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Research Article

Convergence Theorems for Common Fixed Points of a Finite Family of Relatively Nonexpansive Mappings in Banach Spaces

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We establish some strong convergence theorems for a common fixed point of a finite family of relatively nonexpansive mappings by using a new hybrid iterative method in mathematical programming and the generalized projection method in a Banach space. Our results improve and extend the corresponding results by many others.

1. Introduction

Let E be a smooth Banach space and E^* the dual of E. The function $\Phi : E \times E \rightarrow R$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \tag{1}$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E^*. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T (see [1]), if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called nonexpansive, if

$$\|Tx - Ty\| \le \|x - y\|, \tag{2}$$

for all $x, y \in C$, and relatively nonexpansive (see [2]), if $\widehat{F}(T) = F(T)$ and

$$\phi(p, Tx) \le \phi(p, x), \tag{3}$$

for all $x \in C$ and $p \in F(T)$. The iterative methods for approximation of fixed points of nonexpansive mappings, relatively nonexpansive mappings, and other generational nonexpansive mappings have been studied by many researchers; see [3–13].

Actually, Mann [14] firstly introduced Mann iteration process in 1953, which is defined as follows:

$$x_0 = x \in C,$$
$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Tx_n, \quad n \ge 0. \tag{4}$$

It is very useful to approximate a fixed point of a nonexpansive mapping. However, as we all know, it has only weak convergence in a Hilbert space (see [15]). As a matter of fact, the process (3) may fail to converge for a Lipschitz pseudocontractive mapping in a Hilbert space (see [16]). For example, Reich [17] proved that if E is a uniformly convex Banach space with Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (3) converges weakly to a fixed point of T.

Some have made attempts to modify the Mann iteration methods, so that strong convergence is guaranteed. Nakajo and Takahashi [18] proposed the following modification of the Mann iteration method for a single nonexpansive mapping T in a Hilbert space H:

$$x_0 = x \in C,$$
$$y_n = \alpha_n x_n + (1 - \alpha_n) Tx_n,$$

$$
\begin{aligned}
C_n &= \{z \in C : \|z - y_n\| \le \|z - x_n\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{5}
$$

where P_K denotes the metric projection from H onto a closed convex subset of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ defined by (5) converges strongly to $P_{F(T)}x$.

The ideas to generate the process (5) from Hilbert spaces to Banach spaces have been made. By using the properties available on uniformly convex and uniformly smooth Banach spaces, Matsushita and Takahashi [10] presented their idea of the following method for a single relatively nonexpansive mapping T in a Banach space E:

$$
\begin{aligned}
x_0 &= x \in C, \\
y_n &= J^{-1}\left(\alpha_n J x_n + (1 - \alpha_n) J T x_n\right), \\
H_n &= \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\
W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\
x_{n+1} &= \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{6}
$$

where J is the duality mapping on E and $\Pi_{F(T)}x$ is the generalized projection from C onto $F(T)$.

In 2007 and 2008, Plubing and Ungchittrakool [19, 20] improved and generalized the process (6) to the new general process of two relatively nonexpansive mappings in a Banach space:

$$
\begin{aligned}
x_0 &= x \in C, \\
y_n &= J^{-1}\left(\alpha_n J x_n + (1 - \alpha_n) J z_n\right), \\
z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T x_n + \beta_n^{(3)} J S x_n\right), \\
H_n &= \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\
W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\
x_{n+1} &= \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
x_0 &= x \in C, \\
y_n &= J^{-1}\left(\alpha_n J x + (1 - \alpha_n) J z_n\right), \\
z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T x_n + \beta_n^{(3)} J S x_n\right), \\
H_n &= \{z \in C : \phi(z, y_n) \le \phi(z, x_n) \\
&\qquad + \alpha_n \left(\|x\|^2 + 2\langle J x_n - Jx, z\rangle\right)\}, \\
W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\
x_{n+1} &= \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, \ldots.
\end{aligned} \tag{8}
$$

They proved that both iterations (7) and (8) converge strongly to a common fixed point of two relatively nonexpansive mappings S and T provided that the sequences satisfy some appropriate conditions.

