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Research Article

Left and Right Inverse Eigenpairs Problem for κ-Hermitian Matrices

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Left and right inverse eigenpairs problem for κ-hermitian matrices and its optimal approximate problem are considered. Based on the special properties of κ-hermitian matrices, the equivalent problem is obtained. Combining a new inner product of matrices, the necessary and sufficient conditions for the solvability of the problem and its general solutions are derived. Furthermore, the optimal approximate solution and a calculation procedure to obtain the optimal approximate solution are provided.

1. Introduction

Throughout this paper we use some notations as follows. Let $C^{n\times m}$ be the set of all $n\times m$ complex matrices, $UC^{n\times n}$, $HC^{n\times n}$, $SHC^{n\times n}$ denote the set of all $n\times n$ unitary matrices, hermitian matrices, skew-hermitian matrices, respectively. Let $\overline{A}$, A^H, and A^+ be the conjugate, conjugate transpose, and the Moore-Penrose generalized inverse of A, respectively. For $A, B \in C^{n\times m}$, $\langle A, B\rangle = \operatorname{re}(\operatorname{tr}(B^H A))$, where $\operatorname{re}(\operatorname{tr}(B^H A))$ denotes the real part of $\operatorname{tr}(B^H A)$, the inner product of matrices A and B. The induced matrix norm is called Frobenius norm. That is, $\|A\| = \langle A, A\rangle^{1/2} = (\operatorname{tr}(A^H A))^{1/2}$.

Left and right inverse eigenpairs problem is a special inverse eigenvalue problem. That is, giving partial left and right eigenpairs (eigenvalue and corresponding eigenvector), (λ_i, x_i), $i = 1, \ldots, h$; (μ_j, y_j), $j = 1, \ldots, l$, a special matrix set S, finding a matrix $A \in S$ such that

$$A x_i = \lambda_i x_i, \quad i = 1, \ldots, h,$$
$$y_j^T A = \mu_j y_j^T, \quad j = 1, \ldots, l. \tag{1}$$

This problem, which usually arises in perturbation analysis of matrix eigenvalues and in recursive matters, has profound application background [1–6]. When the matrix set S is different, it is easy to obtain different left and right inverse eigenpairs problem. For example, we studied the left and right inverse eigenpairs problem of skew-centrosymmetric matrices and generalized centrosymmetric matrices, respectively [5, 6]. Based on the special properties of left and right eigenpairs of these matrices, we derived the solvability conditions of the problem and its general solutions. In this paper, combining the special properties of κ-hermitian matrices and a new inner product of matrices, we first obtain the equivalent problem, then derive the necessary and sufficient conditions for the solvability of the problem and its general solutions.

Hill and Waters [7] introduced the following matrices.

Definition 1. Let κ be a fixed product of disjoint transpositions, and let K be the associated permutation matrix, that is, $\overline{K} = K^H = K$, $K^2 = I_n$, a matrix $A \in C^{n\times n}$ is said to be κ-hermitian matrices (skew κ-hermitian matrices) if and only if $a_{ij} = \overline{a}_{k(j)k(i)}$ $(a_{ij} = -\overline{a}_{k(j)k(i)})$, $i, j = 1, \ldots, n$. We denote the set of κ-hermitian matrices (skew κ-hermitian matrices) by $KHC^{n\times n}$ $(SKHC^{n\times n})$.

From Definition 1, it is easy to see that hermitian matrices and perhermitian matrices are special cases of κ-hermitian matrices, with $k(i) = i$ and $k(i) = n - i + 1$, respectively. Hermitian matrices and perhermitian matrices, which are one of twelve symmetry patterns of matrices [8], are applied in engineering, statistics, and so on [9, 10].

From Definition 1, it is also easy to prove the following conclusions.

(1) $A \in KHC^{n\times n}$ if and only if $A = KA^H K$.

(2) $A \in SKHC^{n\times n}$ if and only if $A = -KA^H K$.

(3) If K is a fixed permutation matrix, then $KHC^{n\times n}$ and $SKHC^{n\times n}$ are the closed linear subspaces of $C^{n\times n}$ and satisfy

$$C^{n\times n} = KHC^{n\times n} \bigoplus SKHC^{n\times n}. \tag{2}$$

The notation $V_1 \oplus V_2$ stands for the orthogonal direct sum of linear subspace V_1 and V_2.

(4) $A \in KHC^{n\times n}$ if and only if there is a matrix $\widetilde{A} \in HC^{n\times n}$ such that $\widetilde{A} = KA$.

