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*Research Article*

# A Best Proximity Point Result in Modular Spaces with the Fatou Property

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Consider a nonself-mapping $T : A \to B$, where $(A, B)$ is a pair of nonempty subsets of a modular space $X_\rho$. A best proximity point of $T$ is a point $z \in A$ satisfying the condition: $\rho(z - Tz) = \inf\{\rho(x - y) : (x, y) \in A \times B\}$. In this paper, we introduce the class of proximal quasicontraction nonself-mappings in modular spaces with the Fatou property. For such mappings, we provide sufficient conditions assuring the existence and uniqueness of best proximity points.

## 1. Introduction and Preliminaries

Through this paper, we denote by $\mathbb{N}$ the set of positive integers including zero. Let $X$ be a vector space over $\mathbb{R}$. We denote by $0_X$ its zero vector. According to Orlicz [1], a functional $\rho : X \to [0, \infty]$ is said to be modular, if, for any pair $(x, y) \in X^2$, the following conditions are satisfied:

(i) $\rho(x) = 0$ if and only if $x = 0_X$;

(ii) $\rho(-x) = \rho(x)$;

(iii) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ whenever $\alpha, \beta \ge 0$ and $\alpha + \beta = 1$.

If $\rho$ is a modular in $X$, then the set

$$X_\rho := \{x \in X : \rho(\lambda x) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\}, \qquad (1)$$

called a modular space, is a vector space.

As a classical example of modulars, we may give the Orlicz modular defined for every measurable real function $f$ by

$$\rho_\varphi(f) = \int_{\mathbb{R}} \varphi(|f(t)|)\, d\lambda(t), \qquad (2)$$

where $\lambda$ is the Lebesgue measure in $\mathbb{R}$ and $\varphi : \mathbb{R} \to [0, \infty)$ is a function satisfying some conditions. The modular space induced by the Orlicz modular $\rho_\varphi$ is called the Orlicz space. For more examples of modular spaces, we refer the reader to [2–4].

*Definition 1.* Let $X_\rho$ be a modular space.

(1) The sequence $\{x_n\} \subset X_\rho$ is said to be $\rho$-convergent to $x \in X_\rho$ if $\rho(x_n - x) \to 0$, as $n \to \infty$.

(2) The sequence $\{x_n\} \subset X_\rho$ is said to be $\rho$-Cauchy if $\rho(x_n - x_m) \to 0$, as $n, m \to \infty$.

(3) A subset $C$ of $X_\rho$ is called $\rho$-closed if the $\rho$-limit of a $\rho$-convergent sequence of $C$ always belongs to $C$.

(4) A subset $C$ of $X_\rho$ is called $\rho$-complete if any $\rho$-Cauchy sequence in $C$ is $\rho$-convergent and its $\rho$-limit belongs to $C$.

*Definition 2.* The modular $\rho$ has the Fatou property if $\rho(x) \le \liminf_{n\to\infty} \rho(x_n)$ whenever $\{x_n\}$ $\rho$-converges to $x$.

Recently, the existence and uniqueness of best proximity points in metric spaces were investigated by many authors; see [2, 5–14] and references therein. In this paper, we introduce the family of proximal quasicontraction nonself-mappings on modular spaces with the Fatou property. Our main result is a best proximity point theorem providing sufficient conditions assuring the existence and uniqueness of best proximity points for such mappings.

Let $(A, B)$ be a pair of nonempty closed subsets of a modular space $X_\rho$. Through this paper, we will use the following notations:

$$\gamma(A,B) := \inf\{\rho(x-y) : (x,y) \in A \times B\},$$
$$A_0 := \{a \in A : \rho(a-b) = \gamma(A,B) \text{ for some } b \in B\}, \quad (3)$$
$$B_0 := \{b \in B : \rho(a-b) = \gamma(A,B) \text{ for some } a \in A\}.$$

*Definition 3.* Let $T : A \to B$ be a given nonself-mapping. We say that $z \in A_0$ is a best proximity point of $T$ if

$$\rho(z - Tz) = \gamma(A,B). \quad (4)$$

Clearly, from condition (i), if $A = B$, a best proximity point of $T$ will be a fixed point of $T$.

