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Research Article

Some Results on Fixed and Best Proximity Points of Precyclic Self-Mappings

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This paper is devoted to investigating the limit properties of distances and the existence and uniqueness of fixed points, best proximity points and existence, and uniqueness of limit cycles, to which the iterated sequences converge, of single-valued, and so-called, contractive precyclic self-mappings which are proposed in this paper. Such self-mappings are defined on the union of a finite set of subsets of uniformly convex Banach spaces under generalized contractive conditions. Each point of a subset is mapped either in some point of the same subset or in a point of the adjacent subset. In the general case, the contractive condition of contractive precyclic self-mappings is admitted to be point dependent and it is only formulated on a complete disposal, rather than on each individual subset, while it involves a condition on the number of iterations allowed within each individual subset before switching to its adjacent one. It is also allowed that the distances in-between adjacent subsets can be mutually distinct including the case of potential pairwise intersection for only some of the pairs of adjacent subsets.

1. Introduction

A relevant attention has been recently devoted to the research of existence and uniqueness of fixed points of self-mappings as well as to the investigation of associate relevant properties like, for instance, stability of the iterations [1–3]. The extension of such topics to the existence of either fixed points of multivalued self-mappings [1, 4–19], in generalized metric spaces [20, 21], or to the existence of common fixed points of several multivalued mappings or operators is receiving an important attention, for example, [7, 8, 15–19, 22] and references therein. Relevant properties on the existence and uniqueness of fixed points and best proximity points for multivalued cyclic self-mappings have been studied under general contractive-type conditions based on the Hausdorff metric between subsets of a metric space. See, for instance, [4, 7–9], including as a relevant particular case the contractive condition on contractive single-valued self-mappings, [1, 4–10], as well as concerns related to their extension to cyclic self-mappings. See, for instance, [7, 8, 11] and references there in. The various related performed researches include the cases of strict contractive cyclic self-mappings and Meir-Keeler type cyclic contractions [23, 24]. Also, some of the existing background fixed point results on contractive single and multivalued self-mappings, [1, 4, 5, 9, 10, 25–28] and references therein, under some types of contractive conditions, have been revisited and extended in [4]. There is also a wide sample of fixed point type results available on fixed points and asymptotic properties of the iterations for self-mappings satisfying a number of contractive-type conditions while being endowed with partial order conditions. See [18, 19] and references therein.

The main objective of this paper is the investigation of the properties of the distances as well as the existence and uniqueness of fixed points and best proximity points related to contractive so-called single-valued contractive $p(\geq 2)$-precyclic self-mappings $T : \bigcup_{i \in \overline{p}} A_i \rightarrow \bigcup_{i \in \overline{p}} A_i$. Such a concept extends that of contractive $p(\geq 2)$-cyclic self-mappings.

The concept of precyclic self-mapping generalizes that of cyclic self-mappings in the sense that a finite set of consecutive iterations are optionally allowed within a particular subset of the cyclic disposal of interest before a switching of the image of the self-mapping to the adjacent subset of its pre-image in the iterated sequence. It can also eventually happen that some sequence enters a certain subset and the solution remains permanent within such a subset. The precyclic self-mappings are contractive if they are subject to contractive conditions of similar types to those arising in contractive cyclic self-mappings.

Precyclic contractive self-mappings allow the generation of iterated sequences under constraints of the form $T^j(A_i) \subseteq A_i \cup A_{i+1}$ for $j = j(i, x)$ being less than a prescribed positive integer number $\overline{j} = \overline{j}\,(i, x)$; for all $x \in \bigcup_{i\in\overline{p}} A_i$, for all $i \in \overline{p}$ which can be set and point dependent, while $T^{\overline{j}}(A_i) \subseteq A_{i+1}$; for all $i \in \overline{p}$. The ordering of the subsets for switching between pairs of adjacent subsets to perform the p-precyclic self-mapping is, so-called, in the sequel a p-precyclic disposal.

Let $\mathbf{R}_{0+} = \mathbf{R}_+ \cup \{0\}$ be the set of nonnegative real numbers and $\mathbf{Z}_{0+} = \mathbf{Z}_+ \cup \{0\}$ the set of nonnegative integer numbers. Consider a metric space (X, d) endowed with a metric $d : X \times X \to \mathbf{R}_{0+}$ and a finite set of nonempty subsets A_i; $i \in \overline{p} = \{1, 2, \dots p\}$ of X and a so-called $p(\geq 2)$-precyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ such that $T(A_i) \subseteq A_i \cup A_{i+1}$; for all $i \in \overline{p}$, where $A_i = A_j$ for $i, j \in \overline{p}$ under the congruence relation $j \equiv i(\operatorname{mod} p)$, that is $A_{pn+i} = A_i$; for all $n \in \mathbf{Z}_{0+}$, for all $i \in \overline{p}$. Note that the previous concept of precyclic self-mapping generalizes that of a p-cyclic self-mapping which satisfies the stronger constraint $T(A_i) \subseteq A_{i+1}$; for all $i \in \overline{p}$. Let $D_{i,j} = d(A_i, A_j)$ be the distance in-between any two subsets A_i, $A_j \subset X$; for all i, $j \in \overline{p}$. Note that, compared to a cyclic self-mapping, an iterated sequence from a precyclic self-mapping might contain iterated subsequences of finite or infinite cardinals in a single subset $A_i \subset X$ $(i \in \overline{p})$ even if $\bigcap_{i\in\overline{p}} A_i \neq \varnothing$. Also, certain iterated sequences generated from p-precyclic self-mappings can converge to a fixed point, rather than oscillate in-between sets of distinct best proximity points, even if $\bigcap_{i\in\overline{p}} A_i = \varnothing$, provided that the iterated points all stay in the same subset $A_i \subset X$, for some $i \in \overline{p}$, after a finite number of iterations.

For each given $x \in A_i$, define the following nondecreasing strictly ordered (in general, point dependent) *switching sequence* of nonnegative integers:

$$S_i^*(x) = \Big\{ j_{i-1}^*(x) = 0, j_i^*(x), j_{i+1}^*(x), \dots, j_p^*(x), j_{1+p}^*(x), \dots, j_{i+j+\ell p}^*(x), \dots \Big\}; \quad \forall i \in \overline{p} \tag{1}$$

containing the numbers of consecutive iterations within each individual subset $A_i \subseteq X$;for all $i \in \overline{p}$, before switching to the successive adjacent subsets $A_{i+1}, A_{i+2}, \dots, A_{i-1}, A_i, A_{i+1}, A_{i+2}, \dots$, and so forth for $j \in \overline{p}$ of the iterated sequence

$$\begin{aligned} P_i(x) &= \Big\{ \Big\{ T^{j_{i-1}^*(x)} x\,(= x), \\ &\qquad T^{j_{i-1}^*+1(x)} x\,(= Tx), \dots, T^{j_i^*(x)-1} x \Big\} \,(\subseteq A_i), \dots, \\ &\qquad \Big\{ T^{j_{i+j+\ell p}^*(x)} x, \dots, T^{j_{i+j+\ell p+1}^*(x)-1} x \Big\} \left(\subseteq A_{i+j+1}\right), \dots \Big\}. \end{aligned} \tag{2}$$

For all $i \in \overline{p}$ such that

$$j_{i+j+\ell p}^*(x) = \begin{cases} \min\left(k(\geq j+\ell p) \in \mathbf{Z}_{0+} : T^k x \in A_{i+j+1} \setminus A_{i+j}\right); & \forall i \in \overline{p},\ \forall \ell \in \mathbf{Z}_{0+}, \quad \text{if } 0 \leq j \leq p-i-1 \\ \min\left(k(\geq j+\ell p) \in \mathbf{Z}_{0+} : T^k x \in A_{i+j+1-p} \setminus A_{i+j-p}\right); & \forall i \in \overline{p},\ \forall \ell \in \mathbf{Z}_{0+}, \quad \text{if } p \geq j > p-i-1 \end{cases} \tag{3}$$

for any given $x \in A_i$ and, either $j_{k+1}^*(x) - j_k^*(x) < \infty$; for all $k \in \mathbf{Z}_{0+}$ with $\operatorname{card} S_i^*(x) = \chi_0$ (i.e., infinite cardinal of a numerable set) or there is $j_{k^*(x)}^*(x) < \infty$ for some existing finite $k^* = k^*(x) \in \mathbf{Z}_{0+}$ and then $S_i^*(x) = \{j_{i-1}^*(x) = 0, j_i^*(x), j_{i+1}^*(x), \dots, j_{k^*(x)}^*(x) < \infty\}$ is a finite set, that is, $\operatorname{card} S_i^*(x) = k^*(x) - i(x) + 2 < \chi_0$ with $T^{j_{k^*(x)}^*+k} x$ being in the same subset $A_i \subset X$ as $T^{j_{k^*(x)}^*} x$; for all $k(\leq \operatorname{card} S_i^*(x)) \in \mathbf{Z}_{0+}$; for all $i \in \overline{p}$. If $\operatorname{card} S_i^*(x) = 1$ for any $x \in \bigcup_{i\in\overline{p}} A_i$, for all $i \in \overline{p}$, then only one iteration stays at each subset before switching to the adjacent one so that the p-precyclic self-mapping is a standard p-cyclic one. Note, for instance, that if $j_i^*(x) = 1$ in (2) and $x \in A_i$, then $Tx \in A_{i+1}$. If this occurs for each $x \in \bigcup_{i\in\overline{p}} A_i$, then T on $\bigcup_{i\in\overline{p}} A_i$ is a usual p-precyclic self-mapping.