Inspired and motivated by these facts, in this paper, we aim to improve and generalize the process (7) and (8) to the new general process of a finite family of relatively nonexpansive mappings in a Banach space. Let C be a closed convex subset of a Banach space E and let $T_1, T_2, \ldots, T_N : C \to C$ be relatively nonexpansive mappings such that $F := \bigcap_{i=1}^N F(T_i) \ne \phi$. Define $\{x_n\}$ in the following way:

$$
\begin{aligned}
x_0 &= x \in C, \\
y_n &= J^{-1}\left(\alpha_n J x + \beta_n J x_n + \gamma_n J z_n\right), \\
z_n &= J^{-1}\left(\lambda_n^{(0)} J x_n + \sum_{i=1}^{N} \lambda_n^{(i)} J T_i x_n\right), \\
H_n &= \{z \in C : \phi(z, y_n) \le \phi(z, x_n) \\
&\qquad + \alpha_n \left(\|x\|^2 + 2\langle J x_n - Jx, z\rangle\right)\}, \\
W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\
x_{n+1} &= \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{9}
$$

where $\Pi_{H_n \cap W_n}$ is the generalized projection from C onto the intersection set $H_n \cap W_n$; $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n^{(0)}\}$, $\{\lambda_n^{(1)}\}, \ldots, \{\lambda_n^{(N)}\}$ are the sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\sum_{i=0}^N \lambda_n^{(i)} = 1$ for all $n \ge 0$. We prove, under certain appropriate assumptions on the sequences, that $\{x_n\}$ defined by (9) converges strongly to $P_F x$, where P_F is the generalized projection from C to F.

Obviously, the process (9) reduces to become (7) when $N = 2$, $\alpha_n = 0$ and become (8) when $N = 2$, $\beta_n = 0$. So, our results extend and improve the corresponding ones announced by Nakajo and Takahashi [18], Plubtieng and Ungchittrakool [19, 20], Matsushita and Takahashi [10], and Martinez-Yanes and Xu [21].

2. Preliminaries

This section collects some definitions and lemmas which will be used in the proofs for the main results in the next section.

Throughout this paper, let E be a real Banach space. Let J denote the normalized duality mapping from E into 2^{E^*} given by

$$
J(x) = \{f \in E^*, \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\|\}, \quad \forall x \in E, \tag{10}
$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

A Banach space E is said to be strictly convex if $\|x + y\|/2 < 1$ for $\|x\| = \|y\| = 1$ and $x \ne y$. It is also said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}(\|x_n + y_n\|/2) = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E, then the Banach space E is said to be smooth provided that $\lim_{t\to 0}((\|x + ty\| - \|x\|)/t)$ exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attainted uniformly for each $x, y \in U$. It is well known that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Some properties of the duality mapping have been given in [22]. A Banach space E is said to have Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [22] for more details.

Let E be a smooth Banach space. The function $\Phi : E \times E \to R$ is defined by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \tag{11}$$

for all $x, y \in E$. It is obvious from the definition of the function ϕ that

(1) $(\|x\| - \|y\|)^2 \le \phi(y,x) \le (\|y\|^2 + \|x\|^2)$,

(2) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jx\rangle$,

(3) $\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \le \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$,

for all $x, y \in E$; see [4, 7, 23] for more details.

Lemma 1 (see [4]). *If E is a strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.*

Lemma 2 (see [23]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a closed convex subset of E. Suppose that E is reflexive, strictly convex, and smooth. Then, for any $x \in E$, there exists a point $x_0 \in C$ such that $\phi(x_0, x) = \min_{y\in C}\phi(y, x)$. The mapping $\Pi_C : E \to C$ defined by $\Pi_C x = x_0$ is called the generalized projection (see [4, 7, 23]).

