

Research Article

Approximate Controllability of Sobolev Type Nonlocal Fractional Stochastic Dynamic Systems in Hilbert Spaces

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We study a class of fractional stochastic dynamic control systems of Sobolev type in Hilbert spaces. We use fixed point technique, fractional calculus, stochastic analysis, and methods adopted directly from deterministic control problems for the main results. A new set of sufficient conditions for approximate controllability is formulated and proved. An example is also given to provide the obtained theory.

1. Introduction

We are concerned with the following nonlocal fractional stochastic system of Sobolev type:

$$\begin{aligned} {}^C D_t^q [Lx(t)] &= Mx(t) + Bu(t) + f(t, x(t)) \\ &\quad + \sigma(t, x(t)) \frac{dw(t)}{dt}, \\ x(0) + g(x(t)) &= x_0, \end{aligned} \quad (1)$$

where ${}^C D_t^q$ is the Caputo fractional derivative of order q , $0 < q \leq 1$, and $t \in J = [0, b]$. Let X and Y be two Hilbert spaces, and the state $x(\cdot)$ takes its values in X . We assume that the operators L and M are defined on domains contained in X and ranges contained in Y , the control function $u(\cdot)$ belongs to the space $L^2_\Gamma(J, U)$, a Hilbert space of admissible control functions with U as a Hilbert space, and B is a bounded linear operator from U into Y . It is also assumed that $f : J \times X \rightarrow Y$, $g : C(J : X) \rightarrow Y$ and $\sigma : J \times X \rightarrow L^0_2$ are appropriate functions; x_0 is Γ_0 measurable X -valued random variables independent of w . Here Γ , Γ_0 , L^0_2 , and w will be specified later.

The field of fractional differential equations and its applications has gained a lot of importance during the past three decades, mainly because it has become a powerful tool in modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering [1–8]. Recently, there has been a significant development in the existence and uniqueness of solutions of initial and boundary value problem for fractional evolution systems [9].

Controllability is one of the important fundamental concepts in mathematical control theory and plays a vital role in both deterministic and stochastic control systems. Since the controllability notion has extensive industrial and biological applications, in the literature, there are many different notions of controllability, both for linear and nonlinear dynamical systems. Controllability of the deterministic and stochastic dynamical control systems in infinite dimensional spaces is well developed using different kinds of approaches. It should be mentioned that the theory of controllability for nonlinear fractional dynamical systems is still in the initial stage. There are few works in controllability problems for different kinds of systems described by fractional differential equations [10, 11].

The exact controllability for semilinear fractional order system, when the nonlinear term is independent of the control function, is proved by many authors [12–15]. In these papers, the authors have proved the exact controllability by assuming that the controllability operator has an induced inverse on a quotient space. However, if the semigroup associated with the system is compact, then the controllability operator is also compact and hence the induced inverse does not exist because the state space is infinite dimensional [16]. Thus, the concept of exact controllability is too strong and has limited applicability, and the approximate controllability is a weaker concept than complete controllability and it is completely adequate in applications for these control systems.

In [17, 18] the approximate controllability of first order delay control systems has been proved when nonlinear term is a function of both state function and control function by assuming that the corresponding linear system is approximately controllable. To prove the approximate controllability of first order system, with or without delay, a relation between the reachable set of a semilinear system and that of the corresponding linear system is proved in [19–23]. There are several papers devoted to the approximate controllability for semilinear control systems, when the nonlinear term is independent of control function [24–27].

Stochastic differential equations have attracted great interest due to their applications in various fields of science and engineering. There are many interesting results on the theory and applications of stochastic differential equations (see [12, 28–32] and the references therein). To build more realistic models in economics, social sciences, chemistry, finance, physics, and other areas, stochastic effects need to be taken into account. Therefore, many real world problems can be modeled by stochastic differential equations. The deterministic models often fluctuate due to noise, so we must move from deterministic control to stochastic control problems.

In the present literature there are only a limited number of papers that deal with the approximate controllability of fractional stochastic systems [33], as well as with the existence and controllability results of fractional evolution equations of Sobolev type [34].

Sakthivel et al. [35] studied the approximate controllability of a class of dynamic control systems described by nonlinear fractional stochastic differential equations in Hilbert spaces. More recent works can be found in [10, 11]. Debbouche et al. [4] established a class of fractional nonlocal nonlinear integrodifferential equations of Sobolev type using new solution operators. Fečkan et al. [36] presented the controllability results corresponding to two admissible control sets for fractional functional evolution equations of Sobolev type in Banach spaces with the help of two new characteristic solution operators and their properties, such as boundedness and compactness.

It should be mentioned that there is no work yet reported on the approximate controllability of Sobolev type fractional deterministic stochastic control systems. Motivated by the above facts, in this paper we establish the approximate controllability for a class of fractional stochastic dynamic systems of Sobolev Type with nonlocal conditions in Hilbert spaces.

The paper is organized as follows: in Section 2, we present some essential facts in fractional calculus, semigroup theory, stochastic analysis, and control theory that will be used to obtain our main results. In Section 3, we state and prove existence and approximate controllability results for Sobolev type fractional stochastic system (1). The last sections deal with an illustrative example and a discussion for possible future work in this direction.