*Definition 4.* A nonself-mapping $T : A \to B$ is said to be a proximal quasicontraction if there exists a number $q \in (0,1)$ such that

$$\left.\begin{array}{l}\rho(u-Tx) = \gamma(A,B)\\ \rho(v-Ty) = \gamma(A,B)\end{array}\right\} \Longrightarrow \rho(u-v) \leq q\max\{\rho(x-y), \rho(x-u), \rho(y-v), \rho(x-v), \rho(y-u)\}, \quad (5)$$

where $x, y, u, v \in A$.

**Lemma 5.** *Let $T : A \to B$ be a nonself-mapping. Suppose that*

(i) $A_0 \neq \emptyset$;

(ii) $T(A_0) \subseteq B_0$.

*Then, for any $a \in A_0$, there exists a sequence $\{x_n\} \subset A_0$ such that*

$$x_0 = a,$$
$$\rho(x_{n+1} - Tx_n) = \gamma(A,B), \quad \forall n \in \mathbb{N}. \quad (6)$$

*Proof.* Let $a \in A_0$. From (ii), we have $Ta \in B_0$. By definition of the set $B_0$, there exists $x_1 \in A_0$ such that $\rho(x_1 - Ta) = \gamma(A,B)$. Again, we have $Tx_1 \in B_0$, which implies that there exists $x_2 \in A_0$ such that $\rho(x_2 - Tx_1) = \gamma(A,B)$. Continuing this process, by induction, we obtain a sequence $\{x_n\} \subset A_0$ satisfying (6). □

*Definition 6.* Under the assumptions of Lemma 5, any sequence $\{x_n\} \subset A_0$ satisfying (6) is called a proximal Picard sequence associated to $a \in A_0$. We denote by $PP(a)$ the set of all proximal sequences associated to $a \in A_0$.

*Definition 7.* Under the assumptions of Lemma 5, we say that $A_0$ is proximal $T$-orbitally $\rho$-complete if every $\rho$-Cauchy sequence $\{x_n\} \in PP(a)$ for some $a \in A_0$ $\rho$-converges to an element in $A_0$.

Let $a \in A_0$ and $\{x_n\} \in PP(a)$. For all $n \in \mathbb{N}$, We denote

$$\delta_p(x_n) := \sup\{\rho(x_{n+s} - x_{n+r}) : r, s \in \mathbb{N}\}. \quad (7)$$

Since $x_0 = a$, we have

$$\delta_p(a) = \sup\{\rho(x_s - x_r) : r, s \in \mathbb{N}\}. \quad (8)$$

## 2. A Best Proximity Point Theorem

The following lemmas will be useful later.

**Lemma 8.** *Let $X_\rho$ be a modular space. Suppose that a nonself-mapping $T : A \to B$, where $(A, B)$ is a pair of subsets of $X$, satisfies the following conditions:*

(i) $\exists a \in A_0 \mid \delta_p(a) < \infty$;

(ii) $T(A_0) \subseteq B_0$;

(iii) *$T$ is proximal quasi-contraction.*

*Then, for any $\{x_n\} \in PP(a)$, one has*

$$\rho(x_n - x_{n+m}) \leq \delta_p(x_n) \leq q^n \delta_p(a), \quad (9)$$

*for any $n \geq 1$ and $m \in \mathbb{N}$.*

*Proof.* Let $\{x_n\} \in PP(a)$ and $(s,r) \in \mathbb{N}^2$. From the definition of $PP(a)$, for all $n \geq 1$, we have

$$\rho(x_{n+s} - Tx_{n-1+s}) = \rho(x_{n+r} - Tx_{n-1+r}) = \gamma(A,B), \quad (10)$$

which implies, since $T$ is a proximal quasi-contraction, that

$$\begin{aligned}\rho(x_{n+s} - x_{n+r}) \leq q\max\{&\rho(x_{n-1+s} - x_{n-1+r}),\\ &\rho(x_{n-1+s} - x_{n+s}),\\ &\rho(x_{n-1+r} - x_{n+r}),\\ &\rho(x_{n-1+s} - x_{n+r}),\\ &\rho(x_{n-1+r} - x_{n+s})\}\\ \leq q\delta_p&(x_{n-1}).\end{aligned} \quad (11)$$