We will establish the formulation in metric spaces (X, d). It might be pointed out that it is usual to state formulations related to differential or dynamic systems and their stability, including those being formulated in a fractional calculus framework, in normed or Banach spaces since their dynamics evolve through time described by their state vectors [14, 29–39]. A possibility to focus on the study of their equilibrium points in a formal and structured fashion as well as their limit solutions, provided that they exist, (for instance, the presence of possible limit cycles) is through fixed point theory since the equilibrium points are fixed points of certain mappings and the limit cycles are repeated portions of limit state space trajectories. See, for instance, [33] and references therein. In the subsequently formulated and proved results, where the convexity of sets of X is required, it will be assumed that $(X, \|\ \|)$ is a normed space with associated metric space (X, d) for a norm-induced metric $d : X \times X \to \mathbf{R}_{0+}$, [29]. If $(X, \|\ \|)$ is a Banch space, then (X, d) is a complete metric space. The converses are not true without invoking additional assumptions. For instance, if (X, d) is a metric space (resp., a complete metric space) endowed with a homogeneous and translation -invariant metric $d : X \times X \rightarrow \mathbf{R}_{0+}$, then the metric-induced norm $\| \|$ exists so that $(X, \| \|)$ is a normed (resp., Banach) space endowed with such a metric-induced norm.

Example 1. For some given $\varepsilon \in \mathbf{R}_{0+}$, define the real intervals

$$A_1 = \mathbf{R}_{\varepsilon+} = \{z \in \mathbf{R} : z \geq \varepsilon\}; \quad A_2 = \mathbf{R}_{\varepsilon-} = \{z \in \mathbf{R} : z \leq -\varepsilon\} \tag{4}$$

and consider the scalar sequence

$$x_{n+1} = \begin{cases} \hat{x}_{n+1}, & \text{if } |\hat{x}_{n+1}| \geq \varepsilon, \\ -\varepsilon \operatorname{sgn} x_n, & \text{otherwise}, \end{cases} \tag{5}$$

where

$$\hat{x}_{n+1} = (-1)^{n+1} a_n x_n; \quad \forall n \in \mathbf{Z}_{0+} \text{ with initial condition satisfying } |x_0| \geq \varepsilon \tag{6}$$

with $\{a_n\} \subset \mathbf{R}_{0+}$. Note that, if $\varepsilon = 0$, then $A_1 \cap A_2 = \{0\}$ and $\{0\}$ can be a candidate for fixed point depending on certain simple contractive or, at least, nonexpansive conditions on the sequence $\{a_n\}$. If $\varepsilon \neq 0$, then the convergence of the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ (but not that of $\{x_n\}$ which is not convergent) can be possible only to best proximity points $\pm\varepsilon$. The self-mapping T on $A_1 \cup A_2$ defining the solution sequence is a 2-cyclic one since the solution points are alternately in A_1 and A_2. However, the modification

$$x_{n+1+i} = \begin{cases} \hat{x}_{n+1+i}, & \text{if } |\hat{x}_{n+1}| \geq \varepsilon, \\ -\varepsilon \nu_{n+i} \operatorname{sgn} x_n, & \text{otherwise}, \end{cases} \tag{7}$$

where

$$\hat{x}_{n+1+i} = (-1)^{n+1} \nu_{n+i} a_n x_n \quad \nu_{n+i} = \begin{cases} (-1)^{n+1}, & \text{if } 0 \leq i \leq j^* = 2, \\ 1, & \text{if } i = j^*. \end{cases} \tag{8}$$

For all $n \in \mathbf{Z}_{0+}$ with initial value $|x_0| \geq \varepsilon$ is a 2-precyclic (but no a 2-cyclic) self-mapping which generates two consecutive iterations in both A_1 and A_2 before switching to the other subset. Several extended variants are possible; that is, $j^* = j_0^* = j_1^* = 2$ is constant in this case. For instance, j^* can be dependent on the solution point $x(n)$ or on the initial condition. If $j^*(x(n_0))$ is infinity, then the trajectory solution remains in either A_1 (resp., in A_2) for $n \geq n_0$ if $x(n_0) \in A_1$ (resp., $x(n_0) \in A_2$). In this case, depending on conditions on the parameterization sequence $\{a_n\}$, the convergence of the solution in one of the subsets could be possible, even if $\varepsilon \neq 0$, when j^* is infinity in at least one of the sets A_1 and A_2 for some subset of values of the solution so that the solution enters such a set and remains in it for all later iterations. If $j^* = 1$ for any point of the solution sequence at any iteration, then the solution trajectory switches in-between the subsets A_1 and A_2 so that the 2-precyclic self-mapping is also a 2-cyclic one.

2. Convergence of Iterated Sequences to Fixed Points

The following assumptions are made.

(1) There are p bounded real functions $K_i : A_i \rightarrow K_i \in (0, \overline{K}_i]$; for all $i \in \overline{p}$ fulfilling $K_i(x) = K_j(x)$ in $\bigcup_{i \in \overline{p}} A_i$ under the congruence relation $j \equiv i \pmod p$ for some $i = i(j) \in \overline{p}$ and any given $j \in \mathbf{Z}_+$ such that

$$d\left(Tx, T^2 x\right) \leq K_i(x)\, d(x, Tx) + \left(1 - K_i(x)\right) \nu_i(x)\, D_i \tag{9}$$

for $x \in A_i$ where $\nu_i : A_i \rightarrow \{0,1\}$ are binary functions; for all $i \in \overline{p}$ such that $\nu_i(x) = 0$ if $Tx \in A_i$ and $\nu_i(x) = 1$ if $Tx \in A_{i+1} \setminus A_i$; for all $x \in A_i$, for all $i \in \overline{p}$. The notation to be used does not distinguish explicitly the cases when the contractive-like functions are real constants or point-dependent functions, but this can be easily deduced from context.

(2) If $D_i > 0$, that is, if $A_i \cap A_{i+1} = \varnothing$, then $\overline{K}_i < 1$; for all $i \in \overline{p}$.

(3) If card $S^*(x) < \chi_0$, then $K_{k^*(x)} \leq 1$, where $k^*(x) \equiv i + j \pmod p$; for all $x \in A_i$, for all $i \in \overline{p}$, provided that A_{i+j} is closed

(4)

$$\limsup_{k \to \infty} \left(\sum_{j=0}^{k} \left[K_{i+j}^{j_{i+j}^*(x)} \right] \right) \leq 1; \quad x \in \bigcup_{i \in \overline{p}} A_i, \ \forall i, j \in \overline{p}. \tag{10}$$

If $(x, y \neq Tx) \in A_i \times A_{i+1}$ for some $i \in \overline{p}$, then the constraint (9) is replaced with its following standard counterpart stated for p-cyclic self-mappings:

$$d(Tx, Ty) \leq K_i(x)\, d(x, y) + \left(1 - K_i(x)\right) D_i. \tag{11}$$

Note that the previous condition holds if $Tx \in A_{i+1}$ and $Ty \in A_{i+2}$ but also if $Tx, Ty \in A_{i+1}$ in its particular version $d(Tx, Ty) \leq K_i(x) d(x, y)$, since $D_i \geq 0$, and that it contains (9) for iterated sequences generated from $T : \bigcup_{i \in \overline{p}} A_i \rightarrow \bigcup_{i \in \overline{p}} A_i$ as a particular case. Note also the following.

