weakly converges to y and if the sequence $\{T(x^k)\}$ strongly converges to z, then $T(y) = z$.

In what follows, only the particular case of demiclosedness at zero will be used, which is the particular case when $z = 0$.

The following lemmas will be needed in the proof of the convergence of the algorithm.

Lemma 2. *Let T be a quasi-nonexpansive mapping. Set $T_\alpha := (1 - \alpha)I + \alpha T$. Then, it is immediate that for all $(x,q) \in H \times \mathrm{Fix}(T)$:*

(1) $\langle x - T(x), x - q\rangle \ge (1/2)\|x - T(x)\|^2$ *and* $\langle x - T(x), q - T(x)\rangle \le (1/2)\|x - T(x)\|^2$;

(2) $\|T_\alpha(x) - q\|^2 \le \|x - q\|^2 - \alpha(1 - \alpha)\|x - T(x)\|$;

(3) $\langle x - T_\alpha(x), x - q\rangle \ge (\alpha/2)\|x - T(x)\|^2$.

Lemma 3 (see [8]). *Assume $\varphi_k \in [0,\infty)$ and $\delta_k \in [0,\infty)$ satisfy*

(1) $\varphi_{k+1} - \varphi_k \le \theta_k(\varphi_k - \varphi_{k-1}) + \delta_k$,

(2) $\sum_{k=1}^{+\infty} \delta_k < \infty$,

(3) $\{\theta_k\} \subset [0,\theta]$, *where* $\theta \in [0,1)$.

Then, the sequence $\{\varphi_k\}$ is convergent with $\sum_{k=1}^{+\infty}[\varphi_{k+1} - \varphi_k]_+ < \infty$, where $[t]_+ := \max\{t, 0\}$ (for any $t \in R$).

3. The Inertial Algorithm and Its Asymptotic Convergence

In what follows, we will focus our attention on the following general two-operator split common fixed-point problem:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q, \tag{9}$$

where $A : H_1 \to H_2$ is a bounded linear operator and $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two quasi-nonexpansive operators with nonempty fixed-point sets $\mathrm{Fix}(U) = C$ and $\mathrm{Fix}(T) = Q$, and denote the solution set of the two-operator SCFP by

$$\Gamma = \{y \in C : Ay \in Q\}. \tag{10}$$

3.1. The Inertial Algorithm. To solve (9), Moudafi [15] proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm:

$$x^{k+1} = U_{\alpha_k}\left(x^k + \gamma\left(A^* T_\beta - I\right)A\left(x^k\right)\right), \quad k \in N, \tag{11}$$

where $\beta \in (0,1)$, $\alpha_k \in (0,1)$ are relaxation parameters and $\gamma > 0$. Inspired by the inertial technique, we introduce the following inertial method and then present its convergence analysis.

Algorithm 4.

Initialization: Let $x^0 \in H^1$ be arbitrary.

Iterative step: For $k \in N$, set $u = I + \gamma\eta A^*(T - I)A$, and let

$$\begin{aligned} y^k &= x^k + \theta_k\left(x^k - x^{k-1}\right) \\ x^{k+1} &= (1 - \alpha_k)\,u\left(y^k\right) + \alpha_k U\left(u\left(y^k\right)\right), \quad k \in N, \end{aligned} \tag{12}$$

where $\eta \in (0,1)$, $\alpha_k \in (0,1)$, and $\gamma \in (0, 1/(\lambda\eta))$, with λ being the spectral radius of the operator A^*A, $\theta_k \in [0,1)$.

3.2. Asymptotic Convergence of the Inertial Algorithm. In this subsection, we establish the asymptotic convergence of Algorithm 4.

