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Research Article

Oscillation for Higher Order Dynamic Equations on Time Scales

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We investigate the oscillation of the following higher order dynamic equation: $\{a_n(t)[(a_{n-1}(t)(\cdots(a_1(t)x^{\Delta}(t))^{\Delta}\cdots)^{\Delta})^{\Delta}]^{\alpha}\}^{\Delta} + p(t)x^{\beta}(t) = 0$, on some time scale $\mathbf{T}$, where $n \geq 2$, $a_k(t)$ $(1 \leq k \leq n)$ and $p(t)$ are positive rd-continuous functions on $\mathbf{T}$ and α, β are the quotient of odd positive integers. We give sufficient conditions under which every solution of this equation is either oscillatory or tends to zero.

1. Introduction

In this paper, we investigate the oscillation of the following higher order dynamic equation:

$$\left\{a_n(t)\left[\left(a_{n-1}(t)\left(\cdots\left(a_1(t)x^{\Delta}(t)\right)^{\Delta}\cdots\right)^{\Delta}\right)^{\Delta}\right]^{\alpha}\right\}^{\Delta} + p(t)x^{\beta}(t) = 0, \tag{E}$$

on some time scale $\mathbf{T}$, where $n \geq 2$, $a_k(t)$ $(1 \leq k \leq n)$ and $p(t)$ are positive rd-continuous functions on $\mathbf{T}$ and α, β are the quotient of odd positive integers. Write

$$S_k(t, x(t)) = \begin{cases} x(t), & \text{if } k = 0, \\ a_k(t)S_{k-1}^{\Delta}(t, x(t)), & \text{if } 1 \leq k \leq n-1, \\ a_n(t)\left[S_{n-1}^{\Delta}(t, x(t))\right]^{\alpha}, & \text{if } k = n, \end{cases} \tag{1}$$

then (E) reduces to the following equation:

$$S_n^{\Delta}(t, x(t)) + p(t)x^{\beta}(t) = 0. \tag{2}$$

Since we are interested in the oscillatory behavior of solutions near infinity, we assume that $\sup \mathbf{T} = \infty$ and $t_0 \in \mathbf{T}$ is a constant. For any $a \in \mathbf{T}$, we define the time scale interval $[a, \infty)_{\mathbf{T}} = \{t \in \mathbf{T} : t \geq a\}$. By a solution of (2), we mean a nontrivial real-valued function $x(t) \in C^1_{\mathrm{rd}}[T_x, \infty)$, $T_x \geq t_0$, which has the property that $S_k(t, x(t)) \in C^1_{\mathrm{rd}}[T_x, \infty)$ for $0 \leq k \leq n$ and satisfies (2) on $[T_x, \infty)$, where C^1_{rd} is the space of differentiable functions whose derivative is rd-continuous. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution $x(t)$ of (2) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory.

The theory of time scale, which has recently received a lot of attention, was introduced by Hilger's landmark paper [1], a rapidly expanding body of the literature that has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus, where a time scale is an nonempty closed subset of the real numbers, and the cases when this time scale is equal to the real numbers or to the integers represent the classical theories of differential or of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [2]). The new theory of the so-called "dynamic equations" not only unifies the theories of differential equations and difference equations, but also extends these classical cases to cases "in between," for example, to the so-called q-difference equations when $\mathbf{T} = \{1, q, q^2, \ldots, q^k, \ldots\}$, which has important applications in quantum theory (see [3]). In this work, knowledge and understanding of time scales and time scale notation are assumed; for an excellent introduction to the calculus on time scales, see Bohner and Peterson [2, 4]. In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to the papers [5–20].

Recently, Erbe et al. in [21–23] considered the third-order dynamic equations

$$\left(a(t)\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right)^{\Delta}+p(t)f(x(t))=0,$$

$$x^{\Delta\Delta\Delta}(t)+p(t)x(t)=0, \tag{3}$$

$$\left(a(t)\left\{\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right\}^{\gamma}\right)^{\Delta}+f(t,x(t))=0,$$

respectively, and established some sufficient conditions for oscillation.

Hassan [24] studied the third-order dynamic equations

$$\left(a(t)\left\{\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right\}^{\gamma}\right)^{\Delta}+f(t,x(\tau(t)))=0 \tag{4}$$

and obtained some oscillation criteria, which improved and extended the results that have been established in [21–23].

2. Main Results

In this section, we investigate the oscillation of (2). To do this, we need the following lemmas.

Lemma 1 (see [25]). *Assume that*

$$\int_{t_0}^{\infty}\left[\frac{1}{a_n(s)}\right]^{1/\alpha}\Delta s=\int_{t_0}^{\infty}\frac{\Delta s}{a_i(s)}=\infty \quad \forall 1\le i\le n-1, \tag{5}$$

and $1\le m\le n$*. Then,*

(1) $\liminf_{t\to\infty}S_m(t,x(t))>0$ *implies* $\lim_{t\to\infty}S_i(t,x(t))=\infty$ *for* $0\le i\le m-1$*;*

(2) $\limsup_{t\to\infty}S_m(t,x(t))<0$ *implies* $\lim_{t\to\infty}S_i(t,x(t))=-\infty$ *for* $0\le i\le m-1$.

Lemma 2 (see [25]). *Assume that* (5) *holds. If* $S_n^{\Delta}(t,x(t))<0$ *and* $x(t)>0$ *for* $t\ge t_0$*, then there exists an integer* $0\le m\le n$ *with* $m+n$ *even such that*

(1) $(-1)^{m+i}S_i(t,x(t))>0$ *for* $t\ge t_0$ *and* $m\le i\le n$*;*

(2) *if* $m>1$*, then there exists* $T\ge t_0$ *such that* $S_i(t,x(t))>0$ *for* $1\le i\le m-1$ *and* $t\ge T$.

Remark 3. Let $a_n(t)=\cdots=a_1(t)=1$, and let $\mathbf{T}$ be the set of integers. Then, Lemmas 1 and 2 are Lemma 1.8.10 and Theorem 1.8.11 of [26], respectively.