(a) A particular pair (x, Tx) can satisfy simultaneously several constraints (9). For instance, assume that $x,\ Tx \in A_i \cap A_{i+1} (\neq \varnothing)$ for some $i \in \overline{p}$. Then

$$d\left(Tx, T^2 x\right) \leq \min\left(K_i(x), K_{i+1}(x)\right) d(x, Tx). \tag{12}$$

(b) If card $S^*(x) < \chi_0$, then there is some set A_j ($j \in \overline{p}$) such that $T^k x \subseteq A_j$; for all $k \geq k_0$ and some finite $k_0 = k_0(x) \in \mathbf{Z}_{0+}$ for each given $x \in \bigcup_{i \in \overline{p}} A_i$. Then, $T : \bigcup_{i \in \overline{p}} A_i \rightarrow \bigcup_{i \in \overline{p}} A_i$ is nonexpansive and $\{T^k x\}_{k \in \mathbf{Z}_{0+}}$ is bounded. Note that Assumption 4 is guaranteed directly by Assumption 3 if card $S^*(x) < \chi_0$. If card $S^*(x) = \chi_0$, then Assumption 3 is not invoked; however, Assumption 4 guarantees that $\{T^k x\}_{k \in \mathbf{Z}_{0+}}$ is bounded with $T^k x \in \bigcup_{i \in \overline{p}} A_i$.

(c) Assumption 4 implies that the p-precyclic self-mapping $T : \bigcup_{i \in \overline{p}} A_i \rightarrow \bigcup_{i \in \overline{p}} A_i$ is asymptotically nonexpansive.

(d) Any p-precyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is also a p-cyclic self-mapping.

In the following, fix(G) denotes the set of fixed points of the self-mapping $G : X \to X$. The following results hold.

Theorem 2. *Let $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ be a p-precyclic self-mapping. Assume also that the constraint (9) is extended for any $x \in A_i$ and $y \in A_j$; for all $i, j \in \overline{p}$ as follows:*

$$d(Tx, Ty) \le \max(K_i(x), K_j(y))\, d(x, y) + (1 - \max(K_i(x), K_j(y)))\, \nu_{ij}(x)\, D_{ij}, \tag{13}$$

where $D_{ij} = d(A_i, A_j)$ and $\nu_{ij}(x) = 1$ if $Tx \in A_{i+1} \setminus A_i$ and $Ty \in A_{j+1} \setminus A_j$ and $\nu_{ij}(x) = 0$, otherwise; for all $x \in A_i$, for all $y \in A_j$.

Then, the following properties hold.

(i) *$T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-cyclic self-mapping if and only if $T(A_i) \subseteq A_{i+1}$; for all $i \in \overline{p}$.*

(ii) *If $D_i > 0$; for all $i \in \overline{p}$ and card $S^*(x) = \chi_0$; for all $x \in \bigcup_{i\in\overline{p}} A_i$, then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ has no fixed point in $\bigcup_{i\in\overline{p}} A_i$.*

(iii) *If card $S^*(x) < \chi_0$ for some $x \in \bigcup_{i\in\overline{p}} A_i$, then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ has a fixed point in a subset $A_j \subset X$, for some $j \in \overline{p}$, to which the iterated sequence $\{x, Tx, \ldots, T^k x, \ldots\}$ converges if (X, d) is complete, $K_j < 1$ and A_j is closed. If $x \in A_i$ for some $i \in \overline{p}$, then the iterated sequence $\{x, Tx, \ldots, T^k x, \ldots\}$ converges to a fixed point $x_{i+j}^* \in A_{i+j}$ of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ for some $j \in \overline{p-i}$ with $K_{i+j} < 1$ such that $k^*(x) \equiv i + j(\mathrm{mod}\, p)$, provided that A_{i+j} is closed, or to a fixed point $x_j^* \in A_j$ for some $j \in \overline{i-1}$ with $K_j < 1$ and $k^*(x) \equiv i + j(\mathrm{mod}\, p)$ provided that A_j is closed. If, furthermore, $x_{i+j}^* \in A_{i+j}$ for $j \in \overline{p-i}$, respectively, $x_j^* \in A_j$ for $j \in \overline{i}$ are convex then the corresponding fixed point is unique.*

Proof. $T(A_i) \cap A_i \ne \emptyset$ for some $i \in \overline{p}$; then $\exists x \in A_i$ such that $Tx \notin A_{i+1}$. Then, $T(A_i) \subseteq A_i \cup A_{i+1}$ is not a cyclic self-mapping. Hence, Property (i) follows.

Note that $D_i > 0$; for all $i \in \overline{p} \Leftrightarrow A_i \cap A_{i+1} = \emptyset$; for all $i \in \overline{p}$. If card $S^*(x) = \chi_0$, then $S^*(x)$ is an infinite sequence of switches in-between adjacent subsets of X which are never intersecting. Thus, $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ has no fixed point in $\bigcup_{i\in\overline{p}} A_i$. Hence, Property (ii) follows.

To prove Property (iii), first note that

$$A_{i+j+\ell p} = A_{i+j} = A_k; \quad \forall \ell \in \mathbf{Z}_{0+};\ \forall i, j \in \overline{p} \tag{14}$$

such that

$$\begin{aligned} i + j + \ell p &\equiv i + j\,(\mathrm{mod}\, p) \\ &\equiv k\,(= i + j - p;\ k \in \mathbf{Z}_+)\,(\mathrm{mod}\, p). \end{aligned} \tag{15}$$

Note also that, if card $S^*(x) < \chi_0$ for some $x \in \bigcup_{i\in\overline{p}} A_i$, then the iterated sequence $S(x) = \{x, Tx, \ldots, T^k x, \ldots\}$ built from such an initial point $x \in \bigcup_{i\in\overline{p}} A_i$ through $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ remains in $A_j \subset X$, for such a $j \in \overline{p}$, for all $k \ge j_{k^*(x)}^*$ and some finite integer $j_{k^*(x)}^*$. Then, the asymptotically nonexpansive self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is asymptotically contractive from Assumption 4 and also contractive after a finite number of iterations since $K_j < 1$. Thus, $S(x)$ is bounded and has a Cauchy convergent subsequence since (X, d) is complete. Since the subset $A_j \subset X$ is nonempty and closed for such $j \in \overline{p}$, there is a fixed point of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ from Banach contraction principle and such a fixed point is unique if the subset is, furthermore, convex. □

Now, the following result is proven for a class of contractive p-precyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$.

Theorem 3. *Assume that $(X, \|\,\|)$ is a normed space with associated metric space (X, d) for a norm-induced metric $d : X \times X \to \mathbf{R}_{0+}$. Let $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ be a p-precyclic self-mapping and $D_i = 0$; for all $i \in \overline{p}$. If $A_i \subset X$ are nonempty, bounded, closed, and convex; for all $i \in \overline{p}$ and $\overline{K} = \prod_{i\in\overline{p}} [\overline{K}_i^{\overline{j}_i^*}] < 1$, where $\overline{j}_i^* = \sup_{x\in A_i} j_i^*(x)$. Then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ has a unique fixed point $x^* \in \bigcap_{i\in\overline{p}} A_i$. If $A_i \subset X$ is not closed for some $i \in \overline{p}$ while $(X, \|\,\|)$ is a Banach space, and then (X, d) is a complete metric space, then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ has a unique fixed point in $\bigcap_{i\in\overline{p}} clA_i$.*

Proof. It follows that

$$\lim_{n\to\infty} d\left(T^{pn\overline{j}^*} x, T^{pn\overline{j}^* - 1} x\right) \le \lim_{n\to\infty} \left(\overline{K}^n\right) d(Tx, x) = 0; \quad \forall x \in \bigcup_{i\in\overline{p}} A_i \tag{16}$$

since $d(x, Tx) < \infty$ for any finite $x \in \bigcup_{i\in\overline{p}} A_i$, since $\bigcup_{i\in\overline{p}} A_i$ is bounded, where

$$\begin{aligned} \overline{j}^* &= \sum_{i=1}^{p} \overline{j}_i^* \\ &= \max_{i\in\overline{p}} \sup_{x\in A_i} \Big(j_i^*(x) + j_{i+1}^*\left(T^{j_i^*+1} x\right) + \cdots + j_p^*\left(T^{\sum_{j=i}^{p-1} j_j^* + 1} x\right) \\ &\quad + j_1^*\left(T^{\sum_{j=i}^{p} j_j^* + 1} x\right) \\ &\quad + \cdots + j_{i-1}^*\left(T^{\sum_{j=i}^{p} j_j^* + \sum_{j=p-i+1}^{p-1} j_j^* + 1} x\right)\Big); \end{aligned} \tag{17}$$

that is, the distance between any two consecutive elements of any such a sequence converges asymptotically to zero. Furthermore, since the subsets $A_i \subset X$ are nonempty, closed, all of them intersect and the composite self-mapping $T^{p\overline{j}^*} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is uniformly Lipschitz -continuous in $\bigcup_{i\in\overline{p}} A_i$, since it is contractive with constant $\overline{K} < 1$, the sequence $\{T^{pn\bar{j}^*}x\}$ converges to $x_i^* = x_i^*(x) \in A_i$ for some $i \in \bar{p}$, which is a unique fixed point of the composite $T^{p\bar{j}^*}: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ in A_i; for all $i \in \bar{p}$. To prove uniqueness, proceed by contradiction. Assume that there are two fixed points $x_i^* = T^{p\bar{j}^*}x_i^*$, $y_i^*(\neq x_i^*) = T^{p\bar{j}^*}y_i^*$ in A_i of $T^{p\bar{j}^*}: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$.

