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Research Article

Global Solvability of Hammerstein Equations with Applications to BVP Involving Fractional Laplacian

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Some sufficient conditions for the nonlinear integral operator of the Hammerstein type to be a diffeomorphism defined on a certain Sobolev space are formulated. The main result assures the invertibility of the Hammerstein operator and in consequence the global solvability of the nonlinear Hammerstein equations. The applications of the result to nonlinear Dirichlet BVP involving the fractional Laplacian and to some specific Hammerstein equation are presented.

1. Introduction

Consider, for any $\sigma \in (1,2]$, an arbitrary real number λ, a given function z, and a nonlinear term h, the Dirichlet boundary value problem involving one-dimensional fractional Laplacian which reads as

$$\lambda(-\Delta)^{\sigma/2}x(t) + h(t, x(t)) = (-\Delta)^{\sigma/2}z(t), \quad t \in (-1, 1), \tag{1}$$

provided with the following Dirichlet exterior boundary condition:

$$x(t) = 0, \quad t \in (-\infty, -1] \cup [1, \infty). \tag{2}$$

The problems with the fractional Laplacian attracted lot of attention in recent years as they naturally arise in various areas of applications to mention only, see [1–5] and references therein:

(i) Probability—Mathematical Finance—as infinitesimal generators of stable Lévy processes,

(ii) Mechanics—Elastostatics—in Signorini obstacle problem originating from linear elasticity,

(iii) Fluid Mechanics—appearing in quasi-geostrophic fractional Navier-Stokes equation,

(iv) Hydrodynamics—describing some porous media flows in the hydrodynamical model.

For fractional derivatives in various senses one can also see the books and articles like [6–8].

The problem (1) can be transformed into the operator equation

$$\lambda x + \left((-\Delta)_D^{\sigma/2}\right)^{-1} h(\cdot, x) = z, \tag{3}$$

where the inverse of the fractional Laplacian with Dirichlet boundary condition (2) is defined by

$$\left((-\Delta)_D^{\sigma/2}\right)^{-1} g(t) = \int_{-1}^{1} G(t, \tau)\, g(\tau)\, d\tau, \tag{4}$$

where the Green function for the Dirichlet fractional Laplace operator is defined, for example, in [2], as

$$G(t, \tau) = c_\sigma |t - \tau|^{\sigma-1} \int_0^{w(t,\tau)} r^{\sigma/2-1}(r+1)^{-1/2} dr, \qquad w(t, \tau) = \left(1 - t^2\right)\left(1 - \tau^2\right)|t - \tau|^{-2}, \tag{5}$$

and the constant c_σ is defined as

$$c_\sigma = \frac{\Gamma(1/2)}{2^\sigma \pi^{1/2} \Gamma^2(\sigma/2)}. \tag{6}$$

It should be underlined that only in the case $\sigma = 2$ the derivative of the Green function is nonsingular, but as soon as $\sigma < 2$

the singularity for the derivative of the Green function G_t appears (cf. [9, 10]) so we should allow in our theory to treat also singular integrals if we want to guarantee the operator on the right hand side of (3) to be a diffeomorphism in H_0^1, which appears to be true for $\sigma \in (1, 2]$.

Consider, to address the solvability of (3), the general equation of the form

$$\mathcal{T} x = z, \tag{7}$$

where in the leading example (3) the operator $\mathcal{T}$, being a sum of the rescaled identity operator $\lambda\mathcal{I}$ and the Hammerstein operator, is expressed as follows:

$$\mathcal{T} x = \lambda x + \left((-\Delta)_D^{\sigma/2}\right)^{-1} h(\cdot, x). \tag{8}$$

The operator $((-\Delta)_D^{\sigma/2})^{-1} h(\cdot, x)$ is the composition of two operators: the linear integral nonlocal operator $((-\Delta)_D^{\sigma/2})^{-1}$—the inverse of the fractional Laplacian equipped with the Green function kernel G given by (5) and the nonlinear Nemitskii operator $x \mapsto h(\cdot, x)$ defined by the nonlinear function h. We will show that (7) is globally solvable. In fact it can be proved that under suitable assumptions the operator $\mathcal{T}$ is the global diffeomorphism on the Sobolev space $H_0^1([-1, 1])$ of absolutely continuous functions; hence, apart from the solvability (7) also the differentiable continuous dependence on data follows.

In the sequel we will therefore consider the nonlinear integral operators of Hammerstein type of the following form:

$$\mathcal{T} x(t) = \lambda x(t) + \int_{-1}^{1} G(t, \tau) h(\tau, x(\tau)) \, d\tau, \tag{9}$$

where $\lambda \in \mathbb{R}$, $t \in [-1, 1]$, $G : P \to \mathbb{R}$, $P = [-1, 1] \times [-1, 1]$, $h : [-1, 1] \times \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$, and $x \in H_0^1$. By H_0^1 we will denote $H_0^1([-1, 1], \mathbb{R}^n)$, the space of absolutely continuous functions defined on $[-1, 1]$ such that $x(-1) = x(1) = 0$, with the square-integrable derivative; that is, $x' \in L^2$, endowed with the norm

$$\|x\|_{H_0^1}^2 = \int_{-1}^{1} \left|x'(t)\right|^2 dt, \tag{10}$$

where $L^2 = L^2([-1, 1], \mathbb{R}^n)$ is the space of square-integrable functions.

Under some appropriate assumptions imposed on the functions G and h to be specified later, it is feasible to formulate some sufficient conditions for the operator $\mathcal{T} : H_0^1 \to H_0^1$ to be a diffeomorphism; that is, $\mathcal{T}(H_0^1) = H_0^1$, and that there exists an inverse operator $\mathcal{T}^{-1}$ while both $\mathcal{T}$, $\mathcal{T}^{-1}$ are Fréchet differentiable at every point from H_0^1. In other words, $\mathcal{T}$ is Fréchet differentiable at every point $x \in H_0^1$ and for every $z \in H_0^1$ there exists a unique solution $x_z \in H_0^1$ to the equation $\mathcal{T}(x) = z$ depending continuously on z and such that the operator $H_0^1 \ni z \to x_z \in H_0^1$ is Fréchet differentiable.