(5) $A \in SKHC^{n\times n}$ if and only if there is a matrix $\widetilde{A} \in SHC^{n\times n}$ such that $\widetilde{A} = KA$.

Proof. (1) From Definition 1, if $A = (a_{ij}) \in KHC^{n\times n}$, then $a_{ij} = \overline{a}_{k(j)k(i)}$, this implies $A = KA^H K$, for $KA^H K = (\overline{a}_{k(j)k(i)})$.

(2) With the same method, we can prove (2). So, the proof is omitted.

(3) (a) For any $A \in C^{n\times n}$, there exist $A_1 \in KHC^{n\times n}$, $A_2 \in SKHC^{n\times n}$ such that

$$A = A_1 + A_2, \tag{3}$$

where $A_1 = (1/2)(A + KA^H K)$, $A_2 = (1/2)(A - KA^H K)$.

(b) If there exist another $\overline{A}_1 \in KHC^{n\times n}$, $\overline{A}_2 \in SKHC^{n\times n}$ such that

$$A = \overline{A}_1 + \overline{A}_2, \tag{4}$$

(3)-(4) yields

$$A_1 - \overline{A}_1 = -\left(A_2 - \overline{A}_2\right). \tag{5}$$

Multiplying (5) on the left and on the right by K, respectively, and according to (1) and (2), we obtain

$$A_1 - \overline{A}_1 = A_2 - \overline{A}_2. \tag{6}$$

Combining (5) and (6) gives $A_1 = \overline{A}_1$, $A_2 = \overline{A}_2$.

(c) For any $A_1 \in KHC^{n\times n}$, $A_2 \in SKHC^{n\times n}$, we have

$$\begin{aligned}\langle A_1, A_2\rangle &= \operatorname{re}\left(\operatorname{tr}\left(A_2^H A_1\right)\right) = \operatorname{re}\left(\operatorname{tr}\left(KA_2^H KKA_1 K\right)\right)\\ &= \operatorname{re}\left(\operatorname{tr}\left(-A_2^H A_1\right)\right) = -\langle A_1, A_2\rangle.\end{aligned} \tag{7}$$

This implies $\langle A_1, A_2\rangle = 0$. Combining (a), (b), and (c) gives (3).

(4) Let $\widetilde{A} = KA$, if $A \in KHC^{n\times n}$, then $\widetilde{A}^H = \widetilde{A} \in HC^{n\times n}$. If $\widetilde{A}^H = \widetilde{A} \in HC^{n\times n}$, then $A = K\widetilde{A}$ and $KA^H K = K\widetilde{A}^H KK = K\widetilde{A} = A \in KHC^{n\times n}$.

(5) With the same method, we can prove (5). So, the proof is omitted. □

In this paper, we suppose that K is a fixed permutation matrix and assume (λ_i, x_i), $i = 1, \ldots, h$, be right eigenpairs of A; (μ_j, y_j), $j = 1, \ldots, l$, be left eigenpairs of A. If we let $X = (x_1, \ldots, x_h) \in C^{n\times h}$, $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_h) \in C^{h\times h}$; $Y = (y_1, \ldots, y_l) \in C^{n\times l}$, $\Gamma = \operatorname{diag}(\mu_1, \ldots, \mu_l) \in C^{l\times l}$, then the problems studied in this paper can be described as follows.

Problem 2. Giving $X \in C^{n\times h}$, $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_h) \in C^{h\times h}$; $Y \in C^{n\times l}$, $\Gamma = \operatorname{diag}(\mu_1, \ldots, \mu_l) \in C^{l\times l}$, find $A \in KHC^{n\times n}$ such that

$$\begin{aligned} AX &= X\Lambda,\\ Y^T A &= \Gamma Y^T. \end{aligned} \tag{8}$$

Problem 3. Giving $B \in C^{n\times n}$, find $\widehat{A} \in S_E$ such that

$$\left\|B - \widehat{A}\right\| = \min_{\forall A \in S_E} \|B - A\|, \tag{9}$$

where S_E is the solution set of Problem 2.

This paper is organized as follows. In Section 2, we first obtain the equivalent problem with the properties of $KHC^{n\times n}$ and then derive the solvability conditions of Problem 2 and its general solution's expression. In Section 3, we first attest the existence and uniqueness theorem of Problem 3 then present the unique approximation solution. Finally, we provide a calculation procedure to compute the unique approximation solution and numerical experiment to illustrate the results obtained in this paper correction.

