

Research Article

Positive Periodic Solution for Second-Order Singular Semipositone Differential Equations

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We study the existence of a positive periodic solution for second-order singular semipositone differential equation by a nonlinear alternative principle of Leray-Schauder. Truncation plays an important role in the analysis of the uniform positive lower bound for all the solutions of the equation. Recent results in the literature (Chu et al., 2010) are generalized.

1. Introduction

In this paper, we study the existence of positive T -periodic solutions for the following singular semipositone differential equation:

$$x'' + h(t)x' + a(t)x = f(t, x, x'), \quad (1)$$

where $h, a \in C(R/TZ, R)$ and the nonlinearity $f \in C((R/TZ) \times (0, +\infty) \times R, R)$ satisfies $f(t, x, x') \geq -M$ for some $M > 0$. In particular, the nonlinearity may have a repulsive singularity at $x = 0$, which means that

$$\lim_{x \rightarrow 0^+} f(t, x, y) = +\infty, \quad \text{uniformly in } (t, y) \in R^2. \quad (2)$$

Electrostatic or gravitational forces are the most important examples of singular interactions.

During the last two decades, the study of the existence of periodic solutions for singular differential equations has attracted the attention of many researchers [1–4]. Some strong force conditions introduced by Gordon [5] are standard in the related earlier works [6, 7]. Compared with the case of a strong singularity, the study of the existence of periodic solutions under the presence of a weak singularity is more recent [2, 8, 9], but has also attracted many researchers. Some classical tools have been used to study singular differential equations in the literature, including the method of upper and lower solutions [10], degree theory [11], some fixed point theorem in cones for completely continuous operators

[12], Schauder's fixed point theorem [8, 9, 13], and a nonlinear Leray-Schauder alternative principle [2, 3, 14, 15].

However the singular differential equations, in which there is the damping term, that is, the nonlinearity is dependent on the derivative, has not attracted much attention in the literature. Several existence results can be found in [14, 16, 17].

The aim of this paper is to further show that the nonlinear Leray-Schauder alternative principle can be applied to (1) in the semipositone cases, that is, $f(t, x, x') \geq -M$ for some $M > 0$.

The remainder of the paper is organized as follows. In Section 2, we state some known results. In Section 3, the main results of this paper are stated and proved. To illustrate our result, we select the following system:

$$x'' + h(t)x' + a(t)x = (1 + |x|^\gamma)(x^{-\alpha} + \mu x^\beta) + e(t), \quad (3)$$

where $\alpha > 1$, $\beta > 0$, $1 > \gamma \geq 0$, $\mu > 0$ is a positive parameter, $e(t)$ is a T -periodic function.

In this paper, let us fix some notations to be used in the following: given $\varphi \in L^1[0, T]$, we write $\varphi > 0$ if $\varphi \geq 0$ for almost everywhere $t \in [0, T]$ and it is positive in a set of positive measure. The usual L^p -norm is denoted by $\|\cdot\|_p$. p^* and p_* the essential supremum and infimum of a given function $p \in L^1[0, T]$, if they exist.

2. Preliminaries

We say that

$$x'' + h(t)x' + a(t)x = 0, \tag{4}$$

associated to the periodic boundary conditions

$$x(0) = x(T), \quad x'(0) = x'(T), \tag{5}$$

is nonresonant when its unique solutions is the trival one. When (4)-(5) is nonresonant, as a consequence of Fredholm's alternative, the nonhomogeneous equation

$$x'' + h(t)x' + a(t)x = l(t) \tag{6}$$

admits a unique T -periodic solution, which can be written as

$$x(t) = \int_0^T G(t,s)l(s) ds, \tag{7}$$

where $G(t,s)$ is the Green's function of problem (4)-(5). Throughout this paper, we assume that the following standing hypothesis is satisfied.

- (A) The Green function $G(t,s)$, associated with (4)-(5), is positive for all $(t,s) \in [0,T] \times [0,T]$.