2. Preliminaries

In this section we give some basic definitions, notations, properties, and lemmas, which will be used throughout the work. In particular, we state main properties of fractional calculus [37–40], well-known facts in semigroup theory [41–43], and elementary principles of stochastic analysis [31, 44].

Definition 1. The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a, b], \mathbb{R}^+)$ is given by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad (2)$$

where Γ is the gamma function. If $a = 0$, we can write $I^\alpha f(t) = (g_\alpha * f)(t)$, where

$$g_\alpha(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases} \quad (3)$$

and as usual, $*$ denotes the convolution of functions. Moreover, $\lim_{\alpha \rightarrow 0} g_\alpha(t) = \delta(t)$, with δ the delta Dirac function.

Definition 2. The Riemann-Liouville derivative of order $n - 1 < \alpha < n$, $n \in \mathbb{N}$, for a function $f \in C([0, \infty))$ is given by

$${}^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0. \quad (4)$$

Definition 3. The Caputo derivative of order $n - 1 < \alpha < n$, $n \in \mathbb{N}$, for a function $f \in C([0, \infty))$ is given by

$${}^C D^\alpha f(t) = {}^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0. \quad (5)$$

Remark 4. The following properties hold (see, e.g., [45]).

(i) If $f \in C^n([0, \infty))$, then

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^n(t), \quad (6)$$

$$t > 0, \quad n - 1 < \alpha < n, \quad n \in \mathbb{N}.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X , then the integrals which appear in Definitions 1–3 are taken in Bochner's sense.

We introduce the following assumptions on the operators L and M .

- (H₁) L and M are linear operators, and M is closed.
- (H₂) $D(L) \subset D(M)$ and L is bijective.
- (H₃) $L^{-1} : Y \rightarrow D(L) \subset X$ is a linear compact operator.

Remark 5. From (H₃), we deduce that L^{-1} is a bounded operator; for short, we denote $C = \|L^{-1}\|$. Note (H₃) also implies that L is closed since L^{-1} is closed and injective; then its inverse is also closed. It comes from (H₁)–(H₃) and the closed graph theorem; we obtain the boundedness of the linear operator $ML^{-1} : Y \rightarrow Y$. Consequently, ML^{-1} generates a semigroup $\{S(t) := e^{ML^{-1}t}, t \geq 0\}$. We suppose that $M_0 := \sup_{t \geq 0} \|S(t)\| < \infty$. According to previous definitions, it is suitable to rewrite problem (1) as the equivalent integral equation

$$\begin{aligned} Lx(t) &= L[x_0 - g(x)] \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \\ &\quad \times [Mx(s) + Bu(s) + f(s, x(s))] ds \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sigma(s, x(s)) dw(s), \end{aligned} \tag{7}$$

provided the integral in (7) exists. Before formulating the definition of mild solution of (1), we first give the following definitions, corollaries, lemmas, and notations.

Let (Ω, Γ, P) be a complete probability space equipped with a normal filtration $\Gamma_t, t \in J$ satisfying the usual conditions (i.e., right continuous and Γ_0 containing all P -null sets). We consider four real separable spaces X, Y, E , and U and Q -Wiener process on (Ω, Γ_b, P) with the linear bounded covariance operator Q such that $\text{tr } Q < \infty$. We assume that there exists a complete orthonormal system $\{e_n\}_{n \geq 1}$ in E , a bounded sequence of nonnegative real numbers $\{\lambda_n\}$ such that $Qe_n = \lambda_n e_n, n = 1, 2, \dots$, and a sequence $\{\beta_n\}_{n \geq 1}$ of independent Brownian motions such that

$$\langle w(t), e \rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle e_n, e \rangle \beta_n(t), \quad e \in E, t \in J. \tag{8}$$

and $\Gamma_t = \Gamma_t^w$, where Γ_t^w is the sigma algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Q^{1/2}E; X)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}E$ to X with the inner product $\langle \psi, \pi \rangle_{L_2^0} = \text{tr}[\psi Q \pi^*]$. Let $L^2(\Gamma_b, X)$ be the Banach space of all Γ_b -measurable square integrable random variables with values in the Hilbert space X . Let $E(\cdot)$ denote the expectation with respect to the measure P . Let $C(J; L^2(\Gamma, X))$ be the Hilbert space of continuous maps from J into $L^2(\Gamma, X)$ satisfying $\sup_{t \in J} E\|x(t)\|^2 < \infty$. Let $H_2(J; X)$ be a closed subspace of $C(J; L^2(\Gamma, X))$ consisting of measurable and Γ_t -adapted X -valued process $x \in C(J; L^2(\Gamma, X))$ endowed

with the norm $\|x\|_{H_2} = (\sup_{t \in J} E\|x(t)\|_X^2)^{1/2}$. For details, we refer the reader to [35, 44] and references therein.

The following results will be used through out this paper.

Lemma 6 (see [33]). *Let $G : J \times \Omega \rightarrow L_2^0$ be a strongly measurable mapping such that $\int_0^b E\|G(t)\|_{L_2^0}^p dt < \infty$. Then*

$$E\left\|\int_0^t G(s)dw(s)\right\|^p \leq L_G \int_0^t E\|G(s)\|_{L_2^0}^p ds \tag{9}$$

for all $0 \leq t \leq b$ and $p \geq 2$, where L_G is the constant involving p and b .