Lemma 4. *Assume that* (5) *holds. Furthermore, suppose that*

$$\int_{t_0}^{\infty}\frac{1}{a_{n-1}(u)}\left\{\int_{u}^{\infty}\left[\frac{1}{a_n(s)}\int_{s}^{\infty}p(v)\Delta v\right]^{1/\alpha}\Delta s\right\}\Delta u=\infty. \tag{6}$$

If $x(t)$ *is an eventually positive solution of* (2)*, then there exists* $T\ge t_0$ *sufficiently large such that*

(1) $S_n^{\Delta}(t,x(t))<0$ *for* $t\ge T$*;*

(2) *either* $S_i(t,x(t))>0$ *for* $t\ge T$ *and* $0\le i\le n$ *or* $\lim_{t\to\infty}x(t)=0$.

Proof. Pick $t_1\ge t_0$ so that $x(t)>0$ on $[t_1,\infty)_{\mathbf{T}}$. It follows from (2) that

$$S_n^{\Delta}(t,x(t))=-p(t)x^{\beta}(t)<0 \quad \text{for } t\ge t_1. \tag{7}$$

By Lemma 2, we see that there exists an integer $0\le m\le n$ with $m+n$ even such that $(-1)^{m+i}S_i(t,x(t))>0$ for $t\ge t_1$ and $m\le i\le n$, and $x(t)$ is eventually monotone.

We claim that $\lim_{t\to\infty}x(t)\neq 0$ implies $m=n$. If not, then $S_{n-1}(t,x(t))<0(t\ge t_1)$ and $S_{n-2}(t,x(t))>0$ $(t\ge t_1)$, and there exist $t_2\ge t_1$ and a constant $c>0$ such that $x(t)\ge c$ on $[t_2,\infty)_{\mathbf{T}}$. Integrating (2) from t into ∞, we get that for $t\ge t_2$

$$-a_n(t)\left[S_{n-1}^{\Delta}(t,x(t))\right]^{\alpha}=-S_n(t,x(t))\le -c^{\beta}\int_t^{\infty}p(v)\Delta v. \tag{8}$$

Thus,

$$S_{n-1}(t,x(t))\le -c^{\beta/\alpha}\times\int_t^{\infty}\left[\frac{1}{a_n(s)}\int_s^{\infty}p(v)\Delta v\right]^{1/\alpha}\Delta s \quad \text{for } t\ge t_2. \tag{9}$$

Again, integrating the above inequality from t_2 into t, we obtain that for $t\ge t_2$

$$\begin{aligned}S_{n-2}(t,x(t)) &\le S_{n-2}(t_2,x(t_2))\\ &\quad -c^{\beta/\alpha}\int_{t_2}^{t}\frac{1}{a_{n-1}(u)}\left\{\int_u^{\infty}\left[\frac{1}{a_n(s)}\times\int_s^{\infty}p(v)\Delta v\right]^{1/\alpha}\Delta s\right\}\Delta u.\end{aligned} \tag{10}$$

It follows from (6) that $\lim_{t\to\infty}S_{n-2}(t,x(t))=-\infty$, which is a contradiction to $S_{n-2}(t,x(t))>0$ $(t\ge t_1)$. The proof is completed. □

Lemma 5. *Assume that* $x(t)$ *is an eventually positive solution of* (2) *such that* $S_n^{\Delta}(t,x(t))<0$ *for* $t\ge T\ge t_0$ *and* $S_i(t,x(t))>0$ *for* $t\ge T$ *and* $0\le i\le n$*. Then,*

$$S_i(t,x(t))\ge S_n^{1/\alpha}(t,x(t))B_{i+1}(t,T) \quad \text{for } 0\le i\le n-1,\quad t\ge T, \tag{11}$$

and there exist $T_1>T$ *and a constant* $c>0$ *such that*

$$x(t)\le cB_1(t,T) \quad \text{for } t\ge T_1, \tag{12}$$

where

$$B_i(t,T)=\begin{cases}\int_T^t\left[\dfrac{1}{a_n(s)}\right]^{1/\alpha}\Delta s, & \text{if } i=n,\\ \int_T^t \dfrac{B_{i+1}(s,T)}{a_i(s)}\Delta s, & \text{if } 1\le i\le n-1.\end{cases}\tag{13}$$

Proof. Since $S_n^{\Delta}(t,x(t))<0$ $(t\ge T)$, it follows that $S_n(t,x(t))$ is strictly decreasing on $[T,\infty)_{\mathbf{T}}$. Then, for $t\ge T$,

$$\begin{aligned}S_{n-1}(t,x(t))&\ge S_{n-1}(t,x(t))-S_{n-1}(T,x(T))\\&=\int_T^t\left[\frac{S_n(s,x(s))}{a_n(s)}\right]^{1/\alpha}\Delta s\\&\ge S_n^{1/\alpha}(t,x(t))B_n(t,T)\\ S_{n-2}(t,x(t))&\ge S_{n-2}(t,x(t))-S_{n-2}(T,x(T))\\&=\int_T^t\frac{S_{n-1}(s,x(s))}{a_{n-1}(s)}\Delta s\\&\ge S_n^{1/\alpha}(t,x(t))B_{n-1}(t,T)\\&\vdots\\ S_1(t,x(t))&\ge S_1(t,x(t))-S_1(T,x(T))\\&=\int_T^t\frac{S_2(s,x(s))}{a_2(s)}\Delta s\ge S_n^{1/\alpha}(t,x(t))B_2(t,T)\\ S_0(t,x(t))&\ge x(t)-x(T)\\&=\int_T^t\frac{S_1(s,x(s))}{a_1(s)}\Delta s\ge S_n^{1/\alpha}(t,x(t))B_1(t,T).\end{aligned}\tag{14}$$