Then,

$$\begin{aligned} d(x_i^*, y_i^*) &= d\left(T^{pn\bar{j}^*}x_i^*, T^{pn\bar{j}^*}y_i^*\right) \\ &\leq d\left(T^{pn\bar{j}^*+1}x_i^*, T^{pn\bar{j}^*}x_i^*\right) \\ &\quad + d\left(T^{pn\bar{j}^*+1}x_i^*, T^{pn\bar{j}^*+1}y_i^*\right) \\ &\quad + d\left(T^{pn\bar{j}^*+1}y_i^*, T^{pn\bar{j}^*}y_i^*\right) \\ &\leq \overline{K}^n\left(d(Tx_i^*, x_i^*) + d(Tx_i^*, Ty_i^*) + d(Ty_i^*, y_i^*)\right) \\ &= \overline{K}^n d(x_i^*, y_i^*) < d(x_i^*, y_i^*); \quad \forall n \in \mathbf{Z}_+ \end{aligned} \tag{18}$$

which leads to the contradiction that $x_i^* = y_i^*$. Thus, there is a unique fixed point of $T^{p\bar{j}^*}: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ in A_i. Also, one gets from (16) that

$$\begin{aligned} &\lim_{n\to\infty} d\left(T^{pn\bar{j}^*+j}x, T^{pn\bar{j}^*+j-1}x\right) \\ &\leq \lim_{n\to\infty}\left(\overline{K}^n\right)\left(\prod_{\ell=1}^{j}\left[\overline{K}_\ell\right]\right) d(Tx, x) = 0; \quad \forall j \in \bar{p} \end{aligned} \tag{19}$$

for any $j \in \bar{p}$. As a result, $\lim_{n\to\infty} d(T^{pn\bar{j}^*+j}x, T^{pn\bar{j}^*+j-1}x) = 0$ and $\{T^{pn\bar{j}^*+j}x\} \to x_i^* = T^{p\bar{j}^*}x_i^*$ with some unique $x_i^* \in A_i$ for some $i \in \bar{p}$; for all $j \in \bar{p}$. Now, assume that there are two distinct fixed points $x_i^* \in A_i$ and $x_j^*(\neq x_i^*) \in A_j(\neq A_i)$ of for some $i, j(\neq i) \in \bar{p}$. Since $A_i \subseteq X$ is nonempty, closed, and convex for any $i \in \bar{p}$, then $A_i \cap A_j$ is nonempty, closed, and convex; for all $i, j(\neq i) \in \bar{p}$. From the convexity of $A_i \cap A_j$, there is $z(\neq x_i^*, x_j^*) \in A_i \cap A_j$ such that $d(x_i^*, z) = \lambda d(x_i^*, x_j^*)$ with some real constant $\lambda \in (0,1)$ and $x_i^*, z \in A_i$. Then,

$$\begin{aligned} &d\left(x_i^*, T^{pn\bar{j}^*}z\right) \\ &= d\left(T^{pn\bar{j}^*}x_i^*, T^{pn\bar{j}^*}z\right) \leq \overline{K}^n d(x_i^*, z) = \lambda\, \overline{K}^n d\left(x_i^*, x_j^*\right) \end{aligned} \tag{20}$$

so that $d(x_i^*, T^{pn\bar{j}^*}z) \to 0$ as $n \to \infty$ and, since $T^{p\bar{j}^*}: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is uniformly Lipschitz continuous in $\bigcup_{i\in\bar{p}} A_i$, the sequence $\{T^{pn\bar{j}^*}z\}_{n\in\mathbf{Z}_{0+}}$ converges to $x_i^* \in A_i$. But $z \in A_j$ so that we can repeat the previous reasoning with $d(x_j^*, z) = \lambda' d(x_i^*, x_j^*)$, $x_j^*, z \in A_j$ and some real constant $\lambda' \in (0,1)$ to conclude that $d(x_j^*, T^{pn\bar{j}^*}z) \to 0$ as $n \to \infty$ and $\{T^{pn\bar{j}^*}z\}_{n\in\mathbf{Z}_{0+}}$ converges to $x_j^*(\neq x_i^*) \in A_j$ which is a contradiction to its convergence to $x_i^* \in A_i$. Then, there is a unique fixed point in the nonempty, closed, and convex set $A_i \cap A_j$. By extending the same reasoning to any pair of subsets A_i and A_j of X, one concludes that the composite self-mapping $T^{p\bar{j}^*}: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ has a unique fixed point $x^* \in \{x^*\} = \mathrm{Fix}(T^{p\bar{j}^*}) \subseteq \bigcap_{i\in\bar{p}} A_i$.

It remains to be proved that the unique fixed point of the composite mapping is also the unique fixed point of $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$. Since the subsets $A_i \subseteq X$ intersect, one gets from (16) that

$$\begin{aligned} &\left(d\left(T^{2p\bar{j}^*+1}x^*, T^{2p\bar{j}^*}x^*\right) \leq K'\overline{K} d\left(T^{p\bar{j}^*}x^*, x^*\right) = 0\right) \\ &\Longrightarrow \left(d\left(T\left(T^{2p\bar{j}^*}\right)x^*, x^*\right) = d(Tx^*, x^*) = 0\right) \\ &\Longrightarrow (Tx^* = x^*) \Longleftrightarrow \left(x^* \in \mathrm{Fix}\left(T^{p\bar{j}^*}\right)\right) \\ &\Longrightarrow (x^* \in \mathrm{Fix}(T)) \end{aligned} \tag{21}$$

since $T^{p\bar{j}^*}x^* = x^*$, where $K' = \max_{i\in\bar{p}} K_i$. Also,

$$\begin{aligned} &(x^* \in \mathrm{Fix}(T)) \\ &\Longrightarrow \left(d(Tx^*, x^*) = d(x^*, x^*) = d\left(T^2x^*, Tx^*\right)\right. \\ &\qquad \left.= \cdots = d\left(T^{n+1}x^*, T^nx^*\right) = 0;\ \forall n \in \mathbf{Z}\right) \\ &\Longleftrightarrow (x^* \in \mathrm{Fix}(T^n);\ \forall n \in \mathbf{Z}_+) \Longrightarrow \left(x^* \in \mathrm{Fix}\left(T^{p\bar{j}^*}\right)\right) \end{aligned} \tag{22}$$

so that $x^* \in \mathrm{Fix}(T)$ and it is the unique fixed point of $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$. If some A_i $i \in \bar{p}$ is not closed, then all Cauchy sequences have a limit in X if (X, d) is complete so that there is still a unique fixed point in $\bigcap_{i\in\bar{p}} clA_i$ of $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ and $T^{p\bar{j}^*}: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$. □

The following result is now proven for a class of nonexpansive p-precyclic self-mapping $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$.