Lemma 3 (see [7]). *Let C be a closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \ge 0, \quad \forall y \in C. \tag{12}$$

Lemma 4 (see [7]). *Let E be a reflexive, strictly convex, and smooth Banach space and let C be a closed convex subset of E and $x \in E$. Then, $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x)$ for all $y \in C$.*

Lemma 5 (see [24]). *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \le r\}$ a closed ball of E. Then, there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \nu z\|^2 \le \lambda\|x\|^2 + \mu\|y\|^2 + \nu\|z\|^2 - \lambda\mu g(\|x - y\|), \tag{13}$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \nu \in [0, 1]$ with $\lambda + \mu + \nu = 1$.

Lemma 6 (see [19]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a closed convex subset of E. Then, for points $w, x, y, z \in E$ and a real number $a \in R$, the set $K := \{v \in C : \phi(v, y) \le \phi(v, x) + \langle v, Jz - Jw\rangle + a\}$ is closed and convex.*

3. Main Results

In this section, we will prove the strong convergence theorem for a common fixed point of a finite family of relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming. Let us prove a proposition first.

Proposition 7. *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \le r\}$ a closed ball of E. Then, there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\left\|\sum_{i=1}^{n}\lambda_i x_i\right\|^2 \le \sum_{i=1}^{n}\lambda_i\|x_i\|^2 - \frac{1}{n^2}g\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_i - x_j\|\right), \tag{14}$$

for all $n \ge 3$, $x_i \in B_r(0)$ and $\lambda_i \in [0, 1]$ with $\sum_{i=1}^{n}\lambda_i = 1$, $i = 1, 2, \ldots, n$.

Proof. If $\lambda_3 + \lambda_4 \cdots + \lambda_n \ne 0$, using Lemma 5 and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
&\left\|\sum_{i=1}^{n}\lambda_i x_i\right\|^2 \\
&= \Big\|\lambda_1 x_1 + \lambda_2 x_2 + (\lambda_3 + \cdots + \lambda_n) \\
&\quad \times\left(\frac{\lambda_3 x_3}{\lambda_3 + \cdots + \lambda_n} + \cdots + \frac{\lambda_n x_n}{\lambda_3 + \cdots + \lambda_n}\right)\Big\|^2 \\
&\le \lambda_1\|x_1\|^2 + \lambda_2\|x_2\|^2 + (\lambda_3 + \cdots + \lambda_n) \\
&\quad \times\left\|\frac{\lambda_3 x_3}{\lambda_3 + \cdots + \lambda_n} + \cdots + \frac{\lambda_n x_n}{\lambda_3 + \cdots + \lambda_n}\right\|^2 \\
&\quad - \lambda_1\lambda_2 g(\|x_1 - x_2\|) \\
&\le \sum_{i=1}^{n}\lambda_i\|x_i\|^2 - \lambda_1\lambda_2 g(\|x_1 - x_2\|).
\end{aligned} \tag{15}$$

If $\lambda_3 + \lambda_4 \cdots + \lambda_n = 0$, the last inequality above also holds obviously. By the same argument in the proof above, we obtain

$$\left\|\sum_{i=1}^{n}\lambda_i x_i\right\|^2 \le \sum_{i=1}^{n}\lambda_i\|x_i\|^2 - \lambda_i\lambda_j g(\|x_i - x_j\|), \tag{16}$$

for all $i, j \in \{1, 2, \ldots, n\}$. Then,

$$n^2 \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|^2 \leq n^2 \sum_{i=1}^{n} \lambda_i \|x_i\|^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j g\left(\|x_i - x_j\|\right) \leq n^2 \sum_{i=1}^{n} \lambda_i \|x_i\|^2 - g\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j \|x_i - x_j\|\right). \tag{17}$$