On the other hand, we have that for $t\ge T$,

$$\begin{aligned}S_{n-1}(t,x(t))&=\int_T^t\left[\frac{S_n(s,x(s))}{a_n(s)}\right]^{1/\alpha}\Delta s+S_{n-1}(T,x(T))\\&\le S_{n-1}(T,x(T))+S_n^{1/\alpha}(T,x(T))B_n(t,T).\end{aligned}\tag{15}$$

Thus, there exist $T_1>T$ and a constant $b_1>0$ such that

$$S_{n-1}(t,x(t))\le b_1B_n(t,T)\quad \text{for } t\ge T_1.\tag{16}$$

Again,

$$\begin{aligned}S_{n-2}(t,x(t))&=S_{n-2}(T_1,x(T_1))+\int_{T_1}^t\frac{S_{n-1}(s,x(s))}{a_{n-1}(s)}\Delta s\\&\le S_{n-2}(T_1,x(T_1))+b_1\int_T^t\frac{B_n(s,T)}{a_{n-1}(s)}\Delta s.\end{aligned}\tag{17}$$

Thus, there exists a constant $b_2>0$ such that

$$\begin{aligned}S_{n-2}(t,x(t))&\le b_2\int_T^t\frac{B_n(s,T)}{a_{n-1}(s)}\Delta s\\&=b_2B_{n-1}(t,T)\quad\text{for } t\ge T_1.\end{aligned}\tag{18}$$

Again,

$$\begin{aligned}S_{n-3}(t,x(t))&=S_{n-3}(T_1,x(T_1))+\int_{T_1}^t\frac{S_{n-2}(s,x(s))}{a_{n-2}(s)}\Delta s\\&\le S_{n-3}(T_1,x(T_1))+b_2\int_T^t\frac{B_{n-1}(s,T)}{a_{n-2}(s)}\Delta s.\end{aligned}\tag{19}$$

Thus, there exists a constant $b_3>0$ such that

$$\begin{aligned}S_{n-3}(t,x(t))&\le b_3\int_T^t\frac{B_{n-1}(s,T)}{a_{n-2}(s)}\Delta s\\&=b_3B_{n-2}(t,T)\quad\text{for } t\ge T_1.\end{aligned}\tag{20}$$

The rest of the proof is by induction. The proof is completed. □

Lemma 6 (see [2]). *Let $f:\mathbf{R}\to\mathbf{R}$ be continuously differentiable and suppose that $g:\mathbf{T}\to\mathbf{R}$ is delta differentiable. Then, $f\circ g$ is delta differentiable and the formula*

$$(f\circ g)^{\Delta}(t)=g^{\Delta}(t)\int_0^1 f'\left(hg(t)+(1-h)g^{\sigma}(t)\right)dh.\tag{21}$$

Lemma 7 (see [27]). *If A,B are nonnegative and $\lambda>1$, then*

$$A^{\lambda}-\lambda AB^{\lambda-1}+(\lambda-1)B^{\lambda}\ge 0.\tag{22}$$

Now, we state and prove our main results.

Theorem 8. *Suppose that* (5) *and* (6) *hold. If there exists a positive nondecreasing delta differentiable function θ such that for all sufficiently large $T\in[t_0,\infty)_{\mathbf{T}}$ and for any positive constants c_1,c_2, there is a $T_1>T$ such that*

$$\limsup_{t\to\infty}\int_{T_1}^t\left[\theta(s)p(s)-\frac{\theta^{\Delta}(s)}{B_1^{\alpha}(s,T)}\delta_1(t,T,c_1,c_2)\right]\Delta s=\infty,\tag{23}$$

where

$$\delta_1(t,T,c_1,c_2)=\begin{cases}c_1, & \text{if } \alpha<\beta,\\ 1, & \text{if } \alpha=\beta,\\ c_2B_1^{\alpha-\beta}(t,T), & \text{if } \alpha>\beta,\end{cases}\tag{24}$$

and $B_1(t,T)$ is as in Lemma 5. Then, every solution of (2) *is either oscillatory or tends to zero.*

Proof. Assume that (2) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbf{T}}$. Then, without loss of generality, there is a $t_1\ge t_0$, sufficiently large, such that $x(t)>0$ for $t\ge t_1$. Therefore, we get from Lemma 4 that there exists $t_2\ge t_1$ such that

(i) $S_n^{\Delta}(t,x(t))<0$ for $t\ge t_2$;

(ii) either $S_i(t,x(t))>0$ for $t\ge t_2$ and $0\le i\le n$ or $\lim_{t\to\infty}x(t)=0$.

Let $S_i(t,x(t))>0$ for $t\geq t_2$ and $0\leq i\leq n$. Consider

$$w(t)=\theta(t)\frac{S_n(t,x(t))}{x^\beta(t)}\quad \text{for } t\geq t_2. \tag{25}$$

It follows from Lemma 6 that

$$\left(x^\beta\right)^\Delta(t)=\beta x^\Delta(t)\int_0^1\left(hx(t)+(1-h)x(t)^\sigma\right)^{\beta-1}dh>0 \quad \text{for } t\geq t_2. \tag{26}$$

Then,

$$\begin{aligned} w^\Delta &= \left[\frac{\theta}{x^\beta}\right]^\Delta S_n^\sigma(\cdot,x)+\frac{\theta}{x^\beta}S_n^\Delta(\cdot,x)\\ &= \left[\frac{\theta^\Delta}{(x^\beta)^\sigma}-\frac{\theta\left(x^\beta\right)^\Delta}{x^\beta(x^\beta)^\sigma}\right]S_n^\sigma(\cdot,x)-\theta p\\ &\leq \frac{\theta^\Delta}{x^\beta}S_n(\cdot,x)-\theta p. \end{aligned} \tag{27}$$

From (11) and (27), we get

$$w^\Delta(t)\leq\frac{\theta^\Delta(t)}{B_1^\alpha(t,t_2)}x^{\alpha-\beta}(t)-\theta(t)p(t)\quad \text{for } t\geq t_2. \tag{28}$$

Now, we consider the following three cases.