Corollary 4. *Assume the hypothesis of Theorem 3 with the modified weaker condition $\overline{K} = \prod_{i\in\bar{p}}[\overline{K}_i^{\bar{j}_i^*}] \leq 1$ of nonexpansive self-mapping $T: \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$. Then, $\{T^nx\}_{n\in\mathbf{Z}_{0+}}$ is bounded; for all $x \in \bigcup_{i\in\bar{p}} A_i$ there is a subsequence $\{T^{p\bar{j}^* n_k}x\}_{n_k\in Z_s\subseteq\mathbf{Z}_{0+}} \subseteq \{T^nx\}_{n\in\mathbf{Z}_{0+}}$, for some strictly ordered subset Z_s of $\mathbf{Z}_{0+}$, such that $d(T^{p\bar{j}^* n_k}x, T^{p\bar{j}^* n_{k+1}}x) \to C = C(x)$ as $k \to \infty$; for all $x \in \bigcup_{i\in\bar{p}} A_i$ for some real $C \in \mathbf{R}_{0+}$. If, in addition, $\overline{K}_i \leq 1$; for all $i \in \bar{p}$, then $d(T^{n+1}x, T^nx) \to C = C(x)$ as $n \to \infty$; for all $x \in \bigcup_{i\in\bar{p}} A_i$.*

Proof. $x \in \bigcup_{i\in\bar{p}} A_i$ is always finite since $\bigcup_{i\in\bar{p}} A_i$ is bounded and then $\{T^nx\}_{n\in\mathbf{Z}_{0+}} \subseteq \bigcup_{i\in\bar{p}} A_i$ is a bounded sequence; for all $x \in \bigcup_{i\in\bar{p}} A_i$. Also, $d(T^{p\bar{j}^* n}x, T^{p\bar{j}^* n+1}x) \leq d(x, Tx)$; for all $n \in \mathbf{Z}_{0+}$, for all $x \in \bigcup_{i\in\overline{p}} A_i$ since $\overline{K} = \prod_{i\in\overline{p}}[\overline{K}_i^{\overline{j}_i^*}] \leq 1$. Thus, $\limsup_{n\to\infty} d(x, T^{n+1}x) < \infty$; for all $x \in \bigcup_{i\in\overline{p}} A_i$ since $\{T^n x\}_{n\in\mathbf{Z}_{0+}}$ is bounded and $d(T^n x, T^{n+1}x) < \infty$; for all $n \in \mathbf{Z}_{0+}$ from the properties of distances since $x \in \bigcup_{i\in\overline{p}} A_i$ is finite. Thus, there is a subsequence $\{T^{p\overline{j}^* n_k} x\}_{n_k\in Z_s\subseteq\mathbf{Z}_{0+}} \subseteq \{T^k x\}_{k\in\mathbf{Z}_{0+}}$ for which $d(T^{p\overline{j}^* n_k} x, T^{p\overline{j}^* n_{k+1}} x)$ converges as $k \to \infty$. If, in addition, $\overline{K}_i \leq 1$; for all $i \in \overline{p}$, then $d(T^{n+1}x, T^n x) \leq (\max_{i\in\overline{p}} K_i) d(x, Tx) \leq d(x, Tx)$; for all $x \in \bigcup_{i\in\overline{p}} A_i$, so that $d(T^{n+1}x, T^n x) \to C = C(x)$ as $n \to \infty$; for all $x \in \bigcup_{i\in\overline{p}} A_i$. □

The subsequent result is related to convergence to a unique fixed point in one of the subsets $A_i \subseteq X$ $(i \in \overline{p})$ of the precyclic disposal even if the subsets do not intersect, provided that the self-mapping is asymptotically contractive in one of the subsets.

Corollary 5. *Assume the hypothesis of Theorem 3 with the subsequent further hypothesis on the p-precyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$:*

(1) $\bigcap_{i\in\overline{p}} A_i = \varnothing$; $\operatorname{card} S_i^*(x) < \chi_0$

(2) $\exists j \in J_0 = J_0(x) \subseteq \overline{p}$ *(nonnecessarily unique; that is, J_0 can have a cardinal greater than one) such that* $\limsup_{\ell\to\infty} K_j\ (T^{i+j+\ell p+1}x) < 1$ *for some given initial point $x \in A_i$.*

Then, the iterated sequence $P_i(x)$, (2), converges to a fixed point in $A_k \subset X$ for a unique $k = \min(j \in J_{i\prec} : j \in J_0) \in \overline{p}$, where

$$\begin{aligned} J_{i\prec} &= J_{i\prec}(x) \\ &= \{i\,(\prec i+1), i+1\,(\prec i+2), i+2\,(\prec i+3), \ldots, \\ &\qquad p-1\,(\prec p), p\,(\prec, 1), 1\,(\prec 2), \ldots, \ i-1 \prec (i)\} \end{aligned} \tag{23}$$

is a strictly ordered finite set of card $J_{i\prec} = p$*, containing all the p elements of the set $\overline{p}$, with the strict order relation $\prec$ defined by its enumeration order defined by $a \prec b$ for any $a, b \in J$ if a precedes b in the previous enumeration definition of the set J.*

Proof. Since $\operatorname{card} S_i^*(x) < \chi_0$; $\exists j \in \overline{p}$ such that for the given $x \in A_i$ $j^*_{i+j+\ell p+1}(x) \to \infty$ as $\ell \to \infty$ if $0 \leq j \leq p-i-1$, or $j^*_{k+\ell p}(x) \to \infty$ as $\ell \to \infty$ if $2p \geq j > p-i-1$ for any nonnegative integer $k > i-1$ such that $j \equiv k-i-1 (\operatorname{mod} p)$. This is obvious since, if such a$j \in \overline{p}$ does not exist, then the iterated sequence $P_i(x)$, (2), with starting point $x \in A_i$ for the given $i \in \overline{p}$ has infinitely many switches in-between consecutive adjacent subsets $A_i \subset X$; then the switching sequence $S_i^*(x)$ associated with such an iterated sequence $P_i(x)$ is finite so that $\operatorname{card} S_i^*(x) = \chi_0$.

From the extra previous hypothesis 2, there is a nonempty set $J_0(x) \subseteq \overline{p}$ for which $\limsup_{\ell\to\infty} K_j(T^{i+j+\ell p+1}x) < 1$ for the given $x \in A_i$. Note that the p-precyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is asymptotically contractive, for the given $x \in A_i$ on $A_k \subset X$, where $k = \min(j \in J_{i\prec}(x) : j \in J_0(x)) \in \overline{p}$ is unique (even if $J_0(x)$ is of cardinal greater than one) from the fact that $J_{i\prec}(x)$ in (23) is of finite cardinal, strictly ordered, then with unique minimal and maximal elements which are then a unique $i \in \overline{p}$ minimum and maximum $i-1 \in \overline{p-1} \cup \{0\}$, respectively. From the previous part of this proof, it also turns out that $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is, furthermore, (strictly) contractive on such a subset $A_k \subset X$ for any infinite subsequence $P_i(T^{j^*_{i+j+\ell p}(x)}x) \subseteq P_i(x)$. Therefore, such a subsequence $P_i(T^{j^*_{i+j+\ell p}(x)}x)$ is itself a Cauchy sequence fulfilling $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ and then being bounded and convergent in A_k (since $\{T^n x\}_{n\geq j^*_{i+j+\ell p}(x)}$ satisfies a Lipschitz-continuous property) to a fixed point $x^* \in A_k$, since A_k is nonempty, bounded (then x is bounded), and closed, which is unique since A_k is convex. Since the sequence $P_i(x)$ contains, by construction, all the elements of the subsequence alter a finite number of iterations, it also converges to the same unique fixed point x^*. □

Remark 6. (1) Note that Corollary 5 is stated for a certain iterated sequence being built from a starting point since the contractive conditions a point-to-point condition. Point-to-point contractivity-type conditions have been also used in the literature for the characterization of fixed point properties of contractive self-mappings. See, for instance, [40, 41] and references therein. It can be generalized directly under generalization for any starting point in any of the subsets or in some subset of its union. It can be reformulated, in particular, if $\limsup_{\ell\to\infty} K_j\ (T^{i+j+\ell p+1}x) < 1$; for all $x \in A_i$. In such a case, it follows the convergence of the sequence of iterates to the same unique fixed point in A_k built from any initial point $x \in A_i$.

(2) Note that the uniqueness of the final set A_k from the initial set A_i, such that $x \in A_i$, arises from the fact that the first subset where the iterations remain after a finite number of them is the relevant one for the final reached limit if the precyclic self-mapping stops to transfer the iterated sequence to the next adjacent subset.

(3) Note that, if the convexity assumption is only made on the subset A_k, then Corollary 5 still holds.

(4) Note also that, if the convexity assumption on the subsets is removed in Corollary 5, then the existence of the fixed point is still proven although that one is not necessarily unique.

The subsequent result is a consequence of direct proof of Corollaries 4 and 5.

Corollary 7. *Assume the hypothesis of Corollary 5 under the weaker* $\limsup_{\ell\to\infty} K_j\ (T^{i+j+\ell p+1}x) \leq 1$ *condition of asymptotic nonexpansivness of the self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ in some of the subsets $A_k \subseteq X$ for some given initial point $x \in A_i$. Then, $d(T^{n+1}x, T^n x) \to C = C(x)$ and $\{T^n x\}_{n\geq n_0} \subseteq A_k$ for some finite $n_0 \in \mathbf{Z}_{0+}$ and a unique $k \in \overline{p}$ as defined in Corollary 5.*

Some extensions of Corollary 7 can be directly obtained from Remark 6 (3)-(4).