In other words, the strict antimaximum principle holds for (4)-(5).

Definition 1. We say that (4) admits the antimaximum principle if (6) has a unique T -periodic solution for any $l \in \mathbb{C}(\mathbb{R}/T\mathbb{Z})$ and the unique T -periodic solution $x_l(t) > 0$ for all t if $l > 0$.

Under hypothesis (A), we denote

$$A = \min_{0 \leq s, t \leq T} G(t,s), \quad B = \max_{0 \leq s, t \leq T} G(t,s), \quad \iota = \frac{A}{B}. \tag{8}$$

Thus $B > A > 0$ and $0 < \iota < 1$. We also use $w(t)$ to denote the unique periodic solution of (6) with $l(t) = 1$ under condition (5), that is, $w(t) = (\mathcal{L}1)(t)$. In particular, $TA \leq w(t) \leq TB$.

With the help of [18, 19], the authors give a sufficient condition to ensure that (4) admits the antimaximum principle in [14]. In order to state this result, let us define the functions

$$\sigma(h)(t) = \exp\left(\int_0^t h(s) ds\right), \tag{9}$$

$$\sigma_1(h)(t) = \sigma(h)(T) \int_0^t \sigma(h)(s) ds + \int_t^T \sigma(h)(s) ds.$$

Lemma 2 (see [14, Corollary 2.6]). *Assume that $a \not\equiv 0$ and the following two inequalities are satisfied:*

$$\int_0^T a(s)\sigma(h)(s)\sigma_1(-h)(s) ds \geq 0, \tag{10}$$

$$\sup_{0 \leq t \leq T} \left\{ \int_t^{t+T} \sigma(-h)(s) ds \int_t^{t+T} [a(s)]_+ \sigma(h)(s) ds \right\} \leq 4,$$

where $[a(s)]_+ = \max\{a(s), 0\}$. Then the Green's function $G(t,s)$, associated with (5), is positive for all $(t,s) \in [0,T] \times [0,T]$.

Next, recall a well-known nonlinear alternative principle of Leray-Schauder, which can be found in [20] and has been used by Meehan and O'Regan in [4].

Lemma 3. *Assume Ω is an open subset of a convex set K in a normed linear space X and $p \in \Omega$. Let $T : \overline{\Omega} \rightarrow K$ be a compact and continuous map. Then one of the following two conclusions holds:*

- (I) T has at least one fixed point in $\overline{\Omega}$.
- (II) There exists $x \in \partial\Omega$ and $0 < \lambda < 1$ such that $x = \lambda Tx + (1 - \lambda)p$.

In applications below, we take $K = C_T^1 = \{x : x, x' \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})\} \subset X$ with the norm $\|x\| = \max_{t \in [0,T]} |x(t)|$ and define $\Omega = \{x \in C_T^1 : \|x\| < r\}$.

3. Main Results

In this section, we prove a new existence result of (1).

Theorem 4. *Suppose that (4) satisfies (A) and*

$$a(t) > 0. \tag{11}$$

Furthermore, assume that there exist three constants $M, R_0, r > Mw^*/\iota$ such that:

- (H₁) $F(t,x,y) = f(t,x,y) + M \geq 0$ for all $(t,x,y) \in [0,T] \times (0,r] \times \mathbb{R}$.
- (H₂) $f(t,x,y) \geq g_0(x)$ for $(t,x,y) \in [0,T] \times (0,R_0) \times \mathbb{R}$, where the nonincreasing continuous function $g_0(x) > 0$ satisfies $\lim_{x \rightarrow 0^+} g_0(x) = +\infty$ and $\lim_{x \rightarrow 0^+} \int_x^{R_0} g_0(u) du = +\infty$.
- (H₃) $0 \leq F(t,x,y) \leq (g(x) + h(x))\varrho(|y|)$, for all $(t,x,y) \in [0,T] \times (0,r] \times \mathbb{R}$, where $g(\cdot) > 0$ is nonincreasing in $(0,r]$ and $h(\cdot)/g(\cdot) \geq 0, \varrho(\cdot) \geq 0$ are nondecreasing in $(0,r]$.
- (H₄)

$$\frac{r}{g(r - Mw^*)(1 + h(r)/g(r))\varrho((r + M)L)} > w^*, \tag{12}$$

where

$$L = \frac{2 \int_0^T a(t)\sigma(h)(t) dt}{\min_{0 \leq t \leq T} \sigma(h)(t)}. \tag{13}$$