Lemma 5 (Opial [16]). *Let H be a Hilbert space and let $\{x^k\}$ be a sequence in H such that there exists a nonempty set $S \subset H$ satisfying*

(1) *for every x^*, $\lim_k \|x^k - x^*\|$ exists,*

(2) *any weak cluster point of the sequence $\{x^k\}$ belongs to S. Then, there exists $z \in S$ such that $\{x^k\}$ weakly converges to z.*

Theorem 6. *Given a bounded linear operator $A : H_1 \to H_2$, let $U : H_1 \to H_1$ be a quasi-nonexpansive operator with nonempty $\mathrm{Fix}(U) = C$ and let $T : H_2 \to H_2$ be a quasi-nonexpansive operator with nonempty $\mathrm{Fix}(T) = Q$. Assume that $U - I$ and $T - I$ are demiclosed at 0. If $\Gamma \neq \emptyset$, then any sequence $\{x^k\}$ generated by Algorithm 4 weakly converges to a split common fixed point, provided that we choose θ_k satisfying $\theta_k \in [0, \overline{\theta}_k]$ with $\overline{\theta}_k := \min\{\theta, 1/(k\|x^k - x^{k-1}\|)^2\}$, $\theta \in [0,1)$. $\gamma \in (0, 1/(\lambda\eta))$ and $\alpha_k \in (\delta, 1-\delta)$ for a small enough $\delta > 0$.*

Proof. Taking $z \in \Gamma$, and using (2) in Lemma 2, we obtain

$$\begin{aligned}\left\|x^{k+1} - z\right\|^2 &= \left\|(1-\alpha_k)\, u\left(y^k\right) + \alpha_k U\left(u\left(y^k\right)\right) - z\right\|^2 \\ &\leq \left\|u\left(y^k\right) - z\right\|^2 - \alpha_k(1-\alpha_k)\left\|U\left(u\left(y^k\right)\right) - u\left(y^k\right)\right\|^2.\end{aligned} \tag{13}$$

On the other hand, we have

$$\begin{aligned}\left\|u\left(y^k\right) - z\right\|^2 &= \left\|y^k + \gamma\eta A^*(T-I)\left(Ay^k\right) - z\right\|^2 \\ &= \left\|y^k - z\right\|^2 + \gamma^2\eta^2\left\|A^*(T-I)\left(Ay^k\right)\right\|^2 \\ &\quad + 2\gamma\eta\left\langle y^k - z, A^*(T-I)\left(Ay^k\right)\right\rangle \\ &\leq \left\|y^k - z\right\|^2 + \lambda\gamma^2\eta^2\left\|(T-I)\left(Ay^k\right)\right\|^2 \\ &\quad + 2\gamma\eta\left\langle Ay^k - Az, (T-I)\left(Ay^k\right)\right\rangle,\end{aligned} \tag{14}$$

that is,

$$\begin{aligned}\left\|u(y^k) - z\right\|^2 &\leq \left\|y^k - z\right\|^2 + \lambda\gamma^2\eta^2\left\|(T-I)\left(Ay^k\right)\right\|^2 \\ &\quad + 2\gamma\eta\left\langle Ay^k - Az, (T-I)\left(Ay^k\right)\right\rangle.\end{aligned} \tag{15}$$

Now, by setting $v := 2\gamma\eta\langle Ay^k - Az, (T-I)(Ay^k)\rangle$ and using (1) of Lemma 2, we obtain

$$\begin{aligned}v &= 2\gamma\eta\left\langle Ay^k - Az, (T-I)\left(Ay^k\right)\right\rangle \\ &= 2\gamma\eta\left\langle Ay^k - Az + (T-I)\left(Ay^k\right) \right. \\ &\qquad \left. - (T-I)\left(Ay^k\right), (T-I)\left(Ay^k\right)\right\rangle \\ &= 2\gamma\eta\left(\left\langle T\left(Ay^k\right) - Az, (T-I)\left(Ay^k\right)\right\rangle \right. \\ &\qquad \left. - \left\|(T-I)\left(Ay^k\right)\right\|^2\right) \\ &\leq 2\gamma\eta\left(\frac{1}{2}\left\|(T-I)\left(Ay^k\right)\right\|^2 - \left\|(T-I)\left(Ay^k\right)\right\|^2\right) \\ &\leq -\gamma\eta\left\|(T-I)\left(Ay^k\right)\right\|^2.\end{aligned} \tag{16}$$