3. Convergence to Best Proximity Points

The following preliminary technical result concerning the convergence for distances in-between consecutive iterates through the self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is now given in the case that the subsets $A_i \subset X$ do not necessarily intersect.

Theorem 8. *Assume that the following constraints hold.*

(1) $D_i \neq 0$; *for all* $i \in \overline{p}$.

(2) $K_i \in (0,\infty)$ *is constant within each subset* $A_i \subset X$ *and the switching sequences*

$$S_i^* = \left\{j_{i-1}^* = 0, j_i^*, j_{i+1}^*, \dots, j_p^*, j_{1+p}^*, \dots, j_{i+j+\ell p}^*, \dots\right\} = \left\{j_{i-1}^* = 0, j_i^*, S_{i+1}^* \setminus j_i^*\right\}; \quad \forall i \in \overline{p} \tag{24}$$

are not point dependent on the given $x \in A_i$; *for all* $i \in \overline{p}$ *so that, in addition,*

$$\overline{j}_i^* = \overline{j}_{i+np}^* = \sup_{x\in A_i} j_i^*(x) = j_{i+np}^* \quad \forall n \in \mathbf{Z}_{0+}. \tag{25}$$

(3)

$$K = \prod_{i\in\overline{p}} \left[K_i^{\overline{j}_i^*}\right] < 1. \tag{26}$$

(4)

$$D_0 := \frac{\sum_{j=1}^{p}\left(\prod_{i=j+1}^{p}\left[K_i^{j_i^*}\right]\right)\left(1-K_j\right)D_j}{1-\prod_{i=1}^{p}\left[K_i^{j_i^*}\right]}. \tag{27}$$

Then, the following properties hold for any $x \in A_i$:

$$\limsup_{n\to\infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+j}x, T^{n(\sum_{i=1}^{p} j_i^*)+j-1}x\right) \le K_i^{j-1} D_0 \quad \text{for } j = 1,2,\dots,j_i^* \tag{28}$$

$$\begin{aligned} D_{i+k} &\le \limsup_{n\to\infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{\ell=i}^{k} j_\ell^*+j+1}x, T^{n(\sum_{i=1}^{p} j_i^*)+j_i^*+j}x\right) \\ &\le \left(\prod_{\ell=i}^{k}\left[K_\ell\right]\right) D_0 + \sum_{j=i}^{k}\left(\prod_{\ell=j+1}^{k}\left[K_\ell^{j_\ell^*}\right]\right)\left(1-K_j\right)D_j; \end{aligned} \tag{29}$$

for all $x \in \bigcup_{i\in\overline{p}} A_i$, $j = 0,1,\dots,j_{k+1}^*-1$, *for all* $k \in \overline{p}$.

If, in addition, Assumption 5

(5)

$$\begin{aligned} D_1 &= K_1^{j_1^*-1} D_0; \\ D_k &= K_k^{j_k^*-1}\left(\prod_{i=1}^{k-1}\left[K_i^{j_i^*}\right]\right) D_0 \ge \prod_{i=1}^{k-1}\left[K_i^{j_i^*}\right] D_k \quad \text{for } k \ge 2 \end{aligned} \tag{30}$$

Holds; then,

$$\limsup_{n\to\infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*}x, T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*-1}x\right) \le D_k \tag{31}$$

$$\exists \lim_{n\to\infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*+1}x, T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*}x\right) = D_k; \quad \forall k \in \overline{p} \tag{32}$$

$$\begin{aligned} &\limsup_{n\to\infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k+1} j_i^*+j+1}x, T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k+1} j_i^*+j}x\right) \\ &= K_k^{1+j-j_k^*} D_k; \quad j = 1,\dots,j_{k+1}^*-1, \ \forall k \in \overline{p}; \ \forall x \in \bigcup_{i\in\overline{p}} A_i. \end{aligned} \tag{33}$$

Proof. Take any $x \in A_1$. Thus, one gets from (9)

$$\begin{aligned} &d\left(T^2x, Tx\right) \le K_1 d(x,Tx) \\ &d\left(T^{j_1^*}x, T^{j_1^*-1}x\right) \le K_1^{j_1^*-1} d(x,Tx) \\ &D_1 \le d\left(T^{j_1^*+1}x, T^{j_1^*}x\right) \le K_1^{j_1^*} d(x,Tx) + \left(1-K_1\right)D_1 \\ &d\left(T^{j_1^*+2}x, T^{j_1^*+1}x\right) \le K_2 d\left(T^{j_1^*+1}x, T^{j_1^*}x\right) \\ &\qquad \le K_2\left[K_1^{j_1^*} d(x,Tx) + \left(1-K_1\right)D_1\right] \\ &d\left(T^{j_1^*+j_2^*}x, T^{j_1^*+j_2^*-1}x\right) \le K_2^{j_2^*-1}\left[K_1^{j_1^*} d(x,Tx) + \left(1-K_1\right)D_1\right] \\ &D_2 \le d\left(T^{j_1^*+j_2^*+1}x, T^{j_1^*+j_2^*}x\right) \\ &\qquad \le K_2^{j_2^*}\left[K_1^{j_1^*} d(x,Tx) + \left(1-K_1\right)D_1\right] + \left(1-K_2\right)D_2 \\ &d\left(T^{\sum_{i=1}^{p} j_i^*}x, T^{\sum_{i=1}^{p} j_i^*-1}x\right) \\ &\qquad \le K_p^{j_p^*-1}\left(\left(\prod_{i=1}^{p-1}\left[K_i^{j_i^*}\right]\right) d(x,Tx) + \sum_{j=1}^{p-1}\left(\prod_{i=j+1}^{p-1}\left[K_i^{j_i^*}\right]\right)\left(1-K_j\right)D_j\right) \\ &D_p \le d\left(T^{\sum_{i=1}^{p} j_i^*+1}x, T^{\sum_{i=1}^{p} j_i^*}x\right) \le Kd(x,Tx) + M, \end{aligned} \tag{34}$$

where

$$K = \prod_{i=1}^{p} \left[K_i^{j_i^*} \right] < 1;$$
$$M = \sum_{j=1}^{p} \left(\prod_{i=j+1}^{p} \left[K_i^{j_i^*} \right] \right) \left(1 - K_j \right) D_j. \tag{35}$$

Then, since $K < 1$,

$$D_p \le d\left(T^{n(\sum_{i=1}^{p} j_i^*)+1} x, T^{n(\sum_{i=1}^{p} j_i^*)} x \right)$$
$$\le K^n d(x, Tx) + \frac{1 - K^n}{1 - K} M,$$
$$d\left(T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*+1} x, T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*} x \right)$$
$$\le K_1^{j_1^*} \left(K^n d(x, Tx) + \frac{1 - K^n}{1 - K} M \right) + \left(1 - K_1 \right) D_1 \tag{36}$$

with $T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*+1} x \in A_2$. Now, since

$$D_0 = \frac{\sum_{j=1}^{p} \left(\prod_{i=j+1}^{p} \left[K_i^{j_i^*} \right] \right) \left(1 - K_j \right) D_j}{1 - \prod_{i=1}^{p} \left[K_i^{j_i^*} \right]} = \frac{M}{1 - K}, \tag{37}$$

it follows that

$$D_p \le \limsup_{n \to \infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+1} x, T^{n(\sum_{i=1}^{p} j_i^*)} x \right) \le D_0$$
$$\limsup_{n \to \infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+2} x, T^{n(\sum_{i=1}^{p} j_i^*)+1} x \right) \le K_1 D_0$$
$$\limsup_{n \to \infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*-1} x, T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*-2} x \right) \le K_1^{j_1^*-2} D_0$$
$$\limsup_{n \to \infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*} x, T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*-1} x \right) \le K_1^{j_1^*-1} D_0$$
$$D_1 \le \limsup_{n \to \infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*+1} x, T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*} x \right)$$
$$\le K_1^{j_1^*} D_0 + \left(1 - K_1 \right) D_1. \tag{38}$$

Thus, if $D_1 = K_1^{j_1^*-1} D_0$, then one gets from the previous relationships

$$\exists \lim_{n \to \infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*+1} x, T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*} x \right) = D_1$$
$$\limsup_{n \to \infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*} x, T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*-1} x \right) \le D_1$$
$$\exists \lim_{n \to \infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*+1} x, T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*} x \right)$$
$$\le K_1^{j_1^*} D_0 + \left(1 - K_1 \right) D_1 = D_1 \tag{39}$$
$$\limsup_{n \to \infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+1+j_1^*+j} x, T^{n(\sum_{i=1}^{p} j_i^*)+j_1^*+j} x \right)$$
$$\le K_1^{j+j_1^*} D_0 = K_1^{1+j} D_1; \quad j = 1, \dots, j_2^* - 1.$$