It should be underlined that integral operators and integral equations are most commonly considered in the space of square-integrable functions. Under suitable conditions one usually proves some existence and uniqueness theorems for integral equations. In this paper the integral operator $\mathcal{T}$ is defined on the space H_0^1. In the proof of Lemma 12 we have used the compactness of the embedding of the space H_0^1 into the space of continuous functions C. This compact embedding implies that every weakly convergent sequence in H_0^1 is uniformly convergent in C in the supremum norm. Apparently in the case of L^2 space such an implication does not hold. Therefore, one cannot prove, at least with the method applied herein, that the operator $\mathcal{T} : L^2 \to L^2$ is a diffeomorphism.

Integral equations originate from models appearing in various fields of science including elasticity, plasticity, heat and mass transfer, epidemics, fluid dynamics, and oscillation theory; see, for example, books by Corduneanu [11] and by Gripenberg et al. [12]. Various kinds of integral operators considered therein include those of Fedholm, Hammerstein, Volterra and Wiener-Hopf type. Recall that we will establish global solvability of integral equations of Hammerstein type by stating sufficient conditions for Hammerstein operator to be a diffeomorphism. For references on Hammerstein equations see, for example, among others, [13–19] and references therein. Interest in Hammerstein equation, being the special case of Fredholm equation, stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems, whose linear parts possess the inverse defined via the Green's function, can, as a rule, be transformed into equation involving Hammerstein integral operator. Among these, we mention the problem of the forced oscillations of finite amplitude of a pendulum; see, for example, [20] or for the BVP's on real line of Hammerstein and Wiener-Hopf type, see, for example, [19], or for optimal problems for Hammerstein and Volterra equations, see, for example, [17].

2. Global Diffeomorphism by Use of Mountain Pass Theorem

Let X be a real Banach space and let $\psi : X \to \mathbb{R}$ be a C^1-mapping. A sequence $\{x_k\}_{k\in\mathbb{N}}$ is referred to as a Palais-Smale sequence for functional ψ if for some $M > 0$, any $k \in \mathbb{N}$, $|\psi(x_k)| \leq M$ and $\psi'(x_k) \to 0$ as $k \to \infty$. We say that ψ satisfies Palais-Smale condition if any Palais-Smale sequence possesses a convergent subsequence. Moreover, a point $x^* \in X$ is called a critical point of ψ if $\psi'(x^*) = 0$. In such a case $\psi(x^*)$ is referred to as a critical value of ψ.

Let us introduce the following sets used in the Mountain Pass Theorem:

$$W_e = \left\{U \subset X : U \text{ is open}, \ 0 \in U, \ e \notin \overline{U}\right\}, \tag{11}$$

for any $e \in X$ such that $e \neq 0$ and

$$B_\rho = \left\{x \in X : \|x\|_X < \rho\right\}, \tag{12}$$

for any $\rho > 0$.

In the proof of the forthcoming diffeomorphism theorem the well-known variational Mountain Pass Theorem is used as the main tool. For more details we refer the reader to vast literature on the subject, for example, among others [21, 22].

Theorem 1 (Mountain Pass Theorem). *Let $\psi : X \to \mathbb{R}$ be a C^1-mapping satisfying Palais-Smale condition and let $\psi(0) = 0$. If*

(i) *there are some constants $\rho, \alpha > 0$ such that $\psi|_{\partial B_\rho} \geq \alpha$,*

(ii) *there is a point $e \in X \setminus \overline{B_\rho}$ such that $\psi(e) \leq 0$,*

then $c = \sup_{U \in W_e} \inf_{x \in \partial U} \psi(x)$ is the critical value of ψ and $c \geq \alpha$.

Applying the above theorem it is possible, as was done in [23], to prove the following theorem on a global diffeomorphism.

Theorem 2. *Let X be a real Banach space and let H be a real Hilbert space. If $\mathcal{T} : X \to H$ is a C^1-mapping such that*

(a1) *for any $x \in X$ the equation $\mathcal{T}'(x)h = g$ possesses a unique solution for any $g \in H$,*

(b1) *for any $y \in H$ the functional*

$$\Psi_y : X \ni x \longrightarrow \frac{1}{2}\left\| \mathcal{T}x - y \right\|_H^2 \in \mathbb{R}^+ = [0, \infty) \tag{13}$$

satisfies Palais-Smale condition, then $\mathcal{T}$ is a diffeomorphism.

Remark 3. By (a1) and the bounded inverse theorem, for any $x \in X$, there exists $\gamma_x > 0$ such that

$$\left\| \mathcal{T}'(x) h \right\|_H \geq \gamma_x \|h\|_X, \tag{14}$$

for any $h \in X$. Therefore, the above theorem is equivalent in other notations to Theorem 3.1 in [23].

3. Auxiliary Facts and Used Assumptions

The presentation of the proof of the main result of this paper, which formulates sufficient conditions for $\mathcal{T} : H_0^1 \to H_0^1$ defined by (9) to be a diffeomorphism, we precede with a few lemmas.

Lemma 4. *For any $x \in H_0^1$ one has*

$$\begin{gathered} |x(t)| \leq (t+1)^{1/2} \|x\|_{H_0^1} \quad \text{for } t \in [-1, 1], \\ \int_{-1}^{1} |x(t)|^2 dt \leq 2\|x\|_{H_0^1}^2. \end{gathered} \tag{15}$$

Proof. By the Schwarz inequality, for $t \in [-1, 1]$, one obtains

$$|x(t)| \leq \int_{-1}^{t} \left| x'(\tau) \right| d\tau \leq (t+1)^{1/2} \|x\|_{H_0^1}. \tag{16}$$

Consequently,

$$\int_{-1}^{1} |x(t)|^2 dt \leq \|x\|_{H_0^1}^2 \int_{-1}^{1} (t+1)\, dt = 2\|x\|_{H_0^1}^2, \tag{17}$$

and this is precisely the second assertion of the lemma. □

In what follows, we will use the following assumptions imposed on the functions G and h.