Then (1) has at least one positive periodic solution $u(t)$ with $0 < \|u + Mw\| \leq r$.

Proof. For convinence, let us write $Z(t) = x(t) - Mw(t)$, $Z_n(t) = x_n(t) - Mw(t)$, where $w(t) = (\mathcal{L}1)(t)$. Let

$$A_h = \min_{0 \leq t \leq T} \sigma(h)(t), \quad B_h = \max_{0 \leq t \leq T} \sigma(h)(t), \quad \iota_h = \frac{B_h}{A_h}, \tag{14}$$

$$\overline{M}_h = (r + M)L \cdot \max_{0 \leq t \leq T} |h(t)|. \tag{15}$$

First we show that

$$x'' + h(t)x' + a(t)x = F(t, Z(t), Z'(t)) \tag{16}$$

has a solution x satisfying (5), $0 < \|x\| \leq r$ and $Z(t) > 0$ for $t \in [0, T]$. If this is true, it is easy to see that $Z(t)$ will be a positive solution of (1)–(5) with $0 < \|Z + Mw\| \leq r$.

Choose $n_0 \in \{1, 2, \dots\}$ such that $1/n_0 < r$, and then let $N_0 = \{n_0, n_0 + 1, \dots\}$.

Consider the family of equations

$$x'' + h(t)x' + a(t)x = \lambda F_n(t, Z(t), Z'(t)) + \frac{a(t)}{n}, \tag{17}$$

where $\lambda \in [0, 1]$, $n \in N_0$, $x \in B_r = \{x : \|x\| < r\}$ and $F_n(t, x, y) = F(t, \max\{1/n, x\}, y)$.

A T -periodic solution of (17) is just a fixed of the operator equation

$$x = \lambda T_n(x) + (1 - \lambda)p, \tag{18}$$

where $p = 1/n$ and T_n is a completely continuous operator defined by

$$(T_n x)(t) = \int_0^T G(t, s) F_n(s, Z(s), Z'(s)) ds + \frac{1}{n}, \tag{19}$$

where we have used the fact

$$\int_0^T G(t, s) a(s) ds \equiv 1. \tag{20}$$

We claim that for any T -periodic solution $x_n(t)$ of (17) satisfies

$$\|x'_n\| \leq Lr. \tag{21}$$

Note that the solution $x_n(t)$ of (17) is also satisfies the following equivalent equation

$$\begin{aligned} & (\sigma(h)(t)x'_n)' + a(t)\sigma(h)(t)x_n \\ & = \sigma(h)(t) \left(\lambda F_n(t, Z_n(t), Z'_n(t)) + \frac{a(t)}{n} \right). \end{aligned} \tag{22}$$

Integrating (22) from 0 to T , we obtain

$$\begin{aligned} & \int_0^T a(t)\sigma(h)(t)x_n(t) dt \\ & = \int_0^T \sigma(h)(t) \left(\lambda F_n(t, Z_n(t), Z'_n(t)) + \frac{a(t)}{n} \right) dt. \end{aligned} \tag{23}$$