Now, we present the mild solution of the problem (1).

Definition 7 (compare with [46, 47] and [36, 45]). A stochastic process $x \in H_2(J, X)$ is a mild solution of (1) if, for each control $u \in L_1^2(J, U)$, it satisfies the following integral equation:

$$\begin{aligned} x(t) &= \mathcal{S}(t) L[x_0 - g(x)] \\ &+ \int_0^t (t-s)^{q-1} \mathcal{F}(t-s) [Bu(s) + f(s, x(s))] ds \\ &+ \int_0^t (t-s)^{q-1} \mathcal{F}(t-s) \sigma(s, x(s)) dw(s), \end{aligned} \tag{10}$$

where $\mathcal{S}(t)$ and $\mathcal{F}(t)$ are characteristic operators given by

$$\begin{aligned} \mathcal{S}(t) &= \int_0^{\infty} L^{-1} \xi_q(\theta) S(t^q \theta) d\theta, \\ \mathcal{F}(t) &= q \int_0^{\infty} L^{-1} \theta \xi_q(\theta) S(t^q \theta) d\theta. \end{aligned} \tag{11}$$

Here, $S(t)$ is a C_0 -semigroup generated by the linear operator $ML^{-1} : Y \rightarrow Y$; ξ_q is a probability density function defined on $(0, \infty)$; that is, $\xi_q(\theta) \geq 0, \theta \in (0, \infty)$ and $\int_0^{\infty} \xi_q(\theta) d\theta = 1$.

Lemma 8 (see [45, 48, 49]). *The operators $\{\mathcal{S}(t)\}_{t \geq 0}$ and $\{\mathcal{F}(t)\}_{t \geq 0}$ are strongly continuous; that is, for $x \in X$ and $0 \leq t_1 < t_2 \leq b$, one has $\|\mathcal{S}(t_2)x - \mathcal{S}(t_1)x\| \rightarrow 0$ and $\|\mathcal{F}(t_2)x - \mathcal{F}(t_1)x\| \rightarrow 0$ as $t_2 \rightarrow t_1$.*

We impose the following conditions on data of the problem.

- (i) For any fixed $t \geq 0$, $\mathcal{F}(t)$ and $\mathcal{S}(t)$ are bounded linear operators; that is, for any $x \in X$,

$$\|\mathcal{F}(t)x\| \leq CM_0 \|x\|, \quad \|\mathcal{S}(t)x\| \leq \frac{CM_0}{\Gamma(q)} \|x\|. \tag{12}$$

- (ii) The functions $f : J \times X \rightarrow Y, \sigma : J \times X \rightarrow L_2^0$ and $g : C(J; X) \rightarrow Y$ satisfy linear growth and Lipschitz

conditions. Moreover, there exist positive constants $N_1, N_2 > 0$, $L_1, L_2 > 0$, and $k_1, k_2 > 0$ such that

$$\begin{aligned} \|f(t, x) - f(t, y)\|^2 &\leq N_1 \|x - y\|^2, \\ \|f(t, x)\|^2 &\leq N_2 (1 + \|x\|^2), \\ \|\sigma(t, x) - \sigma(t, y)\|_{L_2^2}^2 &\leq L_1 \|x - y\|^2, \\ \|\sigma(t, x)\|_{L_2^2}^2 &\leq L_2 (1 + \|x\|^2), \\ \|g(x) - g(y)\|^2 &\leq k_1 \|x - y\|^2, \\ \|g(x)\|^2 &\leq k_2 (1 + \|x\|^2). \end{aligned} \quad (13)$$

(iii) The linear stochastic system is approximately controllable on J .

For each $0 \leq t < b$, the operator $\alpha(\alpha I + \Psi_0^b)^{-1} \rightarrow 0$ in the strong operator topology as $\alpha \rightarrow 0^+$, where $\Psi_0^b = \int_0^b (b-s)^{2(q-1)} \mathcal{F}(b-s) B B^* \mathcal{F}^*(b-s) ds$ is the controllability Gramian. Here B^* denotes the adjoint of B , and $\mathcal{F}^*(t)$ is the adjoint of $\mathcal{F}(t)$.

Observe that Sobolev type nonlocal linear fractional deterministic control system

$$\begin{aligned} {}^C D_t^q [Lx(t)] &= Mx(t) + Bu(t), \quad t \in J, \\ x(0) + g(x(t)) &= x_0, \end{aligned} \quad (14)$$

corresponding to (1) is approximately controllable on J iff the operator $\alpha(\alpha I + \Psi_0^b)^{-1} \rightarrow 0$ strongly as $\alpha \rightarrow 0^+$. The approximate controllability for linear fractional deterministic control system (14) is a natural generalization of approximate controllability of linear first order control system ($q = 1$, $g = 0$, and L is the identity) [50].

Definition 9. The system (1) is approximately controllable on J if $\mathfrak{R}(b) = L^2(\Omega, \Gamma_b, X)$, where

$$\mathfrak{R}(b) = \{x(b) = x(b, u) : u \in L_\Gamma^2(J, U)\}. \quad (15)$$

Here $L_\Gamma^2(J, U)$ is the closed subspace of $L_\Gamma^2(J \times \Omega; U)$, consisting of all Γ_t -adapted, U -valued stochastic processes.