(A1) One has the following:

(a) the functions $G(\cdot, \tau)$ and $h(\tau, \cdot)$ are continuous for a.e. $\tau \in [-1, 1]$,

(b) there exists continuous derivative $G_t(\cdot, \tau)$ on $(-1, 1) \setminus \{\tau\}$ for a.e. $\tau \in [-1, 1]$,

(c) there exists derivative $h_x(\tau, \cdot)$ and it is continuous for a.e. $\tau \in [-1, 1]$;

(A2) One has the following:

(a) the function $G(t, \cdot)h(\cdot, x(\cdot))$ is integrable and this integral is locally bounded with respect to $x \in H_0^1$, that is, for every $\rho > 0$ there exists $l_\rho > 0$ such that for any $t \in [-1, 1]$ and $|x(t)| \leq \rho$:

$$\int_{-1}^{1} |G(t, \tau)|\ |h(\tau, x(\tau))|\, d\tau < 2l_\rho, \tag{18}$$

(b) the function $G_t(t, \cdot)h(\cdot, x(\cdot))$ is integrable and for every $\rho > 0$ there exist $l_\rho > 0$ such that

$$\int_{-1}^{1} \left| G_t(t, \tau) \right| |h(\tau, x(\tau))|\, d\tau < 2l_\rho, \tag{19}$$

for $x \in H_0^1$ such that $|x(t)| \leq \rho$ for $t \in [-1, 1]$,

(c) the function $G(t, \cdot)h_x(\cdot, x(\cdot))$ satisfies (A2)(a) with h_x instead of h whereas the function $G_t(t, \cdot)h_x(\cdot, x(\cdot))$ satisfies (A2)(b) with h_x instead of h;

(A3) G satisfies the Dirichlet boundary conditions $G(-1, \tau) = G(1, \tau) = 0$ for a.e. $\tau \in [-1, 1]$;

(A4) $\int_{-1}^{1} |G_t(t, \tau)||h_x(\tau, x(\tau))| d\tau < |\lambda|$ for any $x \in H_0^1$ and $t \in [-1, 1]$;

(A5) One has the following:

(a) $|h(\tau, x)| \leq a(\tau)|x| + b(\tau)$ where $\tau \in [-1, 1]$, $x \in \mathbb{R}^n$, $a, b \in L^2([-1, 1], \mathbb{R}^+)$,

(b) $\|G_t(\cdot, \cdot)a(\cdot)\|_{L^2(P, \mathbb{R})} < \sqrt{2}|\lambda|/4$, $\|G_t(\cdot, \cdot)b(\cdot)\|_{L^2(P, \mathbb{R})} < \infty$.

Remark 5. Besides regularity (A1), (A2), and technical (A3) assumptions, we must finally impose on the functions G and h some growth and quantitative global assumptions: (A4) and (A5).

Lemma 6. *If the functions G and h satisfy (A1)(a), (A1)(b), (A2)(a), (A2)(b), and (A3), then the operator $\mathcal{T}$ is well defined by (9) on the space H_0^1 with values in H_0^1.*

Proof. Let us choose any $x_0 \in H_0^1$. By (A3), $\mathcal{T}x_0(-1) = \mathcal{T}x_0(1) = 0$. It suffices to show that the function

$$y(t) = \mathcal{T}x_0(t) - \lambda x_0(t) = \int_{-1}^{1} G(t, \tau)\, h(\tau, x_0(\tau))\, d\tau \tag{20}$$

is absolutely continuous and $y' \in L^2$. Observe, by (A2)(b) and $y(t_i) = \int_{-1}^{t_i} y'(t)dt$ that one has

$$\begin{aligned}\sum_{i=1}^{N} \left| y(t_{i+1}) - y(t_i) \right| &= \sum_{i=1}^{N} \int_{t_i}^{t_{i+1}} \int_{-1}^{1} \left| G_t(t,\tau) h(\tau, x_0(\tau)) \right| d\tau\, dt \\ &\le 2 l_\rho \sum_{i=1}^{N} \left| t_{i+1} - t_i \right|,\end{aligned} \tag{21}$$

where $-1 \le t_1 < t_2 < \cdots < t_i < t_{i+1} < \cdots < t_N < t_{N+1} \le 1$. As a result, for any $x_0 \in H_0^1$ the function y is absolutely continuous and therefore for almost any $t \in (-1, 1)$ there exists $y'(t)$ and its square integral can be estimated by (A2)(b) as

$$\int_{-1}^{1} \left| y'(t) \right|^2 dt \le \int_{-1}^{1} \left(\int_{-1}^{1} \left| G_t(t,\tau) h(\tau, x_0(\tau)) \right| d\tau \right)^2 dt \le 8 l_\rho^2 \tag{22}$$

so that $y' \in L^2$. □

Now we present some sufficient conditions for $\mathcal{T} : H_0^1 \to H_0^1$ to be Fréchet differentiable.

Lemma 7. *Suppose that functions G and h satisfy (A1)(a), (A1)(c), (A2)(a), (A2)(c), and (A3). Then the operator $\mathcal{T}$ defined by* (9) *is Fréchet differentiable at any $x_0 \in H_0^1$ while for $x \in H_0^1$ and $t \in [-1, 1]$*

$$\mathcal{T}'(x_0)\, x(t) = \lambda x(t) + \int_{-1}^{1} G(t,\tau) h_x(\tau, x_0(\tau))\, x(\tau)\, d\tau. \tag{23}$$