2. Solvability Conditions of Problem 2

We first discuss the properties of $KHC^{n\times n}$

Lemma 4. *Denoting $M = KEKGE$, and $E \in HC^{n\times n}$, one has the following conclusions.*

(1) *If $G \in KHC^{n\times n}$, then $M \in KHC^{n\times n}$.*

(2) *If $G \in SKHC^{n\times n}$, then $M \in SKHC^{n\times n}$.*

(3) *If $G = G_1 + G_2$, where $G_1 \in KHC^{n\times n}$, $G_2 \in SKHC^{n\times n}$, then $M \in KHC^{n\times n}$ if and only if $KEKG_2E = 0$. In addition, one has $M = KEKG_1E$.*

Proof. (1) $KM^H K = KEG^H KEKK = KE(KGK)KE = KEKGE = M$.

Hence, we have $M \in KHC^{n\times n}$.

(2) $KM^H K = KEG^H KEKK = KE(-KGK)KE = -KEKGE = -M$.

Hence, we have $M \in SKHC^{n\times n}$.

(3) $M = KEK(G_1 + G_2)E = KEKG_1E + KEKG_2E$, we have $KEKG_1E \in KHC^{n\times n}$, $KEKG_2E \in SKHC^{n\times n}$ from (1) and (2). If $M \in KHC^{n\times n}$, then $M - KEKG_1E \in KHC^{n\times n}$, while $M - KEKG_1E = KEKG_2E \in SKHC^{n\times n}$. Therefore from the conclusion (3) of Definition 1, we have $KEKG_2E = 0$, that is, $M = KEKG_1E$. On the contrary, if $KEKG_2E = 0$, it is clear that $M = KEKG_1E \in KHC^{n\times n}$. The proof is completed. □

Lemma 5. *Let $A \in KHC^{n\times n}$, if (λ, x) is a right eigenpair of A, then $(\overline{\lambda}, K\overline{x})$ is a left eigenpair of A.*

Proof. If (λ, x) is a right eigenpair of A, then we have

$$Ax = \lambda x. \tag{10}$$

From the conclusion (1) of Definition 1, it follows that

$$KA^H Kx = \lambda x. \tag{11}$$

This implies

$$(K\overline{x})^T A = \overline{\lambda}(K\overline{x})^T. \tag{12}$$

So $(\overline{\lambda}, K\overline{x})$ is a left eigenpair of A. □

From Lemma 5, without loss of the generality, we may assume that Problem 2 is as follows.

$$X \in C^{n\times h}, \qquad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_h) \in C^{h\times h}, \qquad Y = K\overline{X} \in C^{n\times h}, \qquad \Gamma = \overline{\Lambda} \in C^{h\times h}. \tag{13}$$

Combining (13) and the conclusion (4) of Definition 1, it is easy to derive the following lemma.

Lemma 6. *If X, Λ, Y, Γ are given by (13), then Problem 2 is equivalent to the following problem. If X, Λ, Y, Γ are given by (13), find $KA \in HC^{n\times n}$ such that*

$$KAX = KX\Lambda. \tag{14}$$

Lemma 7 (see [11]). *If giving $X \in C^{n\times h}$, $B \in C^{n\times h}$, then matrix equation $\widetilde{A}X = B$ has solution $\widetilde{A} \in HC^{n\times n}$ if and only if*

$$B = BX^+X, \qquad B^H X = X^H B. \tag{15}$$

Moreover, the general solution $\widetilde{A}$ can be expressed as

$$\widetilde{A} = BX^+ + (BX^+)^H (I_n - XX^+) + (I_n - XX^+)\widetilde{G}(I_n - XX^+), \quad \forall \widetilde{G} \in HC^{n\times n}. \tag{16}$$

Theorem 8. *If X, Λ, Y, Γ are given by (13), then Problem 2 has a solution in $KHC^{n\times n}$ if and only if*

$$X^H KX\Lambda = \overline{\Lambda}X^H KX, \quad X\Lambda = X\Lambda X^+ X. \tag{17}$$

Moreover, the general solution can be expressed as

$$A = A_0 + KEKGE, \quad \forall G \in KHC^{n\times n}, \tag{18}$$

where

$$A_0 = X\Lambda X^+ + K(X\Lambda X^+)^H KE, \qquad E = I_n - XX^+. \tag{19}$$

Proof. Necessity: If there is a matrix $A \in KHC^{n\times n}$ such that $(AX = X\Lambda, Y^T A = \Gamma Y^T)$, then from Lemma 6, there exists a matrix $KA \in HC^{n\times n}$ such that $KAX = KX\Lambda$, and according to Lemma 7, we have

$$KX\Lambda = KX\Lambda X^+ X, \qquad (KX\Lambda)^H X = X^H (KX\Lambda). \tag{20}$$

It is easy to see that (20) is equivalent to (17).