Combining the key inequality above with (15) yields

$$\begin{aligned}\left\|x^{k+1} - z\right\|^2 &\leq \left\|y^k - z\right\|^2 \\ &\quad - \gamma\eta(1-\lambda\gamma\eta)\left\|(T-I)\left(Ay^k\right)\right\|^2 \\ &\quad - \alpha_k(1-\alpha_k)\left\|U\left(u\left(y^k\right)\right) - u\left(y^k\right)\right\|^2.\end{aligned} \tag{17}$$

Define the auxiliary real sequence $\varphi_k := (1/2)\|x^k - z\|^2$. Therefore, from (17), we have

$$\begin{aligned}\varphi_{k+1} &\leq \frac{1}{2}\left\|y^k - z\right\|^2 \\ &\quad - \frac{1}{2}\gamma\eta(1-\lambda\gamma\eta)\left\|(T-I)\left(Ay^k\right)\right\|^2 \\ &\quad - \frac{1}{2}\alpha_k(1-\alpha_k)\left\|U\left(u\left(y^k\right)\right) - u\left(y^k\right)\right\|^2.\end{aligned} \tag{18}$$

By deducing, we have

$$\begin{aligned}\frac{1}{2}\left\|y^k - z\right\|^2 &= \frac{1}{2}\left\|x^k + \theta_k\left(x^k - x^{k-1}\right) - z\right\|^2 \\ &= \frac{1}{2}\left\|x^k - z\right\|^2 + \theta_k\left\langle x^k - z, x^k - x^{k-1}\right\rangle \\ &\quad + \frac{\theta_k^2}{2}\left\|x^k - x^{k-1}\right\|^2 \\ &= \varphi_k + \theta_k\left\langle x^k - z, x^k - x^{k-1}\right\rangle \\ &\quad + \frac{\theta_k^2}{2}\left\|x^k - x^{k-1}\right\|^2.\end{aligned} \tag{19}$$

It is easy to check that $\varphi_k = \varphi_{k-1} + \langle x^k - z, x^k - x^{k-1}\rangle - (1/2)\|x^k - x^{k-1}\|^2$.

Hence,

$$\begin{aligned}\frac{1}{2}\left\|y^k - z\right\|^2 &= \varphi_k + \theta_k\left(\varphi_k - \varphi_{k-1}\right) \\ &\quad + \frac{\theta_k + \theta_k^2}{2}\left\|x^k - x^{k-1}\right\|^2.\end{aligned} \tag{20}$$

Putting (20) into (18), we get

$$\varphi_{k+1} \leq \varphi_k + \theta_k (\varphi_k - \varphi_{k-1}) + \frac{\theta_k + \theta_k^2}{2} \left\| x^k - x^{k-1} \right\|^2 - \frac{1}{2}\gamma\eta (1 - \lambda\gamma\eta) \left\| (T - I)(Ay^k) \right\|^2 - \frac{1}{2}\alpha_k (1 - \alpha_k) \left\| U(u(y^k)) - u(y^k) \right\|^2. \tag{21}$$

Since $\gamma \in (0, 1/(\lambda\eta))$, according to $\theta_k^2 \leq \theta_k$, $\alpha_k \in (0,1)$ and (21), we derive

$$\varphi_{k+1} \leq \varphi_k + \theta_k (\varphi_k - \varphi_{k-1}) + \theta_k \left\| x^k - x^{k-1} \right\|^2. \tag{22}$$