This implies immediately that

$$\delta_p(x_n) \leq q\delta_p(x_{n-1}), \quad (12)$$

for all $n \geq 1$. Hence, for any $n \in \mathbb{N}$, we have

$$\delta_p(x_n) \leq q^n \delta_p(a). \quad (13)$$

Using the above inequality, for all $n \geq 1$ and $m \in \mathbb{N}$, we have

$$\rho(x_n - x_{n+m}) \leq \delta_p(x_n) \leq q^n \delta_p(a). \quad (14)$$

□

**Lemma 9.** *Let $(A, B)$ be a pair of subsets of a modular space $X_\rho$. Let $T : A \to B$ be a given nonself-mapping. Suppose that*

(i) *$A_0$ is proximal $T$-orbitally $\rho$-complete;*

(ii) $T(A_0) \subseteq B_0$;

(iii) $\exists a \in A_0$ *such that* $\delta_p(a) < \infty$;

(iv) $T$ *is proximal quasi-contraction;*

(v) $\rho$ *satisfies the Fatou property.*

*Then, any sequence* $\{x_n\} \in PP(a)$ $\rho$*-converges to some* $z \in A_0$ *such that*

$$\rho(x_n - z) \le q^n \delta_p(a), \tag{15}$$

*for all* $n \ge 1$*. Moreover, there exists* $w \in A_0$ *such that*

$$\rho(w - Tz) = \gamma(A, B). \tag{16}$$

*Proof.* Let $\{x_n\} \in PP(a)$. From Lemma 8, we know that $\{x_n\}$ is $\rho$-Cauchy. Since $A_0$ is proximal $T$-orbitally $\rho$-complete, then there exists $z \in A_0$ such that $\{x_n\}$ $\rho$-converges to $z$. Again, by Lemma 8, we have

$$\rho(x_n - x_{n+m}) \le q^n \delta_p(a), \tag{17}$$

for any $n \ge 1$ and $m \in \mathbb{N}$. Letting $m \to \infty$ in the above inequality and using the Fatou property, we obtain

$$\rho(x_n - z) \le q^n \delta_p(a), \tag{18}$$

for all $n \ge 1$. Now, since $Tz \in B_0$, by the definition of $B_0$, there exists some $w \in A_0$ such that $\rho(w - Tz) = \gamma(A, B)$. □

Now, we are ready to state and prove our main result.

**Theorem 10.** *Suppose that the assumptions of the previous lemma are satisfied. Assume* $\rho(z-w) < \infty$ *and* $\rho(a-w) < \infty$*. Then, the* $\rho$*-limit* $z \in A_0$ *of* $\{x_n\} \in PP(a)$ *is a best proximity point of* $T$*. Moreover, if* $u \in A_0$ *is any best proximity point of* $T$ *such that* $\rho(z-u) < \infty$*, then one has* $z = u$*.*

*Proof.* By Lemma 9, we have

$$\rho(w - Tz) = \gamma(A, B). \tag{19}$$

On the other hand, from the definition of $\{x_n\}$, we have

$$\rho(x_1 - Ta) = \gamma(A, B). \tag{20}$$

Since $T$ is proximal quasi-contraction, we get that

$$\begin{aligned}\rho(w - x_1) \le q \max\{&\rho(a - z), \rho(z - w), \\ &\rho(a - x_1), \rho(z - x_1), \rho(a - w)\}.\end{aligned} \tag{21}$$

Using Lemmas 8 and 9, we obtain that

$$\rho(w - x_1) \le \max\left\{q\delta_p(a), q\rho(z - w), q\rho(a - w)\right\}. \tag{22}$$