The following lemma is required to define the control function [35].

Lemma 10. For any $\tilde{x}_b \in L^2(\Gamma_b, X)$, there exists $\tilde{\varphi} \in L_\Gamma^2(\Omega; L^2(0, b; L_2^0))$ such that $\tilde{x}_b = E\tilde{x}_b + \int_0^b \tilde{\varphi}(s) dw(s)$.

Now for any $\alpha > 0$ and $\tilde{x}_b \in L^2(\Gamma_b, X)$, one defines the control function in the following form:

$$\begin{aligned} u^\alpha(t, x) &= B^*(b-t)^{q-1} \mathcal{F}^*(b-t) \\ &\times \left[(\alpha I + \Psi_0^b)^{-1} \{E\tilde{x}_b - \mathcal{S}(b) L[x_0 - g(x)]\} \right. \\ &\quad \left. + \int_0^t (\alpha I + \Psi_0^b)^{-1} \tilde{\varphi}(s) dw(s) \right] \\ &- B^*(b-t)^{q-1} \mathcal{F}^*(b-t) \\ &\times \int_0^t (\alpha I + \Psi_0^b)^{-1} (b-s)^{q-1} \mathcal{F}(b-s) f(s, x(s)) ds \\ &- B^*(b-t)^{q-1} \mathcal{F}^*(b-t) \\ &\times \int_0^t (\alpha I + \Psi_0^b)^{-1} (b-s)^{q-1} \mathcal{F}(b-s) \\ &\quad \times \sigma(s, x(s)) dw(s). \end{aligned} \quad (16)$$

Lemma 11. There exist positive real constants \widehat{M}, \widehat{N} such that, for all $x, y \in H_2$, one has

$$E\|u^\alpha(t, x) - u^\alpha(t, y)\|^2 \leq \widehat{M} E\|x(t) - y(t)\|^2, \quad (17)$$

$$E\|u^\alpha(t, x)\|^2 \leq \widehat{N} \left(\frac{1}{b} + E\|x(t)\|^2 \right). \quad (18)$$

Proof. We start to prove (17). Let $x, y \in H_2$; from Hölder's inequality, Lemma 6, and the assumption on the data, we obtain

$$\begin{aligned} &E\|u^\alpha(t, x) - u^\alpha(t, y)\|^2 \\ &\leq 3E\|B^*(b-t)^{q-1} \mathcal{F}^*(b-t) (\alpha I + \Psi_0^b)^{-1} \\ &\quad \times \mathcal{S}(b) L[g(x(t)) - g(y(t))]\|^2 \\ &\quad + 3E\|B^*(b-t)^{q-1} \mathcal{F}^*(b-t) \\ &\quad \times \int_0^t (\alpha I + \Psi_0^b)^{-1} (b-s)^{q-1} \mathcal{F}(b-s) \\ &\quad \times [f(s, x(s)) - f(s, y(s))]\|^2 ds \\ &\quad + 3E\|B^*(b-t)^{q-1} \mathcal{F}^*(b-t) \\ &\quad \times \int_0^t (\alpha I + \Psi_0^b)^{-1} (b-s)^{q-1} \mathcal{F}(b-s) \\ &\quad \times [\sigma(s, x(s)) - \sigma(s, y(s))]\|^2 dw(s) \|^2 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{3}{\alpha^2} \|B\|^2 (b)^{2q-2} (CM_0)^2 \left(\frac{CM_0}{\Gamma(q)}\right)^2 \|L\|^2 k_1 E \|x(t) - y(t)\|^2 \\
 &\quad + \frac{3}{\alpha^2} \|B\|^2 (b)^{2q-2} \left(\frac{CM_0}{\Gamma(q)}\right)^4 \frac{b^{2q-1}}{(2q-1)} N_1 \\
 &\quad \times \int_0^t E \|x(s) - y(s)\|^2 ds \\
 &\quad + \frac{3}{\alpha^2} \|B\|^2 (b)^{2q-2} \left(\frac{CM_0}{\Gamma(q)}\right)^4 \frac{b^{2q-1}}{(2q-1)} L_1 \\
 &\quad \times \int_0^t E \|x(s) - y(s)\|^2 ds \\
 &\leq \widehat{M} E \|x(t) - y(t)\|^2,
 \end{aligned} \tag{19}$$

where $\widehat{M} = (3/\alpha^2) \|B\|^2 (b)^{2q-2} \{(CM_0)^2 (CM_0/\Gamma(q))^2 \|L\|^2 k_1 + (CM_0/\Gamma(q))^4 (b^{2q-1}/(2q-1)) b [N_1 + L_1]\}$. The proof of the inequality (18) can be established in a similar way to that of (17). \square

3. Approximate Controllability

In this section, we formulate and prove conditions for the existence and approximate controllability results of the nonlocal fractional stochastic dynamic control system of Sobolev type (1) using the contraction mapping principle. For any $\alpha > 0$, define the operator $F_\alpha : H_2 \rightarrow H_2$ by

$$\begin{aligned}
 F_\alpha x(t) &= \mathcal{S}(t) L [x_0 - g(x)] \\
 &\quad + \int_0^t (t-s)^{q-1} \mathcal{F}(t-s) [Bu^\alpha(s, x) + f(s, x(s))] ds \\
 &\quad + \int_0^t (t-s)^{q-1} \mathcal{F}(t-s) \sigma(s, x(s)) dw(s).
 \end{aligned} \tag{20}$$

We state and prove the following lemma, which will be used for the main results.