In the same way, if the constraint $D_1 = K_1^{j_1^*-1} D_0$ is extended to $D_1 = K_k^{j_1^*-1} D_0$; $D_k = K_k^{j_k^*-1} (\prod_{i=1}^{k-1} [K_i^{j_i^*}]) D_0$; for all $k (\ge 2) \in \overline{p}$, then one gets from (34) and (38) that

$$d\left(T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*+1} x, T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*} x \right)$$
$$\le \prod_{i=1}^{k} \left[K_i^{j_i^*} \right] \left(K^n d(x, Tx) + \frac{1 - K^n}{1 - K} M \right) + \left(1 - K_k \right) D_k; \tag{40}$$

for all $k \in \overline{p}$. Thus,

$$D_k \le \limsup_{n \to \infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*+1} x, T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*} x \right)$$
$$\le \lim_{n \to \infty} \left(\prod_{i=1}^{k} \left[K_i^{j_i^*} \right] \left(K^n d(x, Tx) + \frac{1 - K^n}{1 - K} M \right) \right)$$
$$+ \left(1 - K_k \right) D_k$$
$$= \frac{\prod_{i=1}^{k} \left[K_i^{j_i^*} \right]}{1 - K} M + \left(1 - K_k \right) D_k$$
$$= D_0 \left(\prod_{i=1}^{k} \left[K_i^{j_i^*} \right] \right) + \left(1 - K_k \right) D_k$$
$$= K_k D_k + \left(1 - K_k \right) D_k = D_k; \quad \forall k \in \overline{p} \tag{41}$$

with $T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*+1} x \in A_{k+1}$. Then, (31)–(33) follow and the result is proven for $x \in A_1$. Such a choice can be made with no loss in generality since, if, instead, $x \in A_i$ for any given $i \in \overline{p}$, then the previous result still holds with $T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*+1} x \in A_{k+1}$ with $k + 1 + i \equiv j (\mathrm{mod}\, p)$ for a unique integer $j \in \overline{p}$. On the other hand (28)-(29) are a consequence of (36) which is independent of the constraints $D_k = K_k^{j_k^*-1} (\prod_{i=1}^{k-1} [K_i^{j_i^*}]) D_0$; for all $k \in \overline{p}$. □

Remark 9. (1) Note that Theorem 8 requires a set of necessary constraints on the K_i and j_i^*; for all $i \in \overline{p}$ which are induced by Assumptions 4-5 as follows:

$$\sigma_i = \frac{D_{i+1}}{D_i} = K_{i+1}^{j_{i+1}^*-1} K_i \quad (i = 2, 3, \dots, p) \tag{42}$$

$$(1 - K) D_0$$
$$= \left[\sum_{j=2}^{p} \left\{ \left(\prod_{i=j+1}^{p} \left[K_i^{j_i^*} \right] \right) \left(1 - K_j \right) \left(\prod_{k=1}^{j-1} \left[\sigma_k \right] \right) \right\} + \prod_{i=2}^{p} \left[K_i^{j_i^*} \right] \left(1 - K_1 \right) \right] D_1$$

$$= \left[\sum_{j=2}^{p} \left\{ \left(\prod_{i=j+1}^{p} \left[K_i^{j_i^*} \right] \right) \left(1 - K_j\right) \times \left(\prod_{k=1}^{j-1} \left[K_{k+1}^{j_{k+1}^*-1} K_k \right] \right) \right\} + \prod_{i=2}^{p} \left[K_i^{j_i^*} \right] \left(1 - K_1\right) \right] D_1 \tag{43}$$

with $K = \prod_{i=1}^{p} [K_i^{j_i^*}]$.

(2) Note that if $K_i = K < 1$, $j_i^* = 1$ and $D_i = D$; for all $i \in \overline{p}$, then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a contractive p-cyclic and

$$D_0 = \frac{\sum_{i=0}^{p-1} K^i}{1 - K^p} (1 - K) D = \frac{1}{1 - K^p} \frac{1 - K^p}{1 - K} (1 - K) D = D. \tag{44}$$

If (29) is evaluated in this case, then the limit superior is also the limit inferior of the obtained expression leading to the existence of the limit being equal to D, the distance in-between al, the adjacent subsets $D_i \subset X$; for all $i \in \overline{p}$. This is a well- known result for contractive p-cyclic self-mappings $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$, [23, 24, 42].

(3) Note that the first property of (32) is a convergence of the iterated sequences of distances to best proximity points of adjacent subsets provided that Assumptions 1–5 of Theorem 8 hold implying, in particular, a strict contractive constant on a composite self-mapping $T^{\sum_{i=1}^{p} j_i^*} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ (see Assumption 3) and a set of further constraints on the distances in-between adjacent subsets even if such distances are not identical (see Assumptions 4-5). The following result is supported by parallel results in [42] (see also [43]) for cyclic self-mappings. It establishes the asymptotic convergence of the iterated sequences to cycles containing best proximity points in-between adjacent subsets in uniformly convex Banach spaces.

Theorem 10. *Assume that $(X, \|\,\|)$ is a uniformly convex Banach space with the subsets $A_i \subset X$ being all disjoint, bounded, closed, and convex for all $i \in \overline{p}$ and that $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-precyclic self-mapping subject to Assumptions 1–5 of Theorem 8.*

Then, the subsequences $\{T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*} x\}_{n\in\mathbf{Z}_{0+}} \subseteq A_k$ and $\{T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*+1} x\}_{n\in\mathbf{Z}_{0+}} \subseteq A_{k+1}$ converge to respective unique best proximity points x_k^* in A_k and $x_{k+1}^* = Tx_k^*$ in A_{k+1} for each starting point $x \in A_i$; for all $i, k \in \overline{p}$. The sequence $\{T^n x\}_{n\in\mathbf{Z}_{0+}}$ converges asymptotically to a unique cycle of $q = \sum_{i=1}^{p} x_i^*$ points:

$$\widehat{x}^* := \left(\overline{x}_{11}, \overline{x}_{12} = T\,\overline{x}_{11}, \ldots, \overline{x}_{1j_1^*} = T^{j_1^*-1}\overline{x}_{11} = x_1^*, \overline{x}_{21} = Tx_1^*, \ldots, x_{p-1}^*, \overline{x}_{p1} = Tx_{p-1}^*, \ldots, \overline{x}_{pj_p^*} = x_p^*\right) \tag{45}$$

which contains the p best proximity points.

Proof (Outline of Proof). Note that (X, d) is a complete metric space for a $\|\,\|$ (norm-) induced metric $d : X \times X \to \mathbf{R}_{0+}$ since $(X, \|\,\|)$ is a Banach space. Thus, Theorem 8 remains true for such a metric. Since the subsets $A_i \subset X$ are nonempty, bounded, and closed (then compact and also boundedly compact), there exist $x_i^* \in A_i$, $Tx_i^* A_{i+1}$ such that $d(x_i^*, Tx_i^*) = \|x_i^* - Tx_i^*\| = D_i$ for each $i \in \overline{p}$ so that x_i^* and Tx_i^* are the best proximity points in A_i to A_{i+1} and A_{i+1} from to A_i, respectively [42]. If $D_i = 0$ and $j_i^* = 1$, then both of them are confluent in the fixed point $x_i^* = Tx_i^* \in A_i \cap A_{i+1}$; $i \in \overline{p}$. From the relation (32) of Theorem 8, it follows that

$$\exists \lim_{n\to\infty} d\left(T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*+1} x, T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*} x\right) = D_k; \quad \forall x \in A_i;\ \forall i,\ k \in \overline{p}. \tag{46}$$