Case 1. If $\alpha=\beta$, then

$$x^{\alpha-\beta}(t)=1\quad \text{for } t\geq t_2. \tag{29}$$

Case 2. If $\alpha>\beta$, then it follows from (12) that there exist $t_3>t_2$ and a constant $c_2>0$ such that

$$x^{\alpha-\beta}(t)\leq c_2B_1^{\alpha-\beta}(t,t_2)\quad \text{for } t\geq t_3. \tag{30}$$

Case 3. If $\alpha<\beta$, then

$$x(t)\geq x(t_2)\quad \text{for } t\geq t_2. \tag{31}$$

Thus,

$$x^{\alpha-\beta}(t)\leq c_1=x^{\alpha-\beta}(t_2)\quad \text{for } t\geq t_2. \tag{32}$$

From (27)–(32), we obtain

$$w^\Delta(t)\leq\frac{\theta^\Delta(t)}{B_1^\alpha(t,t_2)}\delta_1(t,t_2,c_1,c_2)-\theta(t)p(t)\quad \text{for } t\geq t_3. \tag{33}$$

Integrating the above inequality from t_3 into t, we have

$$\int_{t_3}^t\left[\theta(s)p(s)-\frac{\theta^\Delta(s)}{B_1^\alpha(s,t_2)}\delta_1(s,t_2,c_1,c_2)\right]\Delta s\leq w(t_3)<\infty, \tag{34}$$

which gives a contradiction to (23). The proof is completed. □

Theorem 9. *Suppose that (5) and (6) hold. If there exists a positive nondecreasing delta differentiable function θ such that for all sufficiently large $T\in[t_0,\infty)_{\mathbb{T}}$ and for any positive constants c_1,c_2, there is a $T_1>T$ such that*

$$\limsup_{t\to\infty}\int_{T_1}^t\left[\theta(s)p(s)-\frac{(\alpha/\beta)^\alpha\left(\theta^\Delta(s)\right)^{\alpha+1}a_1^\alpha(s)}{(\alpha+1)^{\alpha+1}\left(B_2(s,T)\theta(s)\delta_2(s,T,c_1,c_2)\right)^\alpha}\right]\Delta s=\infty, \tag{35}$$

where

$$\delta_2(t,T,c_1,c_2)=\begin{cases}c_1 & \text{if } \alpha<\beta,\\ 1, & \text{if } \alpha=\beta,\\ c_2B_1^{(\beta/\alpha)-1}(\sigma(t),T), & \text{if } \alpha>\beta,\end{cases} \tag{36}$$

and $B_1(t,T)$, $B_2(t,T)$ are as in Lemma 5. Then, every solution of (2) is either oscillatory or tends to zero.

Proof. Assume that (2) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1\geq t_0$, sufficiently large, such that $x(t)>0$ for $t\geq t_1$. Therefore, we get from Lemma 4 that there exists $t_2\geq t_1$ such that

(i) $S_n^\Delta(t,x(t))<0$ for $t\geq t_2$;

(ii) either $S_i(t,x(t))>0$ for $t\geq t_2$ and $0\leq i\leq n$ or $\lim_{t\to\infty}x(t)=0$.

Let $S_i(t,x(t))>0$ for $t\geq t_2$ and $0\leq i\leq n$. Note that

$$\begin{aligned} \left(x^\beta\right)^\Delta &= \beta x^\Delta\int_0^1\left(hx+(1-h)x^\sigma\right)^{\beta-1}dh\\ &= \beta x^\Delta\int_0^1\frac{\left(hx+(1-h)x^\sigma\right)^\beta}{hx+(1-h)x^\sigma}dh\\ &\geq \beta x^\Delta\frac{x^\beta}{x^\sigma}. \end{aligned} \tag{37}$$

From (11), we have

$$\begin{aligned} \frac{\left(x^\beta\right)^\Delta}{x^\beta} &\geq \beta\frac{x^\Delta}{x^\sigma}\geq\beta\frac{S_n^{1/\alpha}(\cdot,x)B_2(\cdot,t_2)}{a_1x^\sigma}\\ &\geq \beta\frac{\left(S_n^\sigma(\cdot,x)\right)^{1/\alpha}B_2(\cdot,t_2)}{a_1x^\sigma}\\ &= \beta\frac{\left(w^\sigma\right)^{1/\alpha}}{a_1(\theta^\sigma)^{1/\alpha}}\left(x^\sigma\right)^{(\beta/\alpha)-1}B_2(\cdot,t_2). \end{aligned} \tag{38}$$

Then it follows from (27) that for $t \geq t_2$,

$$\begin{aligned} w^{\Delta} &= \left[\frac{\theta}{x^{\beta}}\right]^{\Delta} S_n^{\sigma}(\cdot, x) + \frac{\theta}{x^{\beta}} S_n^{\Delta}(\cdot, x) \\ &= \left[\frac{\theta^{\Delta}}{(x^{\beta})^{\sigma}} - \frac{\theta (x^{\beta})^{\Delta}}{x^{\beta}(x^{\beta})^{\sigma}}\right] S_n^{\sigma}(\cdot, x) - \theta p \\ &\leq \theta^{\Delta}\frac{w^{\sigma}}{\theta^{\sigma}} - \beta \frac{B_2(\cdot, t_2)\,\theta}{a_1} \frac{(w^{\sigma})^{1+(1/\alpha)}}{(\theta^{\sigma})^{1+(1/\alpha)}} (x^{\sigma})^{(\beta/\alpha)-1} - \theta p. \end{aligned} \tag{39}$$

Now, we consider the following three cases.