Sufficiency: If (17) holds, then (20) holds. Hence, matrix equation $KAX = KX\Lambda$ has solution $KA \in HC^{n\times n}$. Moreover, the general solution can be expressed as follows:

$$KA = KX\Lambda X^+ + (KX\Lambda X^+)^H (I_n - XX^+) + (I_n - XX^+)\widetilde{G}(I_n - XX^+), \quad \forall \widetilde{G} \in HC^{n\times n}. \tag{21}$$

Let

$$A_0 = X\Lambda X^+ + K(X\Lambda X^+)^H KE, \qquad E = I_n - XX^+. \tag{22}$$

This implies $A = A_0 + KE\widetilde{G}E$. Combining the definition of K, E and the first equation of (17), we have

$$\begin{aligned} KA_0^H K &= K(X\Lambda X^+)^H K + X\Lambda X^+ - K(XX^+)^H K (X\Lambda X^+) \\ &= X\Lambda X^+ + K(X\Lambda X^+)^H K - K(X\Lambda X^+)^T KXX^+ \\ &= A_0. \end{aligned} \tag{23}$$

Hence, $A_0 \in KHC^{n\times n}$. Combining the definition of K, E, (13) and (17), we have

$$\begin{aligned} A_0 X &= X\Lambda X^+ X + K(X\Lambda X^+)^H K (I_n - XX^+) X = X\Lambda, \\ Y^T A_0 &= X^H KX\Lambda X^+ + X^H KK(X\Lambda X^+)^H KE \\ &= \overline{\Lambda}X^H KXX^+ + (KX\Lambda X^+ X)^H (I_n - XX^+) \\ &= \overline{\Lambda}X^H KXX^+ + \overline{\Lambda}X^H K (I_n - XX^+) \\ &= \overline{\Lambda}X^H K = \Gamma Y^T. \end{aligned} \tag{24}$$

Therefore, A_0 is a special solution of Problem 2. Combining the conclusion (4) of Definition 1, Lemma 4, and $E = I_n - XX^+ \in HC^{n\times n}$, it is easy to prove that $A = A_0 + KEKGE \in KHC^{n\times n}$ if and only if $G \in KHC^{n\times n}$. Hence, the solution set of Problem 2 can be expressed as (18). □

3. An Expression of the Solution of Problem 3

From (18), it is easy to prove that the solution set S_E of Problem 2 is a nonempty closed convex set if Problem 2 has a solution in $KHC^{n\times n}$. We claim that for any given $B \in R^{n\times n}$, there exists a unique optimal approximation for Problem 3.

Theorem 9. *Giving $B \in C^{n\times n}$, if the conditions of X, Y, Λ, Γ are the same as those in Theorem 8, then Problem 3 has a unique solution $\widehat{A} \in S_E$. Moreover, $\widehat{A}$ can be expressed as*

$$\widehat{A} = A_0 + KEKB_1E, \tag{25}$$

where A_0, E are given by (19) and $B_1 = (1/2)(B + KB^H K)$.

Proof. Denoting $E_1 = I_n - E$, it is easy to prove that matrices E and E_1 are orthogonal projection matrices satisfying $EE_1 = 0$. It is clear that matrices KEK and KE_1K are also orthogonal projection matrices satisfying $(KEK)(KE_1K) = 0$. According to the conclusion (3) of Definition 1, for any $B \in C^{n\times n}$, there exists unique

$$B_1 \in KHC^{n\times n}, \qquad B_2 \in SKHC^{n\times n} \tag{26}$$

such that

$$B = B_1 + B_2, \quad \langle B_1, B_2 \rangle = 0, \tag{27}$$

where

$$B_1 = \frac{1}{2}\left(B + KB^H K\right), \qquad B_2 = \frac{1}{2}\left(B - KB^H K\right). \tag{28}$$