Proof. It is sufficient to show that the operator

$$\mathcal{T}^0(x)(t) = \int_{-1}^{1} G(t,\tau) h(\tau, x(\tau))\, d\tau \tag{24}$$

is Fréchet differentiable. The Mean Value Theorem (cf. [24]) yields, for $t \in [-1, 1]$ and some $\theta \in [0, 1]$,

$$\begin{aligned}&\mathcal{T}^0(x_0 + x)(t) - \mathcal{T}^0(x_0)(t) \\ &\quad = \int_{-1}^{1} \left[G(t,\tau) h(\tau, x_0(\tau) + x(\tau)) - G(t,\tau) h(\tau, x_0(\tau)) \right] d\tau \\ &\quad = \int_{-1}^{1} G(t,\tau) h_x(\tau, x_0(\tau))\, x(\tau)\, d\tau \\ &\qquad + \int_{-1}^{1} \left[\int_{0}^{1} G(t,\tau) h_x(\tau, x_0(\tau) + \theta x(\tau))\, d\theta - G(t,\tau) h_x(\tau, x_0(\tau)) \right] x(\tau)\, d\tau.\end{aligned} \tag{25}$$

From (15) in Lemma 4 one has

$$\begin{aligned}&\left| \int_{-1}^{1} \left[\int_{0}^{1} G(t,\tau) h_x(\tau, x_0(\tau) + \theta x(\tau))\, d\theta - G(t,\tau) h_x(\tau, x_0(\tau)) \right] x(\tau)\, d\tau \right| \\ &\quad \le \sqrt{2} \|x\|_{H_0^1} \int_{-1}^{1} \int_{0}^{1} \left| G(t,\tau) h_x(\tau, x_0(\tau) + \theta x(\tau)) - G(t,\tau) h_x(\tau, x_0(\tau)) \right| d\theta\, d\tau.\end{aligned} \tag{26}$$

Since the strong convergence in H_0^1 implies the uniform convergence in C and since the assumptions (A1)(c) and (A2)(c) of this lemma are satisfied, the Lebesgue Theorem leads, if we take $\|x\|_{H_0^1} \to 0$, to

$$\int_{-1}^{1} \left| G(t,\tau) h_x(\tau, x_0(\tau) + \theta x(\tau)) - G(t,\tau) h_x(\tau, x_0(\tau)) \right| d\tau \longrightarrow 0, \tag{27}$$

and thus,

$$\begin{aligned}&\mathcal{T}^0(x_0 + x)(t) - \mathcal{T}^0(x_0)(t) \\ &\quad = \int_{-1}^{1} G(t,\tau) h_x(\tau, x_0(\tau))\, x(\tau)\, d\tau + o(x),\end{aligned} \tag{28}$$

where $o(x)/\|x\|_{H_0^1} \to 0$ as $\|x\|_{H_0^1} \to 0$, which completes the proof. □

4. Local Solvability: Analysis of Linearized System

Let $x_0 \in H_0^1$ be a fixed but an arbitrary function and $T : H_0^1 \to H_0^1$ be a linear operator defined, for any $x \in H_0^1$ and $t \in [-1, 1]$, by

$$(Tx)(t) = \int_{-1}^{1} G(t,\tau) h_x(\tau, x_0(\tau))\, x(\tau)\, d\tau, \tag{29}$$

where the functions G and h define, respectively, the kernel and the nonlinearity of operator $\mathcal{T}$ defined in (9).

Next, for any $k \in \mathbb{N}$, $t \in [-1, 1]$, and $x \in H_0^1$, consider the following sequence of iterations:

$$\begin{aligned}\left(T^0 x\right)(t) &= x(t), \\ \left(T^1 x\right)(t) &= T\left(T^0 x\right)(t) \\ &= \int_{-1}^{1} G(t,\tau) h_x(\tau, x_0(\tau))\, x(\tau)\, d\tau,\end{aligned}$$

$$\begin{aligned}
\left(T^2 x\right)(t) &= T\left(T^1 x\right)(t) \\
&= \int_{-1}^{1} G(t,\tau)\, h_x\left(\tau, x_0(\tau)\right) \\
&\quad \times \left(T^1 x\right)(\tau)\, d\tau, \\
&\cdots,
\end{aligned} \tag{30}$$

$$\begin{aligned}
\left(T^{k+1} x\right)(t) &= T\left(T^k x\right)(t) \\
&= \int_{-1}^{1} G(t,\tau)\, h_x\left(\tau, x_0(\tau)\right) \\
&\quad \times \left(T^k x\right)(\tau)\, d\tau.
\end{aligned} \tag{31}$$

We will prove the following lemma.

Lemma 8. *Under assumptions (A1)(c), (A2)(c), and (A3) one has the following estimates:*

$$\left|\left(T^k x\right)(t)\right| \le 2^k l_\rho^k M, \quad \text{for } k = 0,1,2,\ldots,\ t \in [-1,1], \tag{32}$$

where $l_\rho > 0$ is defined by (A2)(c),

$$M = \|x\|_\infty := \sup_{t\in[-1,1]} |x(t)|, \tag{33}$$

and $\{T^k x\}$ is a sequence defined iteratively by (30) *and* (31).

Proof. First, from (29)–(31) and the assumptions of the lemma, we obtain subsequently

$$\begin{aligned}
&\left|\left(T^1 x\right)(t)\right| \le 2 l_\rho M, \\
&\left|\left(T^2 x\right)(t)\right| \le \int_{-1}^{1} \left|G(t,\tau)\, h_x\left(\tau, x_0(\tau)\right)\right| \left|\left(T^1 x\right)(\tau)\right| d\tau \\
&\qquad \le 2 l_\rho 2 l_\rho M = 4 l_\rho^2 M, \\
&\left|\left(T^3 x\right)(t)\right| \le \int_{-1}^{1} \left|G(t,\tau)\, h_x\left(\tau, x_0(\tau)\right)\right| \left|\left(T^2 x\right)(\tau)\right| d\tau \\
&\qquad \le 8 l_\rho^3 M.
\end{aligned} \tag{34}$$

To finish the proof we proceed by induction to get estimate (32).

Now, let us consider the linear integral equation

$$\begin{aligned}
&\lambda x(t) + \int_{-1}^{1} G(t,\tau)\, h_x\left(\tau, x_0(\tau)\right) x(\tau)\, d\tau \\
&\quad = y(t) \quad \text{for } t \in [-1,1],
\end{aligned} \tag{35}$$

where $x_0 \in H_0^1$ and $y \in H_0^1$ are fixed. For (35), we will prove the existence and uniqueness result, see Lemma 10. Since, in the proof of this lemma, we will perform spectral analysis we now present some introductory notions and recall some functional analytic theorems and tools on spectral radius.