Case 1. If $\alpha = \beta$, then

$$(x^{\sigma})^{(\beta/\alpha)-1}(t) = 1 \quad \text{for } t \geq t_2. \tag{40}$$

Case 2. If $\alpha > \beta$, then it follows from (12) that there exist $t_3 > t_2$ and a constant c such that

$$x(t) \leq c B_1(t, t_2) \quad \text{for } t \geq t_3. \tag{41}$$

Thus,

$$(x^{\sigma})^{(\beta/\alpha)-1}(t) \geq c_2 B_1^{(\beta/\alpha)-1}(\sigma(t), t_2), \tag{42}$$

with $c_2 = c^{(\beta/\alpha)-1}$.

Case 3. If $\alpha < \beta$, then

$$x(t) \geq x(t_2) \quad \text{for } t \geq t_2. \tag{43}$$

Thus,

$$(x^{\sigma})^{(\beta/\alpha)-1}(t) \geq c_1 = x^{(\beta/\alpha)-1}(t_2). \tag{44}$$

From (39)–(44), we obtain that for $t \geq t_3$,

$$\begin{aligned} w^{\Delta} &\leq \frac{w^{\sigma}}{\theta^{\sigma}}\theta^{\Delta} - \frac{\beta B_2(\cdot, t_2)\,\theta\delta_2(\cdot, t_2, c_1, c_2)}{a_1} \frac{(w^{\sigma})^{1+(1/\alpha)}}{(\theta^{\sigma})^{1+(1/\alpha)}} - \theta p \\ &= -\frac{\beta B_2(\cdot, t_2)\,\theta\delta_2(\cdot, t_2, c_1, c_2)}{a_1} \\ &\quad \times \left\{ \frac{(w^{\sigma})^{1+(1/\alpha)}}{(\theta^{\sigma})^{1+(1/\alpha)}} - \frac{w^{\sigma}}{\theta^{\sigma}} \frac{a_1\theta^{\Delta}}{\beta B_2(\cdot, t_2)\,\theta\delta_2(\cdot, t_2, c_1, c_2)} \right\} - \theta p. \end{aligned} \tag{45}$$

Let

$$A = \frac{w^{\sigma}}{\theta^{\sigma}}, \qquad B = \left[\frac{\alpha a_1 \theta^{\Delta}}{(\alpha+1)\,\beta B_2(\cdot, t_2)\,\theta\delta_2(\cdot, t_2, c_1, c_2)}\right]^{\alpha}, \tag{46}$$

with $\lambda = 1 + 1/\alpha$. By Lemma 7, we have

$$w^{\Delta} \leq \frac{(\alpha/\beta)^{\alpha}(\theta^{\Delta})^{\alpha+1} a_1^{\alpha}}{(\alpha+1)^{\alpha+1}(B_2(\cdot, t_2)\,\theta\delta_2(\cdot, t_2, c_1, c_2))^{\alpha}} - \theta p. \tag{47}$$

Integrating the above inequality from t_3 into t, it follows that

$$\begin{aligned} &\int_{t_3}^{t} \left[\theta(s)\, p(s) \vphantom{\frac{a}{b}}\right. \\ &\quad \left. - \frac{(\alpha/\beta)^{\alpha}(\theta^{\Delta}(s))^{\alpha+1} a_1^{\alpha}(s)}{(\alpha+1)^{\alpha+1}(B_2(s, t_2)\,\theta(s)\,\delta_2(s, t_2, c_1, c_2))^{\alpha}}\right] \Delta s \\ &\leq w(t_3) < \infty, \end{aligned} \tag{48}$$

which gives a contradiction to (35). The proof is completed. □

Remark 10. The trick used in the proofs of Theorems 8 and 9 is from [16].

Theorem 11. *Suppose that* (5) *and* (6) *hold. If for all sufficiently large* $T \in [t_0, \infty)_{\mathbf{T}}$,

$$\int_{T}^{\infty} p(u)\left[\int_{T}^{u} \frac{\Delta s}{a_1(s)}\right]^{\beta} \Delta u = \infty, \tag{49}$$

then every solution of (2) *is either oscillatory or tends to zero.*

Proof. Assume that (2) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbf{T}}$. Then, without loss of generality, there is a $t_1 \geq t_0$, sufficiently large, such that $x(t) > 0$ for $t \geq t_1$. Therefore, we get from Lemma 4 that there exists $t_2 \geq t_1$ such that

(i) $S_n^{\Delta}(t, x(t)) < 0$ for $t \geq t_2$;

(ii) either $S_i(t, x(t)) > 0$ for $t \geq t_2$ and $0 \leq i \leq n$ or $\lim_{t\to\infty} x(t) = 0$.

Let $S_i(t, x(t)) > 0$ for $t \geq t_2$ and $0 \leq i \leq n$. Then, for $t \geq t_2$,

$$\begin{aligned} x(t) &= x(t_2) + \int_{t_2}^{t} \frac{S_1(s, x(s))}{a_1(s)} \Delta s \\ &\geq S_1(t_2, x(t_2)) \int_{t_2}^{t} \frac{\Delta s}{a_1(s)}. \end{aligned} \tag{50}$$

It follows from (2) that

$$-S_n^{\Delta}(t, x(t)) \geq p(t)\left[S_1(t_2, x(t_2)) \int_{t_2}^{t} \frac{\Delta s}{a_1(s)}\right]^{\beta}. \tag{51}$$

Integrating the above inequality from t_2 into ∞, we have

$$S_n(t_2, x(t_2)) \geq S_1^{\beta}(t_2, x(t_2)) \int_{t_2}^{\infty} p(u)\left[\int_{t_2}^{u} \frac{\Delta s}{a_1(s)}\right]^{\beta} \Delta u, \tag{52}$$

which gives a contradiction to (49). The proof is completed. □

Theorem 12. *Suppose that* (5) *and* (6) *hold. If for all sufficiently large* $T \in [t_0, \infty)_{\mathbf{T}}$,

$$\limsup_{t\to\infty} B_1^{\alpha}(t, T)\,\delta_3(t, T, c_1, c_2) \int_{t}^{\infty} p(s)\,\Delta s > 1, \tag{53}$$

where

$$\delta_3(t,T,c_1,c_2) = \begin{cases} c_1, c_1 \text{ is any positive constant,} & \\ \qquad\qquad \text{if } \alpha<\beta, & \\ 1, \qquad \text{if } \alpha=\beta, & \\ c_2 B_1^{\beta-\alpha}(t,T), c_2 \text{ is any positive constant,} & \\ \qquad\qquad \text{if } \alpha>\beta, & \end{cases} \tag{54}$$

and $B_1(t,T)$ is as in Lemma 5, then every solution of (2) is either oscillatory or tends to zero.