By the periodic boundary conditions, we have $x'(t_0) = 0$ for some $t_0 \in [0, T]$. Therefore,

$$\begin{aligned} & |\sigma(h)(t)x'_n(t)| \\ & = \left| \int_{t_0}^t (\sigma(h)(s)x'_n(s))' ds \right| \\ & = \left| \int_{t_0}^t \sigma(h)(s) \left(\lambda F_n(s, Z_n(s), Z'_n(s)) + \frac{a(s)}{n} - a(s)x_n(s) \right) ds \right| \\ & \leq \left| \int_0^T \sigma(h)(s) \left(\lambda F_n(s, Z_n(s), Z'_n(s)) + \frac{a(s)}{n} + a(s)x_n(s) \right) ds \right| \\ & = 2 \int_0^T \sigma(h)(s)a(s)x_n(s) ds \\ & \leq 2r \int_0^T \sigma(h)(s)a(s) ds, \end{aligned} \tag{24}$$

where we have used the assumption (11) and $\|x_n\| < r$. Therefore,

$$\left(\min_{0 \leq t \leq T} \sigma(h)(t) \right) |x'_n(t)| \leq 2r \int_0^T \sigma(h)(s)a(s) ds, \tag{25}$$

which implies that (21) holds. In particular, let $\lambda F_n(t, Z(t), Z'(t)) + a(t)/n = 1$ in (17), we have

$$\|w'(t)\| \leq L. \tag{26}$$

Choose $n_1 \in N_0$ such that $1/n_1 \leq R_1$, and then let $N_1 = \{n_1, n_1 + 1, \dots\}$. The following lemma holds. \square

Lemma 5. *There exists an integer $n_2 > n_1$ large enough such that, for all $n \in N_2 = \{n_2, n_2 + 1, \dots\}$,*

$$Z_n(t) = x_n(t) - Mw(t) \geq \frac{1}{n}. \tag{27}$$

Proof. The lower bound in (27) is established by using the strong force condition of $f(t, x, y)$. By condition (H_2) , there exists $R_1 \in (0, R_0)$ and a continuous function $\tilde{g}_0(x)$ such that

$$F(t, x, y) - a(t)x \geq \tilde{g}_0(x) > \max \{M + \bar{M}, \iota_h r \|a\|_1\} \tag{28}$$

for all $(t, x, y) \in [0, T] \times (0, R_1) \times R$, where $\tilde{g}_0(x)$ satisfies also the strong force condition like in (H_2) .

For $n \in N_1$, let $\alpha_n = \min_{0 \leq t \leq T} Z_n(t)$, $\beta_n = \max_{0 \leq t \leq T} Z_n(t)$.

If $\alpha_n \geq R_1$, due to $n \in N_1$, (27) holds.

If $\alpha_n < R_1$, we claim that, for all $n \in N_1$,

$$\beta_n > R_1. \tag{29}$$

Otherwise, suppose that $\beta_n \leq R_1$ for some $n \in N_1$. Then it is easy to verify

$$F_n(t, Z_n(t), Z'_n(t)) > \iota_h r \|a\|_1. \tag{30}$$

In fact, if $1/n \leq Z_n(t) \leq R_1$, we obtain from (28)

$$\begin{aligned} & F_n(t, Z_n(t), Z'_n(t)) = F(t, Z_n(t), Z'_n(t)) \\ & \geq a(t)Z_n(t) + \tilde{g}_0(Z_n(t)) \\ & \geq \tilde{g}_0(Z_n(t)) \\ & > \iota_h r \|a\|_1. \end{aligned} \tag{31}$$

and, if $Z_n(t) \leq 1/n$, we have

$$\begin{aligned} F_n(t, Z_n(t), Z'_n(t)) &= F\left(t, \frac{1}{n}, Z'_n(t)\right) \geq \frac{a(t)}{n} + \tilde{g}_0\left(\frac{1}{n}\right) \\ &\geq \tilde{g}_0\left(\frac{1}{n}\right) > \iota_h r \|a\|_1. \end{aligned} \tag{32}$$