Lemma 12. *For any $x \in H_2$, $F_\alpha(x)(t)$ is continuous on J in L^2 -sense.*

Proof. Let $0 \leq t_1 < t_2 \leq b$. Then for any fixed $x \in H_2$, from (20), we have

$$E \|(F_\alpha x)(t_2) - (F_\alpha x)(t_1)\|^2 \leq 4 \left[\sum_{i=1}^4 E \|\Pi_i^x(t_2) - \Pi_i^x(t_1)\|^2 \right]. \tag{21}$$

We begin with the first term and get

$$\begin{aligned}
 E \|\Pi_1^x(t_2) - \Pi_1^x(t_1)\|^2 &= E \|(\mathcal{S}(t_2) - \mathcal{S}(t_1))L[x_0 - g(x)]\|^2 \\
 &\leq \|L\|^2 [\|x_0\|^2 + k_2(1 + \|x\|^2)] \\
 &\quad \times E \|\mathcal{S}(t_2) - \mathcal{S}(t_1)\|^2.
 \end{aligned} \tag{22}$$

The strong continuity of $\mathcal{S}(t)$ implies that the right-hand side of the last inequality tends to zero as $t_2 - t_1 \rightarrow 0$.

Next, it follows from Hölder's inequality and assumptions on the data that

$$\begin{aligned}
 &E \|\Pi_2^x(t_2) - \Pi_2^x(t_1)\|^2 \\
 &= E \left\| \int_0^{t_2} (t_2-s)^{q-1} \mathcal{F}(t_2-s) Bu^\alpha(s, x) ds \right. \\
 &\quad \left. - \int_0^{t_1} (t_1-s)^{q-1} \mathcal{F}(t_1-s) Bu^\alpha(s, x) ds \right\|^2 \\
 &\leq E \left\| \int_0^{t_1} (t_1-s)^{q-1} (\mathcal{F}(t_2-s) - \mathcal{F}(t_1-s)) \right. \\
 &\quad \left. \times Bu^\alpha(s, x) ds \right\|^2 \\
 &\quad + E \left\| \int_0^{t_1} ((t_2-s)^{q-1} - (t_1-s)^{q-1}) \mathcal{F}(t_2-s) \right. \\
 &\quad \left. \times Bu^\alpha(s, x) ds \right\|^2 \\
 &\quad + E \left\| \int_{t_1}^{t_2} (t_2-s)^{q-1} \mathcal{F}(t_2-s) Bu^\alpha(s, x) ds \right\|^2 \\
 &\leq \frac{t_1^{2q-1}}{2q-1} \\
 &\quad \times \int_0^{t_1} E \|(\mathcal{F}(t_2-s) - \mathcal{F}(t_1-s)) Bu^\alpha(s, x) ds\|^2 \\
 &\quad + \left(\frac{CM_0}{\Gamma(q)}\right)^2 \|B\|^2 \left(\int_0^{t_1} ((t_2-s)^{q-1} - (t_1-s)^{q-1})^2 ds \right) \\
 &\quad \times \left(\int_0^{t_1} E \|u^\alpha(s, x)\|^2 ds \right) \\
 &\quad + \frac{(t_2-t_1)^{2q-1}}{1-2q} \left(\frac{CM_0}{\Gamma(q)}\right)^2 \|B\|^2 \int_{t_1}^{t_2} E \|u^\alpha(s, x)\|^2 ds.
 \end{aligned} \tag{23}$$

Also, we have

$$\begin{aligned}
 &E \|\Pi_3^x(t_2) - \Pi_3^x(t_1)\|^2 \\
 &= E \left\| \int_0^{t_2} (t_2-s)^{q-1} \mathcal{F}(t_2-s) f(s, x(s)) ds \right. \\
 &\quad \left. - \int_0^{t_1} (t_1-s)^{q-1} \mathcal{F}(t_1-s) f(s, x(s)) ds \right\|^2 \\
 &\leq E \left\| \int_0^{t_1} (t_1-s)^{q-1} (\mathcal{F}(t_2-s) - \mathcal{F}(t_1-s)) \right. \\
 &\quad \left. \times f(s, x(s)) ds \right\|^2
 \end{aligned}$$

$$\begin{aligned}
 & + E \left\| \int_0^{t_1} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) \right. \\
 & \quad \left. \times \mathcal{F}(t_2 - s) f(s, x(s)) ds \right\|^2 \\
 & + E \left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathcal{F}(t_2 - s) f(s, x(s)) ds \right\|^2 \\
 & \leq \frac{t_1^{2q-1}}{2q-1} \\
 & \quad \times \int_0^{t_1} E \left\| (\mathcal{F}(t_2 - s) - \mathcal{F}(t_1 - s)) f(s, x(s)) \right\|^2 ds \\
 & \quad + \left(\frac{CM_0}{\Gamma(q)} \right)^2 \left(\int_0^{t_1} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right)^2 ds \right) \\
 & \quad \times \left(\int_0^{t_1} E \|f(s, x(s))\|^2 ds \right) \\
 & \quad + \frac{(t_2 - t_1)^{2q-1}}{1 - 2q} \left(\frac{CM_0}{\Gamma(q)} \right)^2 \int_{t_1}^{t_2} E \|f(s, x(s))\|^2 ds.
 \end{aligned} \tag{24}$$