Let T be a bounded, continuous operator in a Banach space X. Then we can decompose $\mathbb{C}$ into the resolvent of the operator T defined by

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is bijection on } X\}, \tag{36}$$

and the complementary set—the spectrum of T defined as

$$\sigma(T) = \mathbb{C} \setminus \rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is bijection on } X\}. \tag{37}$$

For any bounded and continuous operator T, we can define the spectral radius of T by the formula

$$r(T) = \lim_{k\to\infty} \sup \left\|T^k\right\|^{1/k}, \tag{38}$$

which must be finite, for example, due to the following estimate:

$$r(T) \le \|T\|. \tag{39}$$

Moreover, we have, following, for example, [25, Theorem VI.6] and [26, Theorem VIII.2.3], the theorem. □

Theorem 9. *For any $|\lambda| > r(T)$, one has $\lambda \in \rho(T)$, which means complementarily that the spectrum of T is contained in the closed ball of radius $r(T)$; that is, $\sigma(T) \subset \overline{B}_{r(T)} = \{\lambda \in \mathbb{C} : |\lambda| \le r(T)\}$.*

Now, we are ready to formulate the lemma on solvability of the linear integral equation (35).

Lemma 10. *For any $x_0 \in H_0^1$, $\rho > 0$ such that $\|x_0\|_\infty \le \rho$ and any $y \in H_0^1$,* (35) *possesses a unique solution in H_0^1 provided that the functions G and h satisfy (A1), (A2), (A3), and weaker, local version of (A4), with constant l_ρ such that*

$$|\lambda| > 2 l_\rho \ge \sup_{t\in[-1,1],|x(t)|<\rho} \int_{-1}^{1} \left|G_t(t,\tau)\, h_x\left(\tau, x(\tau)\right)\right| d\tau. \tag{40}$$

Proof. Our proof starts with observation that (35) can be written in the form

$$\lambda x + T x = y, \tag{41}$$

where

$$(Tx)(t) = \int_{-1}^{1} G(t,\tau)\, h_x\left(\tau, x_0(\tau)\right) x(\tau)\, d\tau. \tag{42}$$

From (29), (30), and (31) it follows that for any $k = 1,2,\ldots$, $t \in [-1,1]$ and $x \in H_0^1$ we have

$$\begin{aligned}
\left(T^{k+1} x\right)(t) &= \int_{-1}^{1} G(t,\tau)\, h_x\left(\tau, x_0(\tau)\right) \left(T^k x\right)(\tau)\, d\tau, \\
\left(T^0 x\right)(t) &= x(t).
\end{aligned} \tag{43}$$

By (A1), (A2), (A3), and inequality (32) from Lemma 8, the analysis similar to that in the proof of Lemma 6 leads, if we apply induction, to the fact that $T^k x \in H_0^1$ for any $x \in H_0^1$ and $k = 1, 2, \ldots$. By (15) from Lemma 4 and (32) from Lemma 8, we get, with $M = \|x\|_\infty$, the following estimate:

$$\begin{aligned}\left\|T^k x\right\|_{H_0^1}^2 &= \int_{-1}^{1} \left|\frac{d}{dt}\left(T^k x\right)(t)\right|^2 dt \\ &= \int_{-1}^{1} \left|\frac{d}{dt}\left[\int_{-1}^{1} G(t,\tau) h_x(\tau, x_0(\tau)) \times \left(T^{k-1}x\right)(\tau)\, d\tau\right]\right|^2 dt \\ &= \int_{-1}^{1} \left|\int_{-1}^{1} G_t(t,\tau) h_x(\tau, x_0(\tau)) \times \left(T^{k-1}x\right)(\tau)\, d\tau\right|^2 dt \\ &\le \int_{-1}^{1} \left[2l_\rho 2^{k-1} l_\rho^{k-1} M\right]^2 dt \\ &= 2^{2k+1} l_\rho^{2k} M^2 \le 2^{2k+2} l_\rho^{2k} \|x\|_{H_0^1}^2,\end{aligned} \tag{44}$$

and hence by an arbitrary choice of $x \in H_0^1$ we get

$$\left\|T^k\right\|^{1/k} \le 2^{1+(1/k)} l_\rho. \tag{45}$$

Consequently,

$$r(T) = \limsup_{k\to\infty} \left\|T^k\right\|^{1/k} \le 2l_\rho \tag{46}$$

which means that the spectral radius $r(T)$ is less or equal to $2l_\rho$. Since, by Theorem 9, $\sigma(T) \subset \overline{B}_{r(T)}$ with $r(T)$ is defined by (46). Then, in particular, for all $\lambda \in \mathbb{R}$ such that $|\lambda| > r(T)$ we have $\lambda \in \rho(T)$. Therefore, we can conclude that, for all $\lambda \in \mathbb{R}$ and $|\lambda| > 2l_\rho$, the operator $T - \lambda I$ is bijective on H_0^1. Thus, for any $|\lambda| > 2l_\rho$, $\|x_0\|_\infty \le \rho$, and $y \in H_0^1$, there exists a unique solution $x \in H_0^1$ to

$$(T + \lambda I)\, x = y, \tag{47}$$

which ends the proof. Indeed, by definition of T, there exists a unique solution to

$$\lambda x(t) + \int_{-1}^{1} G(t,\tau) h_x(\tau, x_0(\tau))\, x(\tau)\, d\tau = y(t). \tag{48}$$

□

5. Palais-Smale Condition Guaranteeing Global Diffeomorphism

Let us consider, for an arbitrary function $y \in H_0^1$, the functional $\Psi_y : H_0^1 \to \mathbb{R}^+$ of the form

$$\begin{aligned}\Psi_y(x) &= \frac{1}{2}\left\|\mathcal{T}x - y\right\|_{H_0^1}^2 \\ &= \frac{1}{2}\int_{-1}^{1} \left|\frac{d}{dt}(\mathcal{T}x(t)) - y'(t)\right|^2 dt \\ &= \frac{1}{2}\int_{-1}^{1} \left|\lambda x'(t) + \int_{-1}^{1} G_t(t,\tau) h(\tau, x(\tau))\, d\tau - y'(t)\right|^2 dt.\end{aligned} \tag{49}$$

To prove the main results of the paper we will need some sufficient conditions under which for any $y \in H_0^1$ the functional Ψ_y is coercive; that is, for any $y \in H_0^1$, $\Psi_y(x) \to \infty$ provided that $\|x\|_{H_0^1} \to \infty$.