Combining Theorem 8, for any $A \in S_E$, we have

$$\begin{aligned}
&\|B - A\|^2 \\
&\quad = \|B - A_0 - KEKGE\|^2 \\
&\quad = \|B_1 + B_2 - A_0 - KEKGE\|^2 \\
&\quad = \|B_1 - A_0 - KEKGE\|^2 + \|B_2\|^2 \\
&\quad = \|(B_1 - A_0)(E + E_1) - KEKGE\|^2 + \|B_2\|^2 \\
&\quad = \|(B_1 - A_0) E - KEKGE\|^2 \\
&\qquad + \|(B_1 - A_0) E_1\|^2 + \|B_2\|^2 \\
&\quad = \|K(E + E_1) K (B_1 - A_0) E - KEKGE\|^2 \\
&\qquad + \|(B_1 - A_0) E_1\|^2 + \|B_2\|^2 \\
&\quad = \|KEK(B_1 - A_0) E - KEKGE\|^2 \\
&\qquad + \|KE_1K(B_1 - A_0) E\|^2 \\
&\qquad + \|(B_1 - A_0)E_1\|^2 + \|B_2\|^2.
\end{aligned} \tag{29}$$

It is easy to prove that $KEKA_0E = 0$ according to the definitions of A_0, E. So we have

$$\begin{aligned}
\|B - A\|^2 = {} & \|KEKB_1E - KEKGE\|^2 + \|KE_1K(B_1 - A_0)E\|^2 \\
& + \|(B_1 - A_0)E_1\|^2 + \|B_2\|^2.
\end{aligned} \tag{30}$$

Obviously, $\min_{A\in S_E}\|B - A\|$ is equivalent to

$$\min_{G\in KHC^{n\times n}} \|KEKB_1E - KEKGE\|. \tag{31}$$

Since $EE_1 = 0, (KEK)(KE_1K) = 0$, it is clear that $G = B_1 + KE_1K\widehat{G}E_1$, for any $\widehat{G} \in KHC^{n\times n}$, is a solution of (31). Substituting this result to (18), we can obtain (25). □

Algorithm 10. (1) Input X, Λ, Y, Γ according to (13). (2) Compute $X^H KX\Lambda$, $\overline{\Lambda}X^H KX$, $X\Lambda X^+X$, $X\Lambda$, if (17) holds, then continue; otherwise stop. (3) Compute A_0 according to (19), and compute B_1 according to (28). (4) According to (25) calculate $\widehat{A}$.

Example 11 ($n = 8$, $h = l = 4$).

$$\begin{gathered}
X = \begin{pmatrix}
0.5661 & -0.2014 - 0.1422i & 0.1446 + 0.2138i & 0.524 \\
-0.2627 + 0.1875i & 0.5336 & -0.2110 - 0.4370i & -0.0897 + 0.3467i \\
-0.4132 + 0.2409i & 0.0226 - 0.0271i & -0.1095 + 0.2115i & -0.3531 - 0.0642i \\
-0.0306 + 0.2109i & -0.3887 - 0.0425i & 0.2531 + 0.2542i & 0.0094 + 0.2991i \\
0.0842 - 0.1778i & -0.0004 - 0.3733i & 0.3228 - 0.1113i & 0.1669 + 0.1952i \\
0.0139 - 0.3757i & -0.2363 + 0.3856i & 0.2583 + 0.0721i & 0.1841 - 0.2202i \\
0.0460 + 0.3276i & -0.1114 + 0.0654i & -0.0521 - 0.2556i & -0.2351 + 0.3002i \\
0.0085 - 0.1079i & 0.0974 + 0.3610i & 0.5060 & -0.2901 - 0.0268i
\end{pmatrix}, \\
\Lambda = \begin{pmatrix}
-0.3967 - 0.4050i & 0 & 0 & 0 \\
0 & -0.3967 + 0.4050i & 0 & 0 \\
0 & 0 & 0.0001 & 0 \\
0 & 0 & 0 & -0.0001i
\end{pmatrix}, \\
K = \begin{pmatrix}
0&0&0&0&0&0&0&1 \\
0&0&0&0&0&0&1&0 \\
0&0&0&0&0&1&0&0 \\
0&0&0&1&0&0&0&0 \\
0&0&0&0&1&0&0&0 \\
0&0&1&0&0&0&0&0 \\
0&1&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0
\end{pmatrix}, \\
Y = K\overline{X}, \qquad \Gamma = \overline{\Lambda}
\end{gathered} \tag{32}$$