So,

$$\left\| \sum_{i=1}^{n} \lambda_i x_i \right\|^2 \leq \sum_{i=1}^{n} \lambda_i \|x_i\|^2 - \frac{1}{n^2} g\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j \|x_i - x_j\|\right). \tag{18}$$

□

Theorem 8. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E and $T_1, T_2, \ldots, T_N : C \to C$ relatively nonexpansive mappings such that $F := \cap_{i=1}^{N} F(T_i) \neq \phi$. The sequence $\{x_n\}$ is given by* (9) *with the following restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$, $0 \leq \alpha_n < 1$, $0 \leq \beta_n < 1$, $0 < \gamma_n \leq 1$ *for all* $n \geq 0$*;*

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\limsup_{n\to\infty} \beta_n < 1$*;*

(c) $\lambda_n^i \in [0, 1]$ *with* $\sum_{i=0}^{N} \lambda_n^i = 1$, $i = 0, 1, 2, \ldots, N$, *for all* $n \geq 0$*;*

(d) $\lim_{n\to\infty} \lambda_n^0 = 0$ *and* $\liminf_{n\to\infty} \lambda_n^i \lambda_n^j > 0$, $i, j = 1, 2, \ldots, N$*; or*

(d′) $\liminf_{n\to\infty} \lambda_n^0 \lambda_n^i > 0$, $i = 1, 2, \ldots, N$.

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_F x$, where Π_F is the generalized projection from C onto F.

Proof. We split the proof into seven steps.

Step 1. Show that P_F is well defined for every $x \in C$.

It is easy to know that $F(T_i)$, $i = 1, 2, \ldots, N$ are closed convex sets and so is F. What is more, F is nonempty by our assumption. Therefore, P_F is well defined for every $x \in C$.

Step 2. Show that H_n and W_n are closed and convex for all $n \geq 0$.

From the definition of W_n, it is obvious W_n is closed and convex for each $n \geq 0$. By Lemma 6, we also know that H_n is closed and convex for each $n \geq 0$.

Step 3. Show that $F \subset H_n \bigcap W_n$ for all $n \geq 0$.

Let $u \in F$ and let $n \geq 0$. Then, by the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \phi(u, z_n) &= \phi\left(u, J^{-1}\left(\lambda_n^{(0)} J x_n + \sum_{i=1}^{N} \lambda_n^{(i)} J T_i x_n\right)\right) \\ &= \|u\|^2 - 2\left\langle u, \lambda_n^{(0)} J x_n + \sum_{i=1}^{N} \lambda_n^{(i)} J T_i x_n \right\rangle + \left\| \lambda_n^{(0)} J x_n + \sum_{i=1}^{N} \lambda_n^{(i)} J T_i x_n \right\|^2 \\ &\leq \|u\|^2 - 2\lambda_n^{(0)} \langle u, J x_n \rangle - 2\sum_{i=1}^{N} \lambda_n^{(i)} \langle u, J T_i x_n \rangle + \lambda_n^{(0)} \|J x_n\|^2 + \sum_{i=1}^{N} \lambda_n^{(i)} \|J T_i x_n\|^2 \\ &= \lambda_n^{(0)} \phi(u, x_n) + \sum_{i=1}^{N} \lambda_n^{(i)} \phi(u, T_i x_n) \\ &\leq \lambda_n^{(0)} \phi(u, x_n) + \sum_{i=1}^{N} \lambda_n^{(i)} \phi(u, x_n) \\ &= \phi(u, x_n), \end{aligned} \tag{19}$$