Furthermore, we use Lemma 6 and previous assumptions; we obtain

$$\begin{aligned}
 & E \left\| \Pi_4^x(t_2) - \Pi_4^x(t_1) \right\|^2 \\
 & = E \left\| \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{F}(t_2 - s) \sigma(s, x(s)) dw(s) \right. \\
 & \quad \left. - \int_0^{t_1} (t_1 - s)^{q-1} \mathcal{F}(t_1 - s) \sigma(s, x(s)) dw(s) \right\|^2 \\
 & \leq E \left\| \int_0^{t_1} (t_1 - s)^{q-1} (\mathcal{F}(t_2 - s) - \mathcal{F}(t_1 - s)) \right. \\
 & \quad \left. \times \sigma(s, x(s)) dw(s) \right\|^2 \\
 & \quad + E \left\| \int_0^{t_1} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) \mathcal{F}(t_2 - s) \right. \\
 & \quad \left. \times \sigma(s, x(s)) dw(s) \right\|^2 \\
 & \quad + E \left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathcal{F}(t_2 - s) \sigma(s, x(s)) dw(s) \right\|^2 \\
 & \leq L_\sigma \frac{t_1^{2q-1}}{2q-1} \\
 & \quad \times \int_0^{t_1} E \left\| (\mathcal{F}(t_2 - s) - \mathcal{F}(t_1 - s)) \sigma(s, x(s)) \right\|^2 ds \\
 & \quad + L_\sigma \left(\int_0^{t_1} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right)^2 ds \right) \\
 & \quad \times \left(\int_0^{t_1} E \left\| \mathcal{F}(t_2 - s) \sigma(s, x(s)) \right\|^2 ds \right)
 \end{aligned}$$

$$\begin{aligned}
 & + L_\sigma \frac{(t_2 - t_1)^{2q-1}}{1 - 2q} \left(\frac{CM_0}{\Gamma(q)} \right)^2 \\
 & \quad \times \int_{t_1}^{t_2} E \left\| \mathcal{F}(t_2 - s) \sigma(s, x(s)) \right\|^2 ds.
 \end{aligned} \tag{25}$$

Hence using the strong continuity of $\mathcal{F}(t)$ and Lebesgue's dominated convergence theorem, we conclude that the right-hand side of the previous inequalities tends to zero as $t_2 - t_1 \rightarrow 0$. Thus, we conclude $F_\alpha(x)(t)$ is continuous from the right of $[0, b)$. A similar argument shows that it is also continuous from the left of $(0, b]$. \square

Theorem 13. Assume hypotheses (i) and (ii) are satisfied. Then the system (1) has a mild solution on J .

Proof. We prove the existence of a fixed point of the operator F_α by using the contraction mapping principle. First, we show that $F_\alpha(H_2) \subset H_2$. Let $x \in H_2$. From (20), we obtain

$$E \|F_\alpha x(t)\|^2 \leq 4 \left[\sup_{t \in J} \sum_{i=1}^4 E \|\Pi_i^x(t)\|^2 \right]. \tag{26}$$

Using assumptions (i)-(ii), Lemma 11, and standard computations yields

$$\begin{aligned}
 & \sup_{t \in J} E \|\Pi_1^x(t)\|^2 \leq C^2 M_0^2 \|L\|^2 \left[\|x_0\|^2 + k_2 (1 + \|x\|^2) \right], \tag{27} \\
 & \sup_{t \in J} \sum_{i=2}^4 E \|\Pi_i^x(t)\|^2 \\
 & \leq \left(\frac{CM_0}{\Gamma(q)} \right)^2 \frac{b^{2q-1}}{2q-1} \|B\|^2 \widehat{N} \left(\frac{1}{b} + \|x\|_{H_2}^2 \right) \\
 & \quad + \left(\frac{CM_0}{\Gamma(q)} \right)^2 \left[\frac{b^{2q-1}}{2q-1} N_2 - \frac{b^{2q-1}}{2q-1} L_2 L_\sigma \right] (1 + \|x\|_{H_2}^2).
 \end{aligned} \tag{28}$$

Hence (26)–(28) imply that $E \|F_\alpha x\|_{H_2}^2 < \infty$. By Lemma 12, $F_\alpha x \in H_2$. Thus for each $\alpha > 0$, the operator F_α maps H_2 into itself. Next, we use the Banach fixed point theorem to prove that F_α has a unique fixed point in H_2 . We

claim that there exists a natural n such that F_α^n is a contraction on H_2 . Indeed, let $x, y \in H_2$; we have

$$\begin{aligned} & E\|(F_\alpha x)(t) - (F_\alpha y)(t)\|^2 \\ & \leq 4 \sum_{i=1}^4 E\|\Pi_i^x(t) - \Pi_i^y(t)\|^2 \\ & \leq 4k_1 C^2 M_0^2 \|L\|^2 E\|x(t) - y(t)\|^2 \\ & \quad + 4\left(\frac{CM_0}{\Gamma(q)}\right)^2 \\ & \quad \times \left[\widehat{M}\|B\|^2 \frac{b^{2q-1}}{2q-1} + \frac{b^{2q-1}}{2q-1} N_1 + \frac{b^{2q-1}}{2q-1} L_1 L_\sigma \right] \\ & \quad \times E\|x(t) - y(t)\|^2. \end{aligned} \tag{29}$$