Again, from the definition of $\{x_n\}$, we have

$$\rho(x_2 - Tx_1) = \gamma(A, B). \tag{23}$$

Since $T$ is proximal quasi-contraction, we get that

$$\begin{aligned}\rho(w - x_2) &\le q \max\{\rho(z - x_1), \rho(z - w), \rho(x_1 - x_2), \\ &\qquad \rho(z - x_2), \rho(x_1 - w)\} \\ &\le q \max\left\{q\delta_p(a), \rho(z - w), \delta_p(x_1), \right. \\ &\qquad \left. q^2\delta_p(a), \rho(x_1 - w)\right\} \\ &\le q \max\left\{q\delta_p(a), \rho(z - w), q\delta_p(a), \right. \\ &\qquad \left. q^2\delta_p(a), \rho(x_1 - w)\right\} \\ &= q \max\left\{q\delta_p(a), \rho(z - w), \rho(x_1 - w)\right\} \\ (\text{from } (22)) &\le \max\left\{q^2\delta_p(a), q\rho(z - w), q^2\rho(a - w)\right\}.\end{aligned} \tag{24}$$

Thus, we proved that

$$\rho(w - x_2) \le \max\left\{q^2\delta_p(a), q\rho(z - w), q^2\rho(a - w)\right\}. \tag{25}$$

Continuing this process, by induction, we get that

$$\rho(w - x_n) \le \max\left\{q^n\delta_p(a), q\rho(z - w), q^n\rho(a - w)\right\}, \tag{26}$$

for all $n \ge 1$. Therefore, we have

$$\limsup_{n\to\infty} \rho(x_n - w) \le q\rho(z - w). \tag{27}$$

Using the Fatou property, we get

$$\rho(z - w) \le q\rho(z - w), \tag{28}$$

which implies, since $q < 1$, that $\rho(z - w) = 0$; that is, $z = w$. Thus, from (19), we get that

$$\rho(z - Tz) = \gamma(A, B). \tag{29}$$

Hence, $z$ is a best proximity point of $T$.

Suppose now that $u \in A_0$ is a best proximity point of $T$ such that $\rho(z-u) < \infty$. Since $T$ is proximal quasi-contraction, we obtain that

$$\begin{aligned}\rho(z - u) &\le q \max\{\rho(z - u), \rho(z - z), \rho(u - u), \\ &\qquad \rho(z - u), \rho(u - z)\} \\ &= q\rho(z - u).\end{aligned} \tag{30}$$

Since $q < 1$, we have $\rho(z - u) = 0$, which implies that $u = z$. □

Consider now the case $A = B$. In this case, a best proximity point of $T : A \to B$ will be a fixed point of the self-mapping $T$.

*Definition 11.* We say that $A$ is $T$-orbitally $\rho$-complete if $\{T^{na}\}$ is a $\rho$-Cauchy for every $a \in A$, then it is $\rho$-convergent to an element of $A$.

Similarly to Ćirić [15] definition, Khamsi [16] introduced the concept of quasicontraction self-mappings in modular spaces.

*Definition 12.* The self-mapping $T : A \to A$ is said to be a quasicontraction if there exists a constant $q \in (0,1)$ such that

$$\rho(Tx - Ty) \le q \max\{\rho(x - y), \rho(x - Tx), \rho(y - Ty), \rho(x - Ty), \rho(y - Tx)\}, \tag{31}$$

for all $x, y \in A$.

From Theorem 10, we can deduce the following result, that is, a slight extension of the fixed point theorem established by Khamsi in [16].

**Corollary 13.** *Consider a self-mapping $T : A \to A$, where $A$ is a nonempty subset of $X_\rho$. Suppose that the following conditions hold:*

(i) *$A$ is $T$-orbitally $\rho$-complete;*

(ii) *$\exists a \in A$ such that* $\sup\{\rho(T^s a - T^r a) : s, r \in \mathbb{N}\} < \infty$*;*

(iii) *$\rho$ satisfies the Fatou property;*

(iv) *$T$ is quasi-contraction.*

*Then, the sequence $\{T^{na}\}$ $\rho$-converges to some $z \in A$. Moreover, if $\rho(z - Tz) < \infty$ and $\rho(a - Tz) < \infty$, then $z$ is a fixed point of $T$. If $u \in A$ is a fixed point of $T$ with $\rho(z - u) < \infty$, then $u = z$.*

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