and then,

$$\begin{aligned} \phi(u, y_n) &= \phi\left(u, J^{-1}\left(\alpha_n J x + \beta_n J x_n + \gamma_n J z_n\right)\right) \\ &= \|u\|^2 - 2\langle u, \alpha_n J x + \beta_n J x_n + \gamma_n J z_n \rangle + \|\alpha_n J x + \beta_n J x_n + \gamma_n J z_n\|^2 \\ &\leq \|u\|^2 - 2\alpha_n \langle u, J x \rangle - 2\beta_n \langle u, J x_n \rangle - 2\gamma_n \langle u, J z_n \rangle + \alpha_n \|x\|^2 + \beta_n \|x_n\|^2 + \gamma_n \|z_n\|^2 \\ &= \alpha_n \phi(u, x) + \beta_n \phi(u, x_n) + \gamma_n \phi(u, z_n) \\ &\leq \alpha_n \phi(u, x) + (1 - \alpha_n) \phi(u, x_n) \\ &= \phi(u, x_n) + \alpha_n (\phi(u, x) - \phi(u, x_n)) \\ &\leq \phi(u, x_n) + \alpha_n \left(\|x\|^2 + 2\langle J x_n - J x, z\rangle\right). \end{aligned} \tag{20}$$

Thus, we have $u \in H_n$. Therefore, we obtain $F \subset H_n$ for all $n \geq 0$.

Next, we prove $F \subset W_n$ for all $n \geq 0$. We prove this by induction. For $n = 0$, we have $F \subset C = W_0$. Assume that $F \subset W_n$. Since x_{n+1} is the projection of x onto $H_n \bigcap W_n$, by Lemma 3, we have

$$\langle x_{n+1} - z, J x - J x_{n+1} \rangle \geq 0, \tag{21}$$

for any $z \in H_n \bigcap W_n$. As $F \subset H_n \bigcap W_n$ by the induction assumption, $F \subset W_n$ holds, in particular, for all $u \in F$. This together with the definition of W_{n+1} implies that $F \subset W_{n+1}$. Hence, $F \subset H_n \bigcap W_n$ for all $n \geq 0$.

Step 4. Show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

In view of (19) and Lemma 4, we have $x_n = P_{W_n} x$, which means that, for any $z \in W_n$,

$$\phi(x_n, x) \leq \phi(z, x). \tag{22}$$

Since $x_{n+1} \in W_n$ and $u \in F \subset W_n$, we obtain

$$\begin{gathered} \phi(x_n, x) \leq \phi(x_{n+1}, x), \\ \phi(x_n, x) \leq \phi(u, x), \end{gathered} \tag{23}$$

for all $n \geq 0$. Consequently, $\lim_{n\to\infty}\phi(x_n, x)$ exists and $\{x_n\}$ is bounded. By using Lemma 4, we have

$$\phi(x_{n+1}, x_n) \leq \phi(x_{n+1}, x) - \phi(x_n, x) \longrightarrow 0, \tag{24}$$

as $n \to \infty$. By using Lemma 2, we obtain $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Step 5. Show that $\|x_n - z_n\| \to 0$ as $n \to \infty$.

From $x_{n+1} = P_{H_n \bigcap W_n} x \in H_n$, we have

$$\begin{aligned} \phi(x_{n+1}, y_n) &\leq \phi(x_{n+1}, x_n) \\ &\quad + \alpha_n \left(\|x\|^2 + 2\langle Jx_n - Jx,\ x_{n+1}\rangle\right) \longrightarrow 0, \end{aligned} \tag{25}$$

as $n \to \infty$. By Lemma 2, we also have $\|x_{n+1} - y_n\| \to 0$, and then,

$$\|x_n - y_n\| \leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \longrightarrow 0, \tag{26}$$

as $n \to \infty$. We observe that

$$\begin{aligned} &\phi(z_n, x_n) \\ &\quad = \phi(z_n, y_n) + \phi(y_n, x_n) + 2\langle z_n - y_n,\ Jy_n - Jx_n\rangle \\ &\quad \leq \phi(z_n, y_n) + \phi(y_n, x_n) + 2\|z_n - y_n\|\|Jy_n - Jx_n\|, \\ &\phi(z_n, y_n) \\ &\quad = \|z_n\|^2 - 2\langle z_n, \alpha_n Jx + \beta_n Jx_n + \gamma_n Jz_n\rangle \\ &\qquad + \|\alpha_n Jx + \beta_n Jx_n + \gamma_n Jz_n\|^2 \\ &\quad \leq \alpha_n \phi(z_n, x) + \beta_n \phi(z_n, x_n). \end{aligned} \tag{27}$$