Lemma 11. *If the functions G and h satisfy (A1)(a), (A1)(b), (A2)(a), (A2)(b), (A3), and (A5), then for any $y \in H_0^1$ the functional Ψ_y is coercive.*

Proof. Since the functional Ψ_y is coercive for any $y \in H_0^1$ if and only if the functional Ψ_y is coercive for $y = 0$, we first observe that the functional Ψ_0 is bounded from below. By the Schwarz inequality and the assumptions of this lemma together with the last estimate from Lemma 4, we obtain

$$\begin{aligned}\Psi_0(x) &= \frac{1}{2}\int_{-1}^{1} \left|\lambda x'(t) + \int_{-1}^{1} G_t(t,\tau) h(\tau, x(\tau))\, d\tau\right|^2 dt \\ &\ge \|x\|_{H_0^1}^2 \left(\frac{1}{2}|\lambda|^2 - \sqrt{2}\,|\lambda|\, \|G_t a\|_{L^2(P,\mathbb{R})}\right) \\ &\quad - \|x\|_{H_0^1} \sqrt{2}\,|\lambda|\, \|G_t b\|_{L^2(P,\mathbb{R})}.\end{aligned} \tag{50}$$

From (A5)(b) and the above estimate it follows that $\Psi_0(x) \to \infty$ if $\|x\|_{H_0^1} \to \infty$. Consequently, for any $y \in H_0^1$ we have $\Psi_y(x) \to \infty$ as $\|x\|_{H_0^1} \to \infty$. □

Lemma 12. *For any $y \in H_0^1$ the functional Ψ_y satisfies Palais-Smale condition provided that assumptions (A1), (A2), (A3), (A4), and (A5) are satisfied.*

Proof. Fix $y \in H_0^1$. Recall that the functional Ψ_y has the form

$$\Psi_y(x) = \frac{1}{2}\int_{-1}^{1} \left|\lambda x'(t) + \int_{-1}^{1} G_t(t,\tau) h(\tau, x(\tau))\, d\tau - y'(t)\right|^2 dt. \tag{51}$$

Straightforward calculation leads to

$$\begin{aligned}\Psi_y(x) = \frac{1}{2}\int_{-1}^{1} \Bigg(&|\lambda|^2 \left|x'(t)\right|^2 \\ &+ 2\left\langle \lambda x'(t), \int_{-1}^{1} G_t(t,\tau) h(\tau, x(\tau))\, d\tau \right\rangle \\ &+ \left|\int_{-1}^{1} G_t(t,\tau) h(\tau, x(\tau))\, d\tau\right|^2 \\ &- 2\left\langle \lambda x'(t), y'(t)\right\rangle\end{aligned}$$

$$-2\left\langle\int_{-1}^{1}G_t(t,\tau)h(\tau,x(\tau))\,d\tau,y'(t)\right\rangle$$
$$+\left|y'(t)\right|^2\Big)\,dt. \quad (52)$$

The functional Ψ_y defined by (49), being a superposition of two C^1-mappings, is also of the same regularity C^1 type and its differential $\Psi_y'(x)$ at $x\in H_0^1$ is given, for $v\in H_0^1$, by

$$\begin{aligned}\Psi_y'(x)v=\int_{-1}^{1}\Big[&|\lambda|^2\left\langle x'(t),v'(t)\right\rangle\\&+\left\langle\lambda v'(t),\int_{-1}^{1}G_t(t,\tau)h(\tau,x(\tau))\,d\tau\right\rangle\\&+\left\langle\lambda x'(t),\int_{-1}^{1}G_t(t,\tau)h_x(\tau,x(\tau))v(\tau)\,d\tau\right\rangle\\&+\left\langle\int_{-1}^{1}G_t(t,\tau)h(\tau,x(\tau))\,d\tau,\right.\\&\qquad\left.\int_{-1}^{1}G_t(t,\tau)h_x(\tau,x(\tau))v(\tau)\,d\tau\right\rangle\\&-\left\langle\lambda v'(t),y'(t)\right\rangle-\left\langle\int_{-1}^{1}G_t(t,\tau)h_x(\tau,x(\tau))\right.\\&\qquad\left.\times v(\tau)\,d\tau,y'(t)\right\rangle\Big]\,dt.\end{aligned}\quad (53)$$

Let $\{x_k\}\subset H_0^1$ be a Palais-Smale sequence for some fixed but an arbitrary $M\geq 0$; that is, $|\Psi_y(x_k)|\leq M$ and $\Psi_y(x_k)\to 0$. Applying Lemma 11 we obtain that Ψ_y is coercive, and hence the sequence $\{x_k\}$ is weakly compact as a bounded sequence in a reflexive space. Passing, if necessary, to a subsequence, one can assume that $x_k\rightharpoonup x_0$ weakly in H_0^1. Moreover, the weak convergence of the sequence $\{x_k\}$ in the space H_0^1 implies the uniform convergence in C; that is, $x_k(t)\rightrightarrows x_0(t)$ uniformly with respect to $t\in[-1,1]$ as well as the weak convergence of its derivatives in L^2; that is, $x_k'\to x_0'$ in L^2 and as being a weakly convergent sequence it has to be bounded. It remains to prove that the sequence $\{x_k\}$ converges to x_0 in H_0^1. By (53), a direct calculation leads to

$$\left\langle\Psi_y'(x_k)-\Psi_y'(x_0),x_k-x_0\right\rangle=|\lambda|^2\|x_k-x_0\|_{H_0^1}^2+\sum_{i=1}^{6}G^i(x_k),\quad (54)$$