The composite self-mapping $T^{\sum_{i=1}^{p} j_i^*} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ Lipschitz with constant $K = \prod_{i\in\overline{p}} [K_i^{\overline{j}_i^*}]$ everywhere in its definition domain $\bigcup_{i\in\overline{p}} A_i$. Thus, the limit of the distance and the distance of limits of the sequences $\{T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*} x\}_{n\in\mathbf{Z}_{0+}} \subseteq A_k$ and $\{T^{n(\sum_{i=1}^{p} j_i^*)+\sum_{i=1}^{k} j_i^*+1} x\}_{n\in\mathbf{Z}_{0+}} \subseteq A_{k+1}$ can be interchanged in (45) if the limits of such sequences exist. But such limits exist for any initial iteration point $x \in A_i$ since both sequences are Cauchy sequences and convergent since $(X, d) \equiv (X, \|\,\|)$ is a uniformly convex Banach space (Lemmas 3.7, 3.8 of [42]) The whole sequences and their limits are within the corresponding subsets since such subsets are closed and $(X, d) \equiv (X, \|\,\|)$ is complete. The uniqueness of the best proximity points in each of the subsets follows from the fact that the subsets $A_i \subset X$; for all $i \in \overline{p}$ are convex. The convergence of the sequences $\{T^n x\}_{n\in\mathbf{Z}_{0+}}$ to a unique cycle $\widehat{x}^*$ of the form (45), containing $q = \sum_{i=1}^{p} x_i^*$ points, follows from the convergence of the above subsequences to unique best proximity points taking part of such a cycle and the fact that $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a single-valued self-mapping. □

Example 11. Consider the following nonlinear difference sequence:

$$x_{2n+i-1} = \left(1 - \alpha_{2n+i-2}\right) x_{2n+i-2} + \omega_{2n+i-2},$$

$$x_{2n+j^*} = x_{2mn+1} = -\beta_{2mn} x_{2mn}; \quad \forall n \in \mathbf{Z}_+ \quad \text{if } x_1 \in A_1 \left(\equiv \mathbf{R}_{0+}\right)$$

$$x_{2n+j^*+1} = x_{2mn+2} = -\beta_{2mn+1} x_{2mn+1},$$
$$x_{2n+i} = \left(1 - \alpha_{2n+i-1}\right) x_{2n+i-1} + \omega_{2n+i-1}; \quad \forall n \in \mathbf{Z}_+$$
$$\text{if } x_1 \in \mathbf{R}_{0-} \left(\equiv -\mathbf{R}_{0+} \equiv A_2\right) \tag{47}$$

for $i \in \overline{j^*}$ with $j^* = j_1^* = j_n^* = j_n^*(m) = 2(m-1)n + 1$, when $x_n \in A_1 \equiv \mathbf{R}_{0-}$, and $j_2^* = 1$ (that is, only one iteration remains in $A_2 \equiv \mathbf{R}_{0-}$ before each switching to $\mathbf{R}_{0+}$) defined for some positive integer $m = m(n)$ under the following constraints:

$\{\omega_n\}$ is a nonnegative real summable sequence

$\{\alpha_n\} \subset [0,1]$ is a real sequence, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\alpha_n \to 0$ as $n \to \infty$

$\{\omega_n\} \subset \mathbf{R}_{0+}$ is a real sequence and $\sum_{n=0}^{\infty} \omega_n < \infty$

$\{\beta_n\} \subset [0, \overline{\beta}]$ is a bounded real sequence fulfilling $\beta_{2mn+2} \le \beta_{2nm+1} x_{2nm+1} / x_{2mn+2}$, for all $n \in \mathbf{Z}_+$ if $x_1 \in \mathbf{R}_{0+}$ and $\beta_{2mn+1} \le \beta_{2mn-1} x_{2mn-1} / x_{2mn+1}$; for all $n \in \mathbf{Z}_+$, otherwise.

Note that the difference equation is generated by a 2-precyclic self-mapping on $\mathbf{R} = \mathbf{R}_{0+} \cup \mathbf{R}_{0-}$ defined by the solution got from any initial condition. If $m = j_n^* = 1$, then the solution point simply alternates in-between the subsets $\mathbf{R}_{0+}$ and $\mathbf{R}_{0-}$ of $\mathbf{R}$ and the mapping becomes a 2-cyclic self-mapping. If $m > 1$, the solution remains $j_n^* = 2(m-1)n + 1$ consecutive iterations in $\mathbf{R}_{0+}$ after entering it before the next switching to $\mathbf{R}_-$ if $m(n)$ is infinity; for all $n \ge n_0$ and some finite $n_0 \in \mathbf{Z}_{0+}$, then the solution remains in $\mathbf{R}_{0+}$ after a given finite iteration. The following nonexpansive condition holds:

$$\left|x_{2n+j^*(n)+1}\right| \le \beta_{2m(n)n+1} \left(\prod_{i=2n-1}^{2m(n)} \left[1 - \alpha_i\right] \right) \left|x_{2n}\right| + \sum_{i=2n-1}^{2m(n)} \prod_{j=i+1}^{2m(n)} \left[1 - \alpha_j\right] \omega_{2i}; \tag{48}$$

for all $n \ge n_1$ since $\{\omega_n\}$ is nonnegative, summable, and then converges to zero, so that it has some strictly decreasing subsequence $\{\omega_n\}$ and then $\omega_{2n} - \omega_{2n-1} < 0$; for all $n \ge n_1$ for some finite positive integer n_1. It follows, since $\alpha_n \to 0$ as $n \to \infty$, that

(a) if $\beta_n \to 1$ as $n \to \infty$, then the solution is weakly 2-precyclic asymptotically nonexpansive since $\alpha_n \to 0$ and $\omega_n \to 0$ as $n \to \infty$ and then $\limsup_{n\to\infty}(|x_{2n+j^*(n)+1}| - |x_{2n}|) \le 0$, $\liminf_{n\to\infty}(|x_{2n+j^*(n)+1}| - |x_{2n}|) \ge 0$ and $\{x_n\}$ converges.

If $j_n^*(x) = 1$ for any $n \in \mathbf{Z}_{0+}$ and $x \in \mathbf{R}$, then the mapping defining the solution is also 2-cyclic asymptotically nonexpansive.

(b) If $\beta_n \to 1$ as $n \to \infty$ and there are no subsequences of $\{\omega_n\}$ and $\{\alpha_n\}$ being simultaneously zero, then the solution is 2-precyclic (2-cyclic if j_n^* is identically unity) weakly asymptotically contractive so that it converges to zero which is the fixed point being a confluent best proximity point of $\mathbf{R}_{0+}$ and $\mathbf{R}_{0-}$.

If $\alpha_n \to \alpha \in (0,1)$ or if $\limsup_{n\to\infty} \alpha_n \le \overline{\alpha} < 1$, then the solution is 2-precyclic (2-cyclic if j_n^* is identically unity) strongly asymptotically contractive so that it converges to zero. If there is some finite positive integer n_0 such that $j_{n_0}^*$ is infinite, then the solution is permanent and nonnegative in $\mathbf{R}_{0+}$ after a finite number of iterations and converges asymptotically to zero.

It is now proven that, in the most general considered case when the solution mapping is weakly 2-precyclic asymptotically nonexpansive, the fixed point (which is also stable equilibrium point and best proximity point on both subsets) is $x = 0$.

Define $\overline{x}_1 = x_1$, if $x_1 \in \mathbf{R}_{0+}$ and $\overline{x}_1 = x_2$, then $\overline{x}_0 \in \mathbf{R}_{0+}$, if $x_1 \in \mathbf{R}_-$ and build the sequence $\{\overline{x}_n\}$ by $\overline{x}_n = x_n$ if $x_n \ge 0$ and $\overline{x}_n = 0$, otherwise. Then, one gets

$$-\infty < -\overline{x}_1 = \lim_{n\to\infty} \left(\overline{x}_{n+1} - \overline{x}_1\right) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \left(\omega_i - \alpha_i \overline{x}_i\right) \right) \tag{49}$$

so that $\sum_{n=1}^{\infty}(\alpha_n \overline{x}_n - \omega_n) = \overline{x}_1 < \infty$. Since $\{\omega_n\}$ is summable and $\sum_{i=1}^{\infty} \alpha_n = \infty$, one gets

$$\left(\min_{n\in\mathbf{Z}_+} \overline{x}_n \right) \left(\sum_{i=1}^{\infty} \alpha_n \right) \le \sum_{i=1}^{\infty} \alpha_n \overline{x}_n = \overline{x}_1 + \sum_{n=1}^{\infty} \omega_n < \infty$$
$$\Longrightarrow \min_{n\in\mathbf{Z}_+} \overline{x}_n = \min_{n\in\mathbf{Z}_+} x_n = 0. \tag{50}$$

Also, $\max_{n\in\mathbf{Z}_-}(|x_n|) \le \lim_{n\to\infty}(\beta_n) \min_{n\in\mathbf{Z}_{0+}} x_n \le \min_{n\in\mathbf{Z}_{0+}} x_n = 0$. As a result, the fixed point, and also equilibrium point, is $x = 0$ and $x_n \to 0$ as $n \to \infty$ for any initial condition $x_1 \in \mathbf{R}$.

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