$B =$

From the first column to the fourth column

$$\left(\begin{array}{cccc} -0.5218+0.0406i & 0.2267-0.0560i & -0.1202+0.0820i & -0.0072-0.3362i \\ 0.3909-0.3288i & 0.2823-0.2064i & -0.0438-0.0403i & 0.2707+0.0547i \\ 0.2162-0.1144i & -0.4307+0.2474i & -0.0010-0.0412i & 0.2164-0.1314i \\ -0.1872-0.0599i & -0.0061+0.4698i & 0.3605-0.0247i & 0.4251+0.1869i \\ -0.1227-0.0194i & 0.2477-0.0606i & 0.3918+0.6340i & 0.1226+0.0636i \\ -0.0893+0.4335i & 0.0662+0.0199i & -0.0177-0.1412i & 0.4047+0.2288i \\ 0.1040-0.2015i & 0.1840+0.2276i & 0.2681-0.3526i & -0.5252+0.1022i \\ 0.1808+0.2669i & 0.2264+0.3860i & -0.1791+0.1976i & -0.0961-0.0117i \end{array}\right. \quad (33)$$

From the fifth column to the eighth column

$$\left.\begin{array}{cccc} -0.2638-0.4952i & -0.0863-0.1664i & 0.2687+0.1958i & -0.2544-0.1099i \\ -0.2741-0.1656i & -0.0227+0.2684i & 0.1846+0.2456i & -0.0298+0.5163i \\ -0.1495-0.3205i & 0.1391+0.2434i & 0.1942-0.5211i & -0.3052-0.1468i \\ -0.2554+0.2690i & -0.4222-0.1080i & 0.2232+0.0774i & 0.0965-0.0421i \\ 0.3856-0.0619i & 0.1217-0.0270i & 0.1106-0.3090i & -0.1122+0.2379i \\ -0.1130+0.0766i & 0.7102-0.0901i & 0.1017+0.1397i & -0.0445+0.0038i \\ 0.1216+0.0076i & 0.2343-0.1772i & 0.5242-0.0089i & -0.0613+0.0258i \\ -0.0750-0.3581i & 0.0125+0.0964i & 0.0779-0.1074i & 0.6735-0.0266i \end{array}\right). \quad (34)$$

It is easy to see that matrices X, Λ, Y, Γ satisfy (17). Hence, there exists the unique solution for Problem 3. Using the software "MATLAB", we obtain the unique solution $\widehat{A}$ of Problem 3.

From the first column to the fourth column

$$\left(\begin{array}{cccc} -0.1983+0.0491i & 0.1648+0.0032i & 0.0002+0.1065i & 0.1308+0.2690i \\ 0.1071-0.2992i & 0.2106+0.2381i & -0.0533-0.3856i & -0.0946+0.0488i \\ 0.1935-0.1724i & -0.0855-0.0370i & 0.0200-0.0665i & -0.2155-0.0636i \\ 0.0085-0.2373i & -0.0843-0.1920i & 0.0136+0.0382i & -0.0328-0.0000i \\ -0.0529+0.1703i & 0.1948-0.0719i & 0.1266+0.1752i & 0.0232-0.2351i \\ 0.0855+0.1065i & 0.0325-0.2068i & 0.2624+0.0000i & 0.0136-0.0382i \\ 0.1283-0.1463i & -0.0467+0.0000i & 0.0325+0.2067i & -0.0843+0.1920i \\ 0.2498+0.0000i & 0.1283+0.1463i & 0.0855-0.1065i & 0.0086+0.2373i \end{array}\right. \quad (35)$$

From the fifth column to the eighth column

$$\left.\begin{array}{cccc} 0.2399-0.1019i & 0.1928-0.1488i & -0.3480-0.2574i & 0.1017-0.0000i \\ -0.1955+0.0644i & 0.2925+0.1872i & 0.3869+0.0000i & -0.3481+0.2574i \\ 0.0074+0.0339i & -0.3132-0.0000i & 0.2926-0.1872i & 0.1928+0.1488i \\ 0.0232+0.2351i & -0.2154+0.0636i & -0.0946-0.0489i & 0.1309-0.2691i \\ -0.0545-0.0000i & 0.0074-0.0339i & -0.1955-0.0643i & 0.2399+0.1019i \\ 0.1266-0.1752i & 0.0200+0.0665i & -0.0533+0.3857i & 0.0002-0.1065i \\ 0.1949+0.0719i & -0.0855+0.0370i & 0.2106-0.2381i & 0.1648-0.0032i \\ -0.0529-0.1703i & 0.1935+0.1724i & 0.1071+0.2992i & -0.1983-0.0491i \end{array}\right). \quad (36)$$

Conflict of Interests

There is no conflict of interests between the authors.

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