Integrating (22) (with $\lambda = 1$) from 0 to T , we deduce that

$$\begin{aligned} 0 &= \int_0^T \left\{ (\sigma(h)(t) x'_n)' + a(t) \sigma(h)(t) x_n \right. \\ &\quad \left. - \sigma(h)(t) \left(F_n(t, Z_n(t), Z'_n(t)) + \frac{a(t)}{n} \right) \right\} dt \\ &= \int_0^T a(t) \sigma(h)(t) x_n dt \\ &\quad - \int_0^T \sigma(h)(t) F_n(t, Z_n(t), Z'_n(t)) dt \\ &\quad - \int_0^T \sigma(h)(t) \frac{a(t)}{n} dt \\ &< \int_0^T a(t) \sigma(h)(t) x_n dt \\ &\quad - \int_0^T \sigma(h)(t) F_n(t, Z_n(t), Z'_n(t)) dt \\ &< 0, \end{aligned} \tag{33}$$

where estimation (30) and the fact $\|x_n\| < r$ are used. This is a contradiction. Hence (29) holds.

Due to $\alpha_n < R_1$, that is, $\alpha_n = \min_{0 \leq t \leq T} [x_n(t) - Mw(t)] = x_n(a_n) - Mw(a_n) < R_1$ for some $a_n \in [0, T]$. By (29), there exists $c_n \in [0, T]$ (without loss of generality, we assume $a_n < c_n$.) such that $x_n(c_n) = Mw(c_n) + R_1$ and $x_n(t) \leq Mw(t) + R_1$ for $a_n \leq t \leq c_n$.

It can be checked that

$$F_n(t, Z_n(t), Z'_n(t)) > a(t) Z_n(t) + M + \overline{M}_h, \tag{34}$$

where \overline{M}_h is defined by (15).

In fact, if $t \in [a_n, c_n]$ is such that $1/n \leq Z_n(t) \leq R_1$, we have

$$\begin{aligned} F_n(t, Z_n(t), Z'_n(t)) &= F(t, Z_n(t), Z'_n(t)) \\ &\geq a(t) Z_n(t) + \tilde{g}_0(x) > \max\{M + \overline{M}, \iota_h r \|a\|_1\} \\ &\geq a(t) Z_n(t) + M + \overline{M}. \end{aligned} \tag{35}$$

and, if $t \in [a_n, c_n]$ is such that $Z_n(t) \leq 1/n$, we have

$$\begin{aligned} F_n(t, Z_n(t), Z'_n(t)) &= F\left(t, \frac{1}{n}, Z'_n(t)\right) \\ &\geq \frac{a(t)}{n} + \tilde{g}_0\left(\frac{1}{n}\right) > \frac{a(t)}{n} + M + \overline{M}_h \\ &\geq a(t) Z_n(t) + M + \overline{M}_h. \end{aligned} \tag{36}$$

So (34) holds.

Using (17) (with $\lambda = 1$) for $x_n(t)$ and the estimation (34), we have, for $t \in [a_n, c_n]$

$$\begin{aligned} Z_n''(t) &= -h(t) Z'_n(t) - a(t) Z_n(t) \\ &\quad - M + F_n(t, Z_n(t), Z'_n(t)) + \frac{a(t)}{n} \\ &> -h(t) Z'_n(t) - a(t) Z_n(t) \\ &\quad - M + a(t) Z_n(t) + M + \overline{M}_h + \frac{a(t)}{n} \\ &\geq -\overline{M}_h - a(t) Z_n(t) - M \\ &\quad + a(t) Z_n(t) + M + \overline{M}_h + \frac{a(t)}{n} \\ &\geq \frac{a(t)}{n} \geq 0. \end{aligned} \tag{37}$$

As $Z'_n(a_n) = 0, Z'_n(t) > 0$ for all $t \in [a_n, c_n]$, so $Z_n(t)$ is strictly increasing on $[a_n, c_n]$. We use ξ_n to denote the inverse function of Z_n restricted to $[a_n, c_n]$.