Proof. Assume that (2) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$. Then, without loss of generality, there is a $t_1 \geq t_0$, sufficiently large, such that $x(t)>0$ for $t \geq t_1$. Therefore, we get from Lemma 4 that there exists $t_2 \geq t_1$ such that

(i) $S_n^{\Delta}(t,x(t))<0$ for $t \geq t_2$;

(ii) either $S_i(t,x(t))>0$ for $t \geq t_2$ and $0 \leq i \leq n$ or $\lim_{t\to\infty} x(t)=0$.

Let $S_i(t,x(t))>0$ for $t \geq t_2$ and $0 \leq i \leq n$. Then, it follows from (2) and (11) that for $t \geq t_2$,

$$\int_t^{\infty} p(s)x^{\beta}(s)\Delta s \leq S_n(t,x(t)) \leq \left[\frac{x(t)}{B_1(t,t_2)}\right]^{\alpha}. \tag{55}$$

Using the fact that $x(t)$ is strictly increasing on $[t_2,\infty)_{\mathbf{T}}$, we obtain

$$x^{\beta}(t)\int_t^{\infty} p(s)\Delta s \leq \left[\frac{x(t)}{B_1(t,t_2)}\right]^{\alpha}. \tag{56}$$

Thus,

$$B_1^{\alpha}(t,t_2)x^{\beta-\alpha}(t)\int_t^{\infty} p(s)\Delta s \leq 1. \tag{57}$$

Now, we consider the following three cases.

Case 1. If $\alpha=\beta$, then

$$x^{\beta-\alpha}(t)=1 \quad \text{for } t \geq t_2. \tag{58}$$

Case 2. If $\alpha>\beta$, then it follows from (12) that there exist $t_3>t_2$ and a constant c such that

$$x(t) \leq cB_1(t,t_2) \quad \text{for } t \geq t_3. \tag{59}$$

Thus,

$$x^{\beta-\alpha}(t) \geq c_2 B_1^{\beta-\alpha}(t,t_2), \tag{60}$$

with $c_2=c^{\beta-\alpha}$.

Case 3. If $\alpha<\beta$, then

$$x(t) \geq x(t_2) \quad \text{for } t \geq t_2. \tag{61}$$

Thus,

$$x^{\beta-\alpha}(t) \geq c_1 = x^{\beta-\alpha}(t_2). \tag{62}$$

From (57)–(62), we obtain that for $t \geq t_3$,

$$B_1^{\alpha}(t,t_2)\delta_3(t,t_2,c_1,c_2)\int_t^{\infty} p(s)\Delta s \leq 1, \tag{63}$$

which gives a contradiction to (53). The proof is completed. □

3. Examples

In this section, we give some examples to illustrate our main results.

Example 1. Consider the following higher order dynamic equation:

$$S_n^{\Delta}(t,x(t))+t^{\gamma}x^{\beta}(t)=0, \tag{64}$$

on an arbitrary time scale $\mathbf{T}$ with $\sup \mathbf{T}=\infty$, where $n \geq 2$, α,β and $S_k(t,x(t))$ $(0 \leq k \leq n)$ are as in (2) with $a_n(t)=t^{\alpha-1}, a_{n-1}(t)=\cdots=a_1(t)=t$, and $\gamma>-1$. Then, every solution of (64) is either oscillatory or tends to zero.

Proof. Note that

$$\begin{aligned} \int_{t_0}^{\infty}\left[\frac{1}{a_n(s)}\right]^{1/\alpha}\Delta s &= \int_{t_0}^{\infty}\left[\frac{1}{s^{\alpha-1}}\right]^{1/\alpha}\Delta s=\infty, \\ \int_{t_0}^{\infty}\frac{\Delta s}{a_i(s)} &= \int_{t_0}^{\infty}\frac{\Delta s}{s}=\infty \quad \text{for } 1 \leq i \leq n-1, \\ \int_{t_0}^{\infty} p(s)\Delta s &= \int_{t_0}^{\infty} s^{\gamma}\Delta s=\infty, \end{aligned} \tag{65}$$

by Example 5.60 in [4]. Pick $t_1>t_0$ such that

$$\int_{t_0}^{t_1}\frac{1}{a_{n-1}(u)}\left\{\int_u^{t_1}\left[\frac{1}{a_n(s)}\right]^{1/\alpha}\Delta s\right\}\Delta u>0. \tag{66}$$

Then,

$$\begin{aligned} &\int_{t_0}^{\infty}\frac{1}{a_{n-1}(u)}\left\{\int_u^{\infty}\left[\frac{1}{a_n(s)}\int_s^{\infty}p(v)\Delta v\right]^{1/\alpha}\Delta s\right\}\Delta u \\ &\quad \geq \left[\int_{t_1}^{\infty}p(v)\Delta v\right]^{1/\alpha} \\ &\quad \times \int_{t_0}^{t_1}\frac{1}{a_{n-1}(u)}\left(\int_u^{t_1}\left[\frac{1}{a_n(s)}\right]^{1/\alpha}\Delta s\right)\Delta u=\infty. \end{aligned} \tag{67}$$