where

$$\begin{aligned}G^1(x_k)=\int_{-1}^{1}\Big\langle&\lambda\left(x_k'(t)-x_0'(t)\right),\\&\int_{-1}^{1}\left[G_t(t,\tau)h(\tau,x_k(\tau))\right.\\&\qquad\left.-G_t(t,\tau)h(\tau,x_0(\tau))\right]d\tau\Big\rangle\,dt,\end{aligned}$$

$$\begin{aligned}G^2(x_k)=\int_{-1}^{1}\Big\langle&\lambda x_k'(t),\int_{-1}^{1}G_t(t,\tau)h_x(\tau,x_k(\tau))\\&\times\left(x_k(\tau)-x_0(\tau)\right)d\tau\\&\times\int_{-1}^{1}G_t(t,\tau)h_x(\tau,x_k(\tau))\Big\rangle\,dt,\end{aligned}$$

$$\begin{aligned}G^3(x_k)=\int_{-1}^{1}\Big\langle&\int_{-1}^{1}G_t(t,\tau)h(\tau,x_k(\tau))\,d\tau,\\&\int_{-1}^{1}G_t(t,\tau)h_x(\tau,x_k(\tau))\\&\times\left(x_k(\tau)-x_0(\tau)\right)d\tau\Big\rangle\,dt,\end{aligned}$$

$$\begin{aligned}G^4(x_k)=-\int_{-1}^{1}\Big\langle&\int_{-1}^{1}\left(G_t(t,\tau)h_x(\tau,x_k(\tau))\right.\\&\left.-G_t(t,\tau)h_x(\tau,x_0(\tau))\right)\\&\times\left(x_k(\tau)-x_0(\tau)\right)d\tau,y'(t)\Big\rangle\,dt,\end{aligned}$$

$$\begin{aligned}G^5(x_k)=-\int_{-1}^{1}\Big\langle&\lambda x_0'(t),\int_{-1}^{1}G_t(t,\tau)h_x(\tau,x_0(\tau))\\&\times\left(x_k(\tau)-x_0(\tau)\right)d\tau\Big\rangle\,dt,\end{aligned}$$

$$\begin{aligned}G^6(x_k)=-\int_{-1}^{1}\Big\langle&\int_{-1}^{1}G_t(t,\tau)h(\tau,x_0(\tau))\,d\tau,\\&\int_{-1}^{1}G_t(t,\tau)h_x(\tau,x_0(\tau))\\&\times\left(x_k(\tau)-x_0(\tau)\right)d\tau\Big\rangle\,dt.\end{aligned}\quad (55)$$

Since $\Psi_y'(z_k)\to 0$ and $x_k\rightharpoonup x_0$ weakly in H_0^1, $\lim_{k\to\infty}\langle\Psi_y'(x_k)-\Psi_y'(x_0),x_k-x_0\rangle=0$. We will prove that, for $i=1,2,\ldots,6$, $\lim_{k\to\infty}G^i(x_k)=0$. By the Schwarz inequality, (A1)(a), and (A2)(b) we get

$$\begin{aligned}\left|G^1(x_k)\right|^2\leq|\lambda|^2\int_{-1}^{1}&\left|x_k'(t)-x_0'(t)\right|^2dt\\\times\int_{-1}^{1}\Big[\int_{-1}^{1}&\left|G_t(t,\tau)h(\tau,x_k(\tau))\right.\\&\left.-G_t(t,\tau)h(\tau,x_0(\tau))\right|d\tau\Big]^2dt.\end{aligned}\quad (56)$$

The first factor above is bounded, whereas the second one, by (A4), is convergent to zero, and therefore, $G^1(x_k)\to 0$ as $k\to\infty$. Next, $G^2(x_k)$ can be estimated by $\varepsilon\lambda^2\sqrt{2}\parallel {x'}_k\|_{L^2}$ if $\parallel x_k-x_0\|_\infty\leq\varepsilon$. Similar estimates can be applied to other terms; thus, one can prove that $G^i(x_k) \to 0$ as $k \to \infty$ for $i = 3,4,5,6$. Hence, from (54), it follows that $x_k \to x_0$ in H_0^1. □

6. Main Results and Applications

Applying formerly presented lemmas and Theorem 2 we prove the main result of this paper.

Theorem 13. *If the functions G and h satisfy assumptions (A1), (A2), (A3), (A4), and (A5), then the nonlinear Hammerstein operator $\mathcal{T} : H_0^1 \to H_0^1$ defined by (9) is a diffeomorphism of H_0^1 on H_0^1.*

Proof. Set $X = H = H_0^1$. From Lemma 10 we infer that the operator $\mathcal{T}$ satisfies assumption (a1) of Theorem 2, while Lemma 12 ascertains that for any $y \in H_0^1$ the functional $\Psi_y(x) = 1/2\|\mathcal{T}x - y\|_{H_0^1}^2$ satisfies Palais-Smale condition so that assumption (b1) of Theorem 2 is fulfilled. Therefore, $\mathcal{T} : H_0^1 \to H_0^1$ defined by (9) is a diffeomorphism. □

Theorem 13 can be formulated in the following equivalent version focusing on the solvability, uniqueness, and continuous dependence issues, following from the diffeomorphism property.

Theorem 14. *If the functions G and h satisfy assumptions of Theorem 13, then for any $z \in H_0^1$ the nonlinear integral equation*

$$\lambda x(t) + \int_{-1}^{1} G(t,\tau) h(\tau, x(\tau))\, d\tau = z(t), \quad t \in [-1,1], \tag{57}$$

possesses a unique solution $x = x_z \in H_0^1$ and moreover the solution operator

$$H_0^1 \ni z \longrightarrow x_z \in H_0^1 \tag{58}$$

is continuously Fréchet differentiable.

Next, we will present the application of our general theorem to the equation involving the fractional Laplacian operator for $n = 1$.