Suppose that (27) does not hold, that is, for some $n \in N_1, Z_n(t) < 1/n < R_1$. Then there would exist $b_n \in (a_n, c_n)$ such that $Z_n(b_n) = 1/n$ and

$$\begin{aligned} Z_n(t) &\leq \frac{1}{n} \quad \text{for } a_n \leq t \leq b_n, \\ \frac{1}{n} &\leq Z_n(t) \leq R_1 \quad \text{for } b_n \leq t \leq c_n. \end{aligned} \tag{38}$$

Multiplying (17) (with $\lambda = 1$) by $Z'_n(t)$ and integrating from b_n to c_n , we obtain

$$\begin{aligned} &\int_{1/n}^{R_1} F(\xi_n(Z), Z, Z') dZ \\ &= \int_{b_n}^{c_n} F(t, Z_n(t), Z'_n(t)) Z'_n(t) dt \\ &= \int_{b_n}^{c_n} F_n(t, Z_n(t), Z'_n(t)) Z'_n(t) dt \\ &= \int_{b_n}^{c_n} \left(x_n''(t) + h(t) x'_n(t) + a(t) x_n(t) - \frac{a(t)}{n} \right) Z'_n(t) dt \end{aligned}$$

$$\begin{aligned}
 &= \int_{b_n}^{c_n} x_n''(t) (x_n'(t) - Mw'(t)) dt + \int_{b_n}^{c_n} h(t) x_n'(t) Z_n'(t) dt \\
 &+ \int_{b_n}^{c_n} \left(a(t) x_n(t) - \frac{a(t)}{n} \right) Z_n'(t) dt.
 \end{aligned} \tag{39}$$

By the facts $\|x_n\| < r$, $\|x_n'\| \leq Lr$, $\|w'\| \leq r$ and the definition of $Z_n(t)$, we can obtain $|Z_n(t)| \leq r + TB$, $|Z_n'(t)| \leq (r+M)L$, together with $\|x_n\| < r$, implies that the second term and the third term are bounded. The first term is

$$\begin{aligned}
 &\frac{[x_n'(c_n)]^2 - [x_n'(b_n)]^2}{2} \\
 &- M(x_n'(c_n)w'(c_n) - x_n'(b_n)w'(b_n)) \\
 &+ M \int_{b_n}^{c_n} x_n'(t)w''(t) dt,
 \end{aligned} \tag{40}$$

which is also bounded. As a consequence, there exists a $B_1 > 0$ such that

$$\int_{1/n}^{R_1} F(\xi_n(Z), Z, Z') dZ \leq B_1. \tag{41}$$

On the other hand, by (H_2) , we can choose $n_2 \in N_1$ large enough such that

$$\int_{1/n}^{R_1} F(\xi_n(Z), Z, Z') dZ \geq \int_{1/n}^{R_1} g_0(Z) dZ > B_1 \tag{42}$$

for all $n \in N_2 = \{n_2, n_2 + 1, \dots\}$. So (27) holds. \square

Furthermore, we can prove $Z_n(t)$ has a uniform positive lower bound δ .

Lemma 6. *There exist a constant $\delta > 0$ such that, for all $n \in N_2$,*

$$Z_n(t) \geq \delta. \tag{43}$$

Proof. Multiplying (17) (with $\lambda = 1$) by $Z_n'(t)$ and integrating from a_n to c_n , we obtain

$$\begin{aligned}
 &\int_{a_n}^{R_1} F(\xi_n(Z), Z, Z') dZ \\
 &= \int_{a_n}^{c_n} F(t, Z_n(t), Z_n'(t)) Z_n'(t) dt \\
 &= \int_{a_n}^{c_n} F_n(t, Z_n(t), Z_n'(t)) Z_n'(t) dt \\
 &= \int_{a_n}^{c_n} \left(x_n''(t) + h(t) x_n'(t) + a(t) x_n(t) - \frac{a(t)}{n} \right) Z_n'(t) dt \\
 &= \int_{a_n}^{c_n} x_n''(t) (x_n'(t) - Mw'(t)) dt + \int_{a_n}^{c_n} h(t) x_n'(t) Z_n'(t) dt \\
 &+ \int_{a_n}^{c_n} \left(a(t) x_n(t) - \frac{a(t)}{n} \right) Z_n'(t) dt.
 \end{aligned} \tag{44}$$