Let $T \in [t_0, \infty)_{\mathbf{T}}$, sufficiently large, and $u_1 > T$ such that $\int_T^{u_1} (1/a_1(s))\Delta s > 1$, then

$$\begin{aligned} &\int_T^{\infty} p(u) \left[\int_T^u \frac{1}{a_1(s)} \Delta s \right]^{\beta} \Delta u \\ &\quad \geq \int_{u_1}^{\infty} p(u) \left[\int_T^u \frac{1}{a_1(s)} \Delta s \right]^{\beta} \Delta u \\ &\quad \geq \int_{u_1}^{\infty} p(u) \Delta u = \infty. \end{aligned} \tag{68}$$

Thus, conditions (5), (6), and (49) are satisfied. By Theorem 11, every solution of (64) is either oscillatory or tends to zero. □

Example 2. Consider the following higher order dynamic equation:

$$S_n^{\Delta}(t, x(t)) + \frac{1}{t^{1+\gamma}} x^{\beta}(t) = 0, \tag{69}$$

on an arbitrary time scale $\mathbf{T}$ with $\sup \mathbf{T} = \infty$, where $n \geq 2$, $S_k(t, x(t))$ $(0 \leq k \leq n)$ are as in (2) with $a_n(t) = 1, a_{n-1}(t) = t^{1/\alpha}, a_{n-2}(t) = \cdots = a_1(t) = t$, $0 < \gamma < \min\{1, \beta\}$, and α, β are the quotient of odd positive integers with $\alpha \geq 1$. Then, every solution of (69) is either oscillatory or tends to zero.

Proof. Note that

$$\begin{aligned} &\int_{t_0}^{\infty} \left[\frac{1}{a_n(s)} \right]^{1/\alpha} \Delta s = \int_{t_0}^{\infty} \Delta s = \infty, \\ &\int_{t_0}^{\infty} \frac{1}{a_{n-1}(s)} \Delta s = \int_{t_0}^{\infty} \frac{1}{s^{1/\alpha}} \Delta s = \infty, \\ &\int_{t_0}^{\infty} \frac{1}{a_i(s)} \Delta s = \int_{t_0}^{\infty} \frac{1}{s} \Delta s = \infty \quad \text{for } 1 \leq i \leq n-2. \end{aligned} \tag{70}$$

Pick $t_1 > t_0$ such that $\int_{t_0}^{t_1} (\Delta u / u^{1/\alpha}) > 0$, then

$$\begin{aligned} &\int_{t_0}^{\infty} \frac{1}{a_{n-1}(u)} \left\{ \int_u^{\infty} \left[\frac{1}{a_n(s)} \int_s^{\infty} p(v) \Delta v \right]^{1/\alpha} \Delta s \right\} \Delta u \\ &\quad = \int_{t_0}^{\infty} \frac{1}{u^{1/\alpha}} \left\{ \int_u^{\infty} \left[\int_s^{\infty} \frac{1}{v^{\gamma+1}} \Delta v \right]^{1/\alpha} \Delta s \right\} \Delta u \\ &\quad \geq \frac{1}{\gamma} \int_{t_0}^{\infty} \frac{1}{u^{1/\alpha}} \left\{ \int_u^{\infty} \left[\int_s^{\infty} \frac{(v^{\gamma})^{\Delta}}{v^{\gamma} (v^{\gamma})^{\sigma}} \Delta v \right]^{1/\alpha} \Delta s \right\} \Delta u \\ &\quad = \frac{1}{\gamma} \int_{t_0}^{\infty} \frac{1}{u^{1/\alpha}} \left[\int_u^{\infty} \left(\frac{1}{s^{\gamma}} \right)^{1/\alpha} \Delta s \right] \Delta u \\ &\quad \geq \frac{1}{\gamma} \int_{t_0}^{t_1} \frac{1}{u^{1/\alpha}} \left[\int_{t_1}^{\infty} \left(\frac{1}{s^{\gamma}} \right)^{1/\alpha} \Delta s \right] \Delta u \\ &\quad = \frac{1}{\gamma} \left[\int_{t_1}^{\infty} \left(\frac{1}{s^{\gamma}} \right)^{1/\alpha} \Delta s \right] \int_{t_0}^{t_1} \frac{\Delta u}{u^{1/\alpha}} \\ &\quad = \infty. \end{aligned} \tag{71}$$

Let $M = \max\{c_1, 1, c_2\}$ with c_1, c_2 being positive constants, $\rho = \min\{\alpha, \beta\}$, and $\gamma < \tau < \min\{1, \beta\}$. Pick $T_1 > T > 0$ such that

$$\frac{1}{t^{\gamma}} \geq \frac{2}{t^{\tau}} \geq \frac{2M}{\left[(1/2)^{n+(1/\alpha)} \left(t - 2^{n-1} T \right) \right]^{\rho}} \quad \text{for } t \geq T_1. \tag{72}$$

Let $\theta(t) = t$, then

$$\begin{aligned} &B_1(t, T) \\ &\quad = \int_T^t \frac{1}{a_1(u_1)} \\ &\qquad \times \left[\int_T^{u_1} \frac{1}{a_2(u_2)} \right. \\ &\qquad \times \left[\cdots \left[\int_T^{u_{n-2}} \frac{1}{a_{n-1}(u_{n-1})} \left[\int_T^{u_{n-1}} \Delta u_n \right]^{1/\alpha} \right. \right. \\ &\qquad \left. \left. \left. \times \Delta u_{n-1} \right] \cdots \right] \Delta u_2 \right] \Delta u_1 \\ &\quad = \int_{2^{n-1}T}^t \frac{1}{u_1} \\ &\qquad \times \left[\int_{2^{n-2}T}^{u_1} \frac{1}{u_2} \left[\cdots \left[\int_{2T}^{u_{n-2}} \left[\frac{1}{u_{n-1}} \int_T^{u_{n-1}} \Delta u_n \right]^{1/\alpha} \right. \right. \right. \\ &\qquad \left. \left. \left. \times \Delta u_{n-1} \right] \cdots \right] \Delta u_2 \right] \Delta u_1 \\ &\quad \geq \left(\frac{1}{2} \right)^{n+(1/\alpha)} \left(t - 2^{n-1} T \right), \\ &\int_{T_1}^t \left[\theta(s) p(s) - \frac{\theta^{\Delta}(s)}{B_1^{\alpha}(s, T)} \delta_1(s, T) \right] \Delta s \\ &\quad = \int_{T_1}^t \left[\frac{1}{s^{\gamma}} - \frac{1}{B_1^{\alpha}(s, T)} \delta_1(s, T, c_1, c_2) \right] \Delta s \\ &\quad \geq \int_{T_1}^t \left[\frac{2}{t^{\tau}} - \frac{M}{\left[(1/2)^{n+(1/\alpha)} \left(t - 2^{n-1} T \right) \right]^{\rho}} \right] \Delta s \\ &\quad \geq \int_{T_1}^t \frac{1}{t^{\tau}} \Delta s. \end{aligned} \tag{73}$$