So,

$$\begin{aligned} \phi(z_n, x_n) &\leq \alpha_n \phi(z_n, x) + \beta_n \phi(z_n, x_n) \\ &\quad + \phi(y_n, x_n) + 2\|z_n - y_n\|\|Jy_n - Jx_n\|. \end{aligned} \tag{28}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\limsup_{n\to\infty}\beta_n < 1$, $\phi(y_n, x_n) \to 0$, and $\|z_n - y_n\|\|Jy_n - Jx_n\| \to 0$ as $n \to \infty$, we have

$$\begin{aligned} \phi(z_n, x_n) &\leq \frac{\alpha_n}{1-\beta_n}\phi(z_n, x) + \frac{1}{1-\beta_n}\phi(y_n, x_n) \\ &\quad + \frac{2}{1-\beta_n}\|z_n - y_n\|\|Jy_n - Jx_n\| \longrightarrow 0, \end{aligned} \tag{29}$$

as $n \to \infty$. Using Lemma 2, we obtain $\|x_n - z_n\| \to 0$ as $n \to \infty$.

Step 6. Show that $\|x_n - T_i x_n\| \to 0$, $i = 1, 2, \ldots, N$.

Since $\{x_n\}$ is bounded and $\phi(p, T_i x_n) \leq \phi(p, x_n)$, where $p \in F$, $i = 1, 2, \ldots, N$, we also obtain that $\{Jx_n\}, \{JT_1 x_n\}, \ldots, \{JT_N x_n\}$ are bounded, and hence, there exists $r > 0$ such that $\{Jx_n\}, \{JT_1 x_n\}, \ldots, \{JT_N x_n\} \subset B_r(0)$. Therefore, Proposition 7 can be applied and we observe that

$$\begin{aligned} &\phi(p, z_n) \\ &\quad = \|p\|^2 - 2\left\langle p, \lambda_n^{(0)} Jx_n + \sum_{i=1}^{N}\lambda_n^{(i)} JT_i x_n \right\rangle \\ &\qquad + \left\|\lambda_n^{(0)} Jx_n + \sum_{i=1}^{N}\lambda_n^{(i)} JT_i x_n\right\|^2 \\ &\quad \leq \|p\|^2 - 2\lambda_n^{(0)}\langle p, Jx_n\rangle - 2\sum_{i=1}^{N}\lambda_n^{(i)}\langle p, JT_i x_n\rangle \\ &\qquad + \lambda_n^{(0)}\|x_n\|^2 + \sum_{i=1}^{N}\lambda_n^{(i)}\|T_i x_n\|^2 \\ &\qquad - \frac{1}{N^2} g\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_n^{(i)}\lambda_n^{(j)}\|JT_i x_n - JT_j x_n\| \right. \\ &\qquad \left. + 2\sum_{i=1}^{N}\lambda_n^{(0)}\lambda_n^{(i)}\|Jx_n - JT_i x_n\|\right) \\ &\quad = \lambda_n^{(0)}\phi(p, x_n) + \sum_{i=1}^{N}\lambda_n^{(i)}\phi(p, T_i x_n) \\ &\qquad - \frac{1}{N^2} g\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_n^{(i)}\lambda_n^{(j)}\|JT_i x_n - JT_j x_n\| \right. \\ &\qquad \left. + 2\sum_{i=1}^{N}\lambda_n^{(0)}\lambda_n^{(i)}\|Jx_n - JT_i x_n\|\right) \\ &\quad \leq \phi(p, x_n) - \frac{1}{N^2} g\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_n^{(i)}\lambda_n^{(j)}\|JT_i x_n - JT_j x_n\| \right. \\ &\qquad \left. + 2\sum_{i=1}^{N}\lambda_n^{(0)}\lambda_n^{(i)}\|Jx_n - JT_i x_n\|\right), \end{aligned} \tag{30}$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous strictly increasing convex function with $g(0) = 0$. And