In the same way as in the proof of (41), one way readily prove that the right-hand side of the above equality is bounded. On the other hand, if $n \in N_2$, by (H_2) ,

$$\begin{aligned}
 &\int_{\alpha_n}^{R_1} F(\xi_n(Z), Z, Z') dZ \\
 &\geq \int_{\alpha_n}^{R_1} g_0(Z) dZ + M(R_1 - \alpha_n) \longrightarrow +\infty
 \end{aligned} \tag{45}$$

if $\alpha_n \rightarrow 0_+$. Thus we know that there exists a constant $\delta > 0$ such that $\alpha_n \geq \delta$. Hence (43) holds.

Next, we will prove (17) has periodic solution $x_n(t)$.

For $\iota > 0$, we can choose $n_3 \in N_2$ such that $1/n_3 < \iota r$, which together with (H_4) imply

$$w^*g(\iota r - Mw^*) \left(1 + \frac{h(r)}{g(r)} \right) \varrho((r+M)L) + \frac{1}{n_3} < r. \tag{46}$$

Let $N_3 = \{n_3, n_3 + 1, \dots\}$. For $n \in N_3$, consider (17).

Next we claim that any fixed point x_n of (18) for any $\lambda \in [0, 1]$ must satisfy $\|x_n\| \neq r$. So, by using the Leray-Schauder alternative principle, (17) (with $\lambda = 1$) has a periodic solution $x_n(t)$. Otherwise, assume that x_n is a fixed point x_n of (18) for some $\lambda \in [0, 1]$ such that $\|x_n\| = r$. Note that

$$\begin{aligned}
 x_n(t) - \frac{1}{n} &= \lambda \int_0^T G(t, s) F_n(s, Z_n(s), Z_n'(s)) ds \\
 &\geq \lambda A \int_0^T F_n(s, Z_n(s), Z_n'(s)) ds \\
 &= \iota B \lambda \int_0^T F_n(s, Z_n(s), Z_n'(s)) ds \\
 &\geq \iota \max_{t \in [0, T]} \left\{ \lambda \int_0^T G(t, s) F_n(s, Z_n(s), Z_n'(s)) ds \right\} \\
 &= \iota \left\| x_n - \frac{1}{n} \right\|.
 \end{aligned} \tag{47}$$

For $n \in N_3$, we have

$$x_n(t) \geq \iota \left\| x_n - \frac{1}{n} \right\| + \frac{1}{n} \geq \iota \left(\|x_n\| - \frac{1}{n} \right) + \frac{1}{n} \geq \iota r. \tag{48}$$

By (27) and assumption (H_3) , for all $t \in [0, T]$ and $n \in N_3$, we have

$$\begin{aligned}
 x_n(t) &= \lambda \int_0^T G(t, s) F_n(s, Z_n(s), Z_n'(s)) ds + \frac{1}{n} \\
 &= \lambda \int_0^T G(t, s) F(s, Z_n(s), Z_n'(s)) ds + \frac{1}{n}
 \end{aligned}$$

$$\begin{aligned}
 &\leq \int_0^T G(t, s) F(s, Z_n(s), Z'_n(s)) ds + \frac{1}{n} \\
 &\leq \int_0^T G(t, s) (g(Z_n(s)) + h(Z_n(s))) \varrho(|Z'_n(s)|) ds \\
 &\quad + \frac{1}{n} \\
 &\leq \int_0^T G(t, s) g(Z_n(s)) \left(1 + \frac{h(Z_n(s))}{g(Z_n(s))}\right) \varrho(|Z'_n(s)|) ds \\
 &\quad + \frac{1}{n} \\
 &\leq \int_0^T G(t, s) g(ir - Mw^*) \left(1 + \frac{h(r)}{g(r)}\right) \varrho((r + M)L) ds \\
 &\quad + \frac{1}{n} \\
 &\leq g(ir - Mw^*) \left(1 + \frac{h(r)}{g(r)}\right) \varrho((r + M)L) w^* + \frac{1}{n_3}.
 \end{aligned} \tag{49}$$