Thus,

$$\limsup_{t \to \infty} \int_{T_1}^t \left[\theta(s) p(s) - \frac{\theta^{\Delta}(s)}{B_1^{\alpha}(s, T)} \delta_1(s, T, c_1, c_2) \right] \Delta s = \infty. \tag{74}$$

So conditions (5), (6), and (23) are satisfied. Then, by Theorem 8, every solution of (69) is either oscillatory or tends to zero. □

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References

[1] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.

[3] V. Kac and P. Cheung, *Quantum Calculus*, Universitext, 2002.

[4] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[5] M. Bohner and S. H. Saker, "Oscillation of second order nonlinear dynamic equations on time scales," *The Rocky Mountain Journal of Mathematics*, vol. 34, no. 4, pp. 1239–1254, 2004.

[6] L. Erbe, "Oscillation results for second-order linear equations on a time scale," *Journal of Difference Equations and Applications*, vol. 8, no. 11, pp. 1061–1071, 2002.

[7] T. S. Hassan, "Oscillation criteria for half-linear dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 176–185, 2008.

[8] R. P. Agarwal, M. Bohner, and S. H. Saker, "Oscillation of second order delay dynamic equations," *The Canadian Applied Mathematics Quarterly*, vol. 13, no. 1, pp. 1–17, 2005.

[9] M. Bohner, B. Karpuz, and Ö. Öcalan, "Iterated oscillation criteria for delay dynamic equations of first order," *Advances in Difference Equations*, vol. 2008, Article ID 458687, 12 pages, 2008.

[10] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation criteria for second-order nonlinear delay dynamic equations," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 1, pp. 505–522, 2007.

[11] Z. Han, B. Shi, and S. Sun, "Oscillation criteria for second-order delay dynamic equations on time scales," *Advances in Difference Equations*, vol. 2007, Article ID 070730, 16 pages, 2007.

[12] Z. Han, S. Sun, and B. Shi, "Oscillation criteria for a class of second-order Emden-Fowler delay dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 2, pp. 847–858, 2007.

[13] T. Sun, H. Xi, X. Peng, and W. Yu, "Nonoscillatory solutions for higher-order neutral dynamic equations on time scales," *Abstract and Applied Analysis*, vol. 2010, Article ID 428963, 16 pages, 2010.

[14] T. Sun, H. Xi, and X. Peng, "Asymptotic behavior of solutions of higher-order dynamic equations on time scales," *Advances in Difference Equations*, vol. 2011, Article ID 237219, 14 pages, 2011.

[15] T. Sun, H. Xi, and W. Yu, "Asymptotic behaviors of higher order nonlinear dynamic equations on time scales," *Journal of Applied Mathematics and Computing*, vol. 37, no. 1-2, pp. 177–192, 2011.

[16] S. R. Grace, R. P. Agarwal, B. Kaymakçalan, and W. Sae-jie, "On the oscillation of certain second order nonlinear dynamic equations," *Mathematical and Computer Modelling*, vol. 50, no. 1-2, pp. 273–286, 2009.

[17] R. P. Agarwal, M. Bohner, T. Li, and C. Zhang, "Oscillation theorems for fourth-order halflinear delay dynamic equations with damping," *Mediterranean Journal of Mathematics*, 2013.

[18] C. Zhang, T. Li, R. P. Agarwal, and M. Bohner, "Oscillation results for fourth-order nonlinear dynamic equations," *Applied Mathematics Letters*, vol. 25, no. 12, pp. 2058–2065, 2012.

[19] Y. Qi and J. Yu, "Oscillation criteria for fourth-order nonlinear delay dynamic equations," *Electronic Journal of Differential Equations*, vol. 2013, Article ID 740568, 11 pages, 2013.

[20] S. R. Grace, S. Sun, and Y. Wang, "On the oscillation of fourth order strongly superlinear and strongly sublinear dynamic equations," *Journal of Applied Mathematics and Computing*, 2013.

[21] L. Erbe, A. Peterson, and S. H. Saker, "Asymptotic behavior of solutions of a third-order nonlinear dynamic equation on time scales," *Journal of Computational and Applied Mathematics*, vol. 181, no. 1, pp. 92–102, 2005.

[22] L. Erbe, A. Peterson, and S. H. Saker, "Hille and Nehari type criteria for third-order dynamic equations," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 112–131, 2007.

[23] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation and asymptotic behavior of a third-order nonlinear dynamic equation," *The Canadian Applied Mathematics Quarterly*, vol. 14, no. 2, pp. 129–147, 2006.

[24] T. S. Hassan, "Oscillation of third order nonlinear delay dynamic equations on time scales," *Mathematical and Computer Modelling*, vol. 49, no. 7-8, pp. 1573–1586, 2009.

[25] T. Sun, W. Yu, and H. Xi, "Oscillatory behavior and comparison for higher order nonlinear dynamic equations on time scales," *Journal of Applied Mathematics & Informatics*, vol. 30, no. 1-2, pp. 289–304, 2012.

[26] R. P. Agarwal, *Difference Equations and Inequalities. Theory, Methods and Applications*, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.

[27] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 2nd edition, 1988.