Hence, we obtain a positive real constant $\gamma(\alpha)$ such that

$$E\|(F_\alpha x)(t) - (F_\alpha y)(t)\|^2 \leq \gamma(\alpha) E\|x(t) - y(t)\|^2, \tag{30}$$

for all $t \in J$ and all $x, y \in H_2$. For any natural number n , it follows from the successive iteration of the previous inequality (30) that, by taking the supremum over J ,

$$\|(F_\alpha^n x)(t) - (F_\alpha^n y)(t)\|_{H_2}^2 \leq \frac{\gamma^n(\alpha)}{n!} \|x - y\|_{H_2}^2. \tag{31}$$

For any fixed $\alpha > 0$, for sufficiently large n , $\gamma^n(\alpha)/n! < 1$. It follows from (31) that F_α^n is a contraction mapping, so that the contraction principle ensures that the operator F_α has a unique fixed point x_α in H_2 , which is a mild solution of (1). \square

Theorem 14. Assume that the assumptions (i)–(iii) hold. Further, if the functions f and σ are uniformly bounded and $\{\mathcal{F}(t) : t \geq 0\}$ is compact, then the system (1) is approximately controllable on J .

Proof. Let x_α be a fixed point of F_α . By using the stochastic Fubini theorem, it can be easily seen that

$$\begin{aligned} x_\alpha(b) &= \tilde{x}_b - \alpha(\alpha I + \Psi)^{-1} (E\tilde{x}_b - \mathcal{S}(b)L[x_0 - g(x)]) \\ & \quad + \alpha \int_0^b (\alpha I + \Psi_s^b)^{-1} (b-s)^{q-1} \mathcal{F}(b-s) f(s, x_\alpha(s)) ds \\ & \quad + \alpha \int_0^b (\alpha I + \Psi_s^b)^{-1} [(b-s)^{q-1} \mathcal{F}(b-s) \sigma \\ & \quad \times (s, x_\alpha(s)) - \tilde{\varphi}(s)] dw(s). \end{aligned} \tag{32}$$

It follows from the assumption on f, g , and σ that there exists $\widehat{D} > 0$ such that

$$\|f(s, x_\alpha(s))\|^2 + \|g(x_\alpha(s))\|^2 + \|\sigma(s, x_\alpha(s))\|^2 \leq \widehat{D} \tag{33}$$

for all $s \in J$. Then there is a subsequence still denoted by $\{f(s, x_\alpha(s)), g(x_\alpha(s)), \sigma(s, x_\alpha(s))\}$ which converges weakly to some $\{f(s), g(s), \sigma(s)\}$ in $Y^2 \times L_2^0$.

From the previous equation, we have

$$\begin{aligned} & E\|x_\alpha(b) - \tilde{x}_b\|^2 \\ & \leq 8E\|\alpha(\alpha I + \Psi_0^b)^{-1} (E\tilde{x}_b - \mathcal{S}(b)Lx_0)\|^2 \\ & \quad + 8E\|\alpha(\alpha I + \Psi_0^b)^{-1}\|^2 \|\mathcal{S}(b)L(g(x_\alpha(s)) - g(s))\|^2 \\ & \quad + 8E\|\alpha(\alpha I + \Psi_0^b)^{-1}\|^2 \|\mathcal{S}(b)Lg(s)\|^2 \\ & \quad + 8E\left(\int_0^b (b-s)^{q-1} \|\alpha(\alpha I + \Psi_s^b)^{-1} \tilde{\varphi}(s)\|_{L_2^0}^2 ds\right) \\ & \quad + 8E\left(\int_0^b (b-s)^{q-1} \|\alpha(\alpha I + \Psi_s^b)^{-1}\| \right. \\ & \quad \left. \times \|\mathcal{F}(b-s)(f(s, x_\alpha(s)) - f(s))\| ds\right)^2 \\ & \quad + 8E\left(\int_0^b (b-s)^{q-1} \|\alpha(\alpha I + \Psi_s^b)^{-1} \mathcal{F}(b-s)f(s)\| ds\right)^2 \\ & \quad + 8E\left(\int_0^b (b-s)^{q-1} \|\alpha(\alpha I + \Psi_s^b)^{-1}\| \right. \\ & \quad \left. \times \|\mathcal{F}(b-s)(\sigma(s, x_\alpha(s)) - \sigma(s))\|_{L_2^0}^2 ds\right) \\ & \quad + 8E \int_0^b (b-s)^{q-1} \\ & \quad \times \|\alpha(\alpha I + \Psi_s^b)^{-1} \mathcal{F}(b-s)\sigma(s)\|_{L_2^0}^2 ds. \end{aligned} \tag{34}$$

On the other hand, by assumption (iii), for all $0 \leq s < b$, the operator $\alpha(\alpha I + \Psi_s^b)^{-1} \rightarrow 0$ strongly as $\alpha \rightarrow 0^+$ and moreover $\|\alpha(\alpha I + \Psi_s^b)^{-1}\| \leq 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of both $\mathcal{S}(t)$ and $\mathcal{F}(t)$ it is implied that $E\|x_\alpha(b) - \tilde{x}_b\|^2 \rightarrow 0$ as $\alpha \rightarrow 0^+$. Hence, we conclude the approximate controllability of (1). \square

In order to illustrate the abstract results of this work, we give the following example.