Therefore,

$$r = \|x\| \leq g(ir - Mw^*) \left(1 + \frac{h(r)}{g(r)}\right) \varrho((r + M)L) w^* + \frac{1}{n_3}. \tag{50}$$

This is a contradiction to the choice of n_3 and the claim is proved.

The fact $\|x_n\| < r$ and $\|x'_n(t)\| < Lr$ show that $\{x_n\}_{n \in N_3}$ is a bounded and equicontinuous family on $[0, T]$. Now Arzela-Ascoli Theorem guarantees that $\{x_n\}_{n \in N_3}$ has a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$, converging uniformly on $[0, T]$ to a function $x \in C[0, T]$. From the fact $\|x_n\| < r$ and $x_n(t) > \delta$, x satisfies $\delta \leq x(t) \leq r$ for all t . Moreover, $\{x_{n_k}\}$ satisfies the integral equation

$$x_{n_k}(t) = \int_0^T G(t, s) F(s, Z_{n_k}(s), Z'_{n_k}(s)) ds + \frac{1}{n_k}. \tag{51}$$

Letting $k \rightarrow \infty$, we arrive at

$$x(t) = \int_0^T G(t, s) F(s, x(s) - Mw(s), x'(s) - Mw'(s)) ds, \tag{52}$$

where the uniform continuity of $F(t, x, y)$ on $[0, T] \times [\delta, r] \times [-(r + M)L, (r + M)L]$ is used. Therefore, x is a positive periodic solution of (16) and $Z(t) = x(t) - Mw(t) \geq \delta$. Thus we complete the prove of Theorem 4. \square

Corollary 7. *Let the nonlinearity in (1) be*

$$f(t, x, y) = (1 + |y|^\gamma) (x^{-\alpha} + \mu x^\beta) + e(t), \tag{53}$$

where $\alpha > 1, \beta > 0, 1 > \gamma \geq 0, \mu > 0$ is a positive parameter, $e(t)$ is a T -periodic function.

- (i) If $\beta + \gamma < 1$, then (1) has at least one positive periodic solution for each $\mu > 0$.
- (ii) If $\beta + \gamma \geq 1$, then (1) has at least one positive periodic solution for each $0 < \mu < \mu_1$, where μ_1 is some positive constant.

Proof. We will apply Theorem 4 with $M = \max_{0 \leq t \leq T} |e(t)|$ and $g(x) = x^{-\alpha}, h(x) = \mu x^\beta + 2M, \varrho(y) = 1 + |y|^\gamma$. Then condition (H₁)–(H₃) are satisfied and existence condition (H₄) becomes

$$\mu < \frac{r(ir - Mw^*)^\alpha - w^* (1 + (r + M)^\gamma L^\gamma) (1 + 2Mr^\alpha)}{w^* (1 + (r + M)^\gamma L^\gamma) r^{\alpha+\beta}}. \tag{54}$$

So (1) has at least one positive periodic solution for

$$\begin{aligned}
 &0 < \mu < \mu_1 \\
 &= \sup_{r > Mw^*/\mu} (r(ir - Mw^*)^\alpha - w^* (1 + (r + M)^\gamma L^\gamma) \\
 &\quad \times (1 + 2Mr^\alpha)) \\
 &\quad \times (w^* (1 + (r + M)^\gamma L^\gamma) r^{\alpha+\beta})^{-1}.
 \end{aligned} \tag{55}$$

Note that $\mu_1 = \infty$ if $\beta + \gamma < 1$ and $\mu_1 < \infty$ if $\beta + \gamma \geq 1$. We have the desired results. \square

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