$$\begin{aligned} &\phi(p, x_n) - \phi(p, z_n) \\ &\quad = \|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2 - \|p\|^2 + 2\langle p, Jz_n\rangle - \|z_n\|^2 \\ &\quad \leq 2\|p\|\|Jz_n - Jx_n\| + \|x_n\|^2 - \|z_n\|^2 \longrightarrow 0, \end{aligned} \tag{31}$$

as $n \to \infty$. From the properties of the mapping g, we have

$$\lim_{n\to\infty} \lambda_n^{(0)} \lambda_n^{(i)} \|x_n - T_i x_n\| = 0,$$
$$\lim_{n\to\infty} \lambda_n^{(i)} \lambda_n^{(j)} \left\|T_i x_n - T_j x_n\right\| = 0, \tag{32}$$

for all $i, j \in \{1, 2, \ldots, N\}$. From the condition (d′), we have $\|x_n - T_i x_n\| \to 0$ immediately, as $n \to \infty$, $i = 1, 2, \ldots, N$; from the condition (d), we can also have $\|x_n - T_i x_n\| \to 0$, as $n \to \infty$, $i = 1, 2, \ldots, N$. In fact, since $\liminf_{n\to\infty} \lambda_n^{(i)} \lambda_n^{(j)} > 0$, it follows that

$$\lim_{n\to\infty} \left\|T_i x_n - T_j x_n\right\| = 0, \tag{33}$$

for all $i, j \in \{1, 2, \ldots, N\}$. Next, we note by the convexity of $\|\cdot\|^2$ and (9) that

$$\begin{aligned}
\phi\left(T_j x_n, z_n\right) &= \left\|T_j x_n\right\|^2 - 2\left\langle T_j x_n, \lambda_n^{(0)} J x_n + \sum_{i=1}^{N} \lambda_n^{(i)} J T_i x_n \right\rangle \\
&\quad + \left\|\lambda_n^{(0)} J x_n + \sum_{i=1}^{N} \lambda_n^{(i)} J T_i x_n \right\|^2 \\
&\le \left\|T_j x_n\right\|^2 - 2\lambda_n^{(0)} \left\langle T_j x_n, J x_n\right\rangle - 2\sum_{i=1}^{N} \lambda_n^{(i)} \left\langle T_j x_n, J T_i x_n\right\rangle \\
&\quad + \lambda_n^{(0)} \left\|x_n\right\|^2 + \sum_{i=1}^{N} \lambda_n^{(i)} \left\|T_i x_n\right\|^2 \\
&= \lambda_n^{(0)} \phi\left(T_j x_n, x_n\right) + \sum_{i=1}^{N} \lambda_n^{(i)} \phi\left(T_j x_n, T_i x_n\right) \longrightarrow 0,
\end{aligned} \tag{34}$$

as $n \to \infty$. By Lemma 2, we have $\lim_{n\to\infty} \|T_i x_n - z_n\| = 0$ and

$$\left\|T_i x_n - x_n\right\| \le \left\|T_i x_n - z_n\right\| + \left\|x_n - z_n\right\| \longrightarrow 0, \tag{35}$$

as $n \to \infty$ for all $i \in \{1, 2, \ldots, N\}$.

Step 7. Show that $x_n \to \Pi_F x$, as $n \to \infty$.