4. Example

Consider the fractional stochastic system with nonlocal condition of Sobolev type

$$\begin{aligned} & \frac{\partial^q}{\partial t^q} [x(z, t) - x_{zz}(z, t)] - \frac{\partial^2}{\partial z^2} x(z, t) \\ & = \mu(z, t) + \hat{f}(t, x(z, t)) + \hat{\sigma}(t, x(z, t)) \frac{d\widehat{w}(t)}{dt}, \end{aligned}$$

$$\begin{aligned}
 x(z, 0) + \sum_{k=1}^m c_k x(z, t_k) &= x_0(z), \quad z \in [0, 1], \\
 x(0, t) = x(1, t) &= 0, \quad t \in J,
 \end{aligned}
 \tag{35}$$

where $0 < q \leq 1$, $0 < t_1 < \dots < t_m < b$ and c_k are positive constants, $k = 1, \dots, m$; the functions $x(t)(z) = x(t, z)$, $f(t, x(t))(z) = \hat{f}(t, x(t, z))$, $\sigma(t, x(t))(z) = \hat{\sigma}(t, x(t, z))$, and $g(x(t))(z) = \sum_{k=1}^m c_k x(z, t_k)$. The bounded linear operator $B : U \rightarrow X$ is defined by $Bu(t)(z) = \mu(z, t)$, $0 \leq z \leq 1$, $u \in U$; $\widehat{w}(t)$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space (Ω, Γ, P) .

Let $X = E = U = L^2[0, 1]$; define the operators $L : D(L) \subset X \rightarrow Y$ and $M : D(M) \subset X \rightarrow Y$ by $Lx = x - x''$ and $Mx = -x''$, where domains $D(L)$ and $D(M)$ are given by

$$\begin{aligned}
 \{x \in X : x, x' \text{ are absolutely continuous, } x'' \in X, \\
 x(0) = x(1) = 0\}.
 \end{aligned}
 \tag{36}$$

Then L and M can be written, respectively, as

$$\begin{aligned}
 Lx &= \sum_{n=1}^{\infty} (1 + n^2) (x, x_n) x_n, \quad x \in D(L), \\
 Mx &= \sum_{n=1}^{\infty} -n^2 (x, x_n) x_n, \quad x \in D(M),
 \end{aligned}
 \tag{37}$$

where $x_n(z) = (\sqrt{2/\pi}) \sin nz$, $n = 1, 2, \dots$ is the orthogonal set of eigen functions of M . Further, for any $x \in X$ we have

$$\begin{aligned}
 L^{-1}x &= \sum_{n=1}^{\infty} \frac{1}{1 + n^2} (x, x_n) x_n, \\
 ML^{-1}x &= \sum_{n=1}^{\infty} \frac{-n^2}{1 + n^2} (x, x_n) x_n, \\
 S(t)x &= \sum_{n=1}^{\infty} \exp\left(\frac{-n^2 t}{1 + n^2}\right) (x, x_n) x_n.
 \end{aligned}
 \tag{38}$$

It is easy to see that L^{-1} is compact and bounded with $\|L^{-1}\| \leq 1$ and ML^{-1} generates the above strongly continuous semigroup $S(t)$ on Y with $\|S(t)\| \leq e^{-t} \leq 1$. Therefore, with the above choices, the system (35) can be written as an abstract formulation of (1) and thus Theorem 13 can be applied to guarantee the existence of mild solution of (35). Moreover, it can be easily seen that Sobolev type deterministic linear fractional control system corresponding to (35) is approximately controllable on J , which means that all conditions of Theorem 14 are satisfied. Thus, fractional stochastic control system of Sobolev type (35) is approximately controllable on J .

5. Conclusion

Sufficient conditions for the approximate controllability of a class of dynamic control systems described by Sobolev

type nonlocal fractional stochastic differential equations in Hilbert spaces are considered. Using fixed point technique, fractional calculations, stochastic analysis, and methods adopted directly from deterministic control problems. In particular, conditions are formulated and proved under the assumption that the approximate controllability of the stochastic control nonlinear dynamical system is implied by the approximate controllability of its corresponding linear part. More precisely, the controllability problem is transformed into a fixed point problem for an appropriate nonlinear operator in a function space. The main used tools are the above required conditions, we guarantee the existence of a fixed point of this operator and study controllability of the considered systems.

Degenerate stochastic differential equations model the phenomenon of convection-diffusion of ideal fluids and therefore arise in a wide variety of important applications, including, for instance, two or three phase flows in porous media or sedimentation-consolidation processes. However, to the best of our knowledge, no results yet exist on approximate controllability for fractional stochastic degenerate systems. Upon making some appropriate assumptions, by employing the ideas and techniques as in this paper, one can establish the approximate controllability results for a class of fractional stochastic degenerate differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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