Example 15. Assume that the nonlinear term h satisfy the Green function G estimates (A1)–(A5). This is the case if, for example, the function h is smooth, that is, C^1, and it satisfies the linear growth conditions (A4)-(A5). Then for any $z \in H_0^1$ and $\sigma \in (1,2]$ there exists a unique solution $x \in H_0^1$ of

$$\begin{aligned}\lambda(-\Delta)^{\sigma/2} x(t) + h(t, x(t)) \\ = (-\Delta)^{\sigma/2} z(t), \quad t \in (-1,1).\end{aligned} \tag{59}$$

By [2, Corollary 3.2] we have for the Green function of $(-\Delta)^{\sigma/2}$ the following estimates:

$$\begin{aligned}c_\alpha \left(\frac{\delta^{\sigma/2}(t)\,\delta^{\sigma/2}(\tau)}{|t-\tau|} \wedge \delta^{(\sigma-1)/2}(t)\,\delta^{(\sigma-1)/2}(\tau) \right) \\ \le G(t,\tau) \le C_\alpha \left(\frac{\delta^{\sigma/2}(t)\,\delta^{\sigma/2}(\tau)}{|t-\tau|} \wedge \delta^{(\sigma-1)/2} \right. \\ \left. \times (t)\,\delta^{(\sigma-1)/2}(\tau) \right),\end{aligned} \tag{60}$$

where $\delta(\tau) = \operatorname{dist}(\tau, \{-1,1\})$, $a \wedge b = \min(a,b)$. It should be noted (cf. [2]) that the Green function for $\sigma \in (1,2]$ is bounded and continuous. For estimates on G_t and regularity see [2, 9, 10]. One can recall or show directly that continuous G behaves like $(\cdot)^{\sigma/2}$ and G_t behaves like $(\cdot)^{\sigma/2-1}$ at the boundary, that is, at -1 or 1, while G is like $(\cdot)^{\sigma-1}$ and G_t is like $(\cdot)^{\sigma-1}$ at $t = \tau$. Therefore, the integrability assumptions are satisfied for mild singularity; that is, only if $\sigma \in (1,2]$ but not in the range of stronger singularity when $\sigma \in (0,1]$.

Finally, we will present the application of the main theorem to some specific nonlinear integral Hammerstein operator this time with smooth kernel.

Example 16. Let us consider the following operator:

$$\begin{aligned}\mathcal{T}x(t) = \lambda x(t) \\ + \int_{-1}^{1} G(t,\tau) \ln\left(1 + B(\tau) x^2(\tau)\right) d\tau, \\ t \in [-1,1],\end{aligned} \tag{61}$$

with functions $B \in C^1([-1,1], \mathbb{R})$ satisfying $B(\tau) > 0$ on $[-1,1]$ and $G \in C^1(P, \mathbb{R})$ with $P = [-1,1]^2$ such that $G(-1,\tau) = G(1,\tau) = 0$ for $\tau \in [-1,1]$.

Since $\ln(1+z^2) \le |z|$, for the function

$$h(\tau, x) = \ln\left(1 + B(\tau) x^2\right) \tag{62}$$

the following estimate holds:

$$|h(\tau, x)| \le \sqrt{B(\tau)}\,|x|. \tag{63}$$

Similarly, since $1 + z^2 \le 2z$ we have the estimate for

$$h_x(\tau, x) = \frac{2B(\tau)\,x}{1 + B(\tau)\,x^2} \tag{64}$$

reading

$$\left|h_x(\tau, x)\right| \le \sqrt{B(\tau)}. \tag{65}$$

Let us define $a(\tau) = \sqrt{B(\tau)}$ and $b(\tau) \equiv 0$. Then $a, b \in L^2([-1,1], \mathbb{R}^+)$ and condition (A5)(a) is fulfilled. Assuming

$$\begin{aligned}\left\|G_t(t,\tau)\,a(\tau)\right\|_{L^2(P,\mathbb{R})} = \left\|G_t(t,\tau)\sqrt{B(\tau)}\right\|_{L^2(P,\mathbb{R})} \\ < \frac{\sqrt{2}}{4}\,|\lambda|,\end{aligned} \tag{66}$$

we can guarantee that assumption (A5)(b) is satisfied.

Consequently, if we assume that

$$\int_{-1}^{1} \left| G_t(t,\tau) \right| \sqrt{B(\tau)} d\tau < |\lambda| , \qquad (67)$$

then condition (A4) holds. Thus, the functions G and h satisfy assumptions (A1)–(A5) and Theorem 14 implies that the equation, for any $z \in H_0^1$,

$$\lambda x(t) + \int_{-1}^{1} G(t,\tau) \ln\left(1 + B(\tau) x^2(\tau)\right) d\tau = z(t) \quad \text{for } t \in [-1,1] \qquad (68)$$

possesses a unique solution $x = x_z \in H_0^1$ and $H_0^1 \ni z \to x_z \in H_0^1$ is continuously Fréchet differentiable.

7. Summary

We have considered the nonlinear integral operator of Hammerstein type $\mathcal{T}$ defined on the Sobolev space H_0^1 with some application to the nonlocal Dirichlet BVP involving the fractional Laplacian. The key point in the proof of the main result of this paper is the application of the theorem on global diffeomorphism. In particular, we have shown that that the assumptions (A1), (A2), (A3), (A4), and (A5) imply some sufficient conditions for the operator $\mathcal{T} : H_0^1 \to H_0^1$ defined by (9) to be a diffeomorphism, compare Theorem 13. Equivalently, we have obtained the existence and uniqueness result for the nonlinear Hammerstein equation (57) and the differentiable dependence of the solution on parameters as well, see Theorem 14. Thus, in other words, our problem is well-posed and robust, compare [27]. It should be emphasized that in the proof of Lemma 12 we have used the compactness of the embedding of the space H_0^1 into the space C and the reflexivity of H_0^1 and these properties are crucial in the method of the proof applied therein. Finally, in Section 6 we have proposed some examples of the nonlinear Hammerstein operators for which Theorems 13 and 14 are applicable, including the one originating from the BVP involving the fractional Laplacian.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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