From the result of Step 6, we know that if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $\{x_{n_k}\} \rightharpoonup \widehat{x} \in C$, then $\widehat{x} \in \cap_{i=1}^{N} \widehat{F}(T_i) = \cap_{i=1}^{N} F(T_i)$. Because E is a uniformly convex and uniformly smooth Banach space and $\{x_n\}$ is bounded, so we can assume $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $\{x_{n_k}\} \rightharpoonup \widehat{x} \in F$ and $\omega = \Pi_F x$. For any $n \ge 1$, from $x_{n+1} = \Pi_{H_n \cap W_n} x$ and $\omega \in F \subset H_n \bigcap W_n$, we have

$$\phi\left(x_{n+1}, x\right) \le \phi\left(\omega, x\right). \tag{36}$$

On the other hand, from weakly lower semicontinuity of the norm, we have

$$\begin{aligned}
\phi\left(\widehat{x}, x\right) &= \|\widehat{x}\|^2 - 2\left\langle \widehat{x}, Jx\right\rangle + \|x\|^2 \\
&\le \liminf_{n\to\infty} \left(\left\|x_{n_k}\right\|^2 - 2\left\langle \left\|x_{n_k}\right\|^2, Jx\right\rangle + \|x\|^2\right) \\
&= \liminf_{n\to\infty} \phi\left(x_{n_k}, x\right) \le \limsup_{n\to\infty} \phi\left(x_{n_k}, x\right) \le \phi\left(\omega, x\right).
\end{aligned} \tag{37}$$

From the definition of $\Pi_F x$, we obtain $\widehat{x} = \omega$, and hence, $\lim_{n\to\infty} \phi(x_{n_k}, x) = \phi(\omega, x)$. So, we have $\lim_{k\to\infty} \|x_{n_k}\| = \|\omega\|$. Using the Kadec-klee property of E, we obtain that $\{x_{n_k}\}$ converges strongly to $\Pi_F x$. Since $\{x_{n_k}\}$ is an arbitrary weakly convergent sequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $\Pi_F x$. □

Corollary 9. *Let C be a nonempty closed convex subset of a Hilbert space H and $T_1, T_2, \ldots, T_N : C \to C$ relatively nonexpansive mappings such that $F := \cap_{i=1}^{N} F(T_i) \ne \phi$. The sequence $\{x_n\}$ is given by* (9) *with the following restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$, $0 \le \alpha_n < 1$, $0 \le \beta_n < 1$, $0 \le \gamma_n \le 1$ *for all $n \ge 0$;*

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\limsup_{n\to\infty} \beta_n < 1$*;*

(c) $\lambda_n^i \in [0, 1]$ *with* $\sum_{i=0}^{N} \lambda_n^i = 1$, $i = 0, 1, 2, \ldots, N$, *for all $n \ge 0$;*

(d) $\lim_{n\to\infty} \lambda_n^0 = 0$ *and* $\liminf_{n\to\infty} \lambda_n^i \lambda_n^j > 0$, $i, j = 1, 2, \ldots, N$*; or*

(d′) $\liminf_{n\to\infty} \lambda_n^0 \lambda_n^i > 0$, $i = 1, 2, \ldots, N$.

Then, the sequence $\{x_n\}$ converges strongly to $P_F x$, where P_F is the metric projection from C onto F.

Proof. It is true because the generalized projection Π_F is just the metric projection P_F in Hilbert spaces. □

Remark 10. The results of Nakajo and Takahashi [18] and Song et al. [11] are the special cases of our results in Corollary 9. And in our results of Theorem 8, if $T_1 = T_2 = \cdots = T_N, \lambda_n^{(0)}$ and $\alpha_n = 0$ for all $n \ge 0$, then, we obtain Theorem 4.1 of Matsushita and Takahashi [10]; if $T_1 = T_2 = \cdots = T_{N-1}$ and $\alpha_n = 0$ for all $n \ge 0$, then, we obtain Theorem 3.1 of Plubtieng and Ungchittrakool [19]; if $T_1 = T_2 = \cdots = T_{N-1}$ and $\beta_n = 0$ for all $n \ge 0$, then, we obtain Theorem 3.2 of Plubtieng and Ungchittrakool [19]. So, our results improve and extend the corresponding results by many others.

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