Evidently,

$$\sum_{k=1}^{+\infty} \theta_k \left\| x^k - x^{k-1} \right\|^2 < \infty, \tag{23}$$

due to $\theta_k \|x^k - x^{k-1}\|^2 \leq 1/k^2$. Let $\delta_k := \theta_k \|x^k - x^{k-1}\|^2$ in Lemma 3. We deduce that the sequence $\{\|x^k - z\|\}$ is convergent (hence, $\{x^k\}$ is bounded). By (23) and Lemma 3, we obtain $\sum_{k=1}^{+\infty} [\|x^k - z\|^2 - \|x^{k-1} - z\|^2]_+ < \infty$. By reason of (21), we have

$$\begin{aligned} &\frac{1}{2}\gamma\eta (1 - \lambda\gamma\eta) \left\| (T - I)(Ay^k) \right\|^2 \\ &\quad \leq \varphi_k - \varphi_{k+1} + \theta_k (\varphi_k - \varphi_{k-1}) + \theta_k \left\| x^k - x^{k-1} \right\|^2, \\ &\frac{1}{2}\alpha_k (1 - \alpha_k) \left\| U(u(y^k)) - u(y^k) \right\|^2 \\ &\quad \leq \varphi_k - \varphi_{k+1} + \theta_k (\varphi_k - \varphi_{k-1}) + \theta_k \left\| x^k - x^{k-1} \right\|^2. \end{aligned} \tag{24}$$

Hence,

$$\begin{aligned} &\sum_{k=1}^{+\infty} \frac{1}{2}\gamma\eta (1 - \lambda\gamma\eta) \left\| (T - I)(Ay^k) \right\|^2 < \infty, \\ &\sum_{k=1}^{+\infty} \frac{1}{2}\alpha_k (1 - \alpha_k) \left\| U(u(y^k)) - u(y^k) \right\|^2 < \infty. \end{aligned} \tag{25}$$

By $\gamma \in (0, 1/(\lambda\eta))$ and the assumption on α_k, we get

$$\left\| (T - I)(Ay^k) \right\|^2 \longrightarrow 0, \tag{26}$$

$$\left\| U(u(y^k)) - u(y^k) \right\|^2 \longrightarrow 0. \tag{27}$$

Denoting by x^* a weak-cluster point $\{x^k\}$, let $\{x^{k_\sigma}\}$ be a subsequence of $\{x^k\}$. Obviously,

$$w - \lim_{\sigma} y^{k_\sigma} = w - \lim_{\sigma} x^{k_\sigma} = x^*. \tag{28}$$

Then, from (26) and the demiclosedness of $T - I$ at 0, we obtain

$$T(Ax^*) = Ax^*, \tag{29}$$

it follows that $Ax^* \in Q$. □

Now, by setting $u_k = y^k + \gamma\eta A^*(T - I)(Ay^k)$, it follows that $w - \lim_\sigma u^{k_\sigma} = x^*$. By the demiclosedness of $U - I$ at 0, from (27), we have

$$U(x^*) = x^*. \tag{30}$$

Hence, $x^* \in C$, and therefore $x^* \in \Gamma$.

Since there is no more than one weak-cluster point, the weak convergence of the whole sequence $\{x^k\}$ follows by applying Lemma 5 with $S = \Gamma$.

Remark 7. Since the current value of $\|x^k - x^{k-1}\|$ is known before choosing the parameter θ_k, then θ_k is well-defined in Theorem 6. In fact, from the process of proof for Theorem 6, we can get the following assert: the convergence result of Theorem 6 always holds provided that we take $\theta_k \in [0, \theta]$, $\theta \in [0, 1)$, for all $k \geq 0$, with

$$\sum_{k=1}^{+\infty} \theta_k \left\| x^k - x^{k-1} \right\|^2 < \infty. \tag{31}$$

To conclude, we have proposed an algorithm for solving the SCFP in the wide class of quasi-nonexpansive operators and proved its convergence in general Hilbert spaces. Next, we will improve the algorithm to assure the strong convergence in infinite Hilbert spaces.

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