Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 202095, 8 pages
http://dx.doi.org/10.1155/2013/202095

Research Article

Strong Convergence Theorems for Semigroups of Asymptotically Nonexpansive Mappings in Banach Spaces

D. R. Sahu,[1] Ngai-Ching Wong,[2,3] and Jen-Chih Yao[3,4]

[1] *Department of Mathematics, Banaras Hindu University, Varanasi 221005, India*
[2] *Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung 804, Taiwan*
[3] *Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung 807, Taiwan*
[4] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Ngai-Ching Wong; wong@math.nsysu.edu.tw

Received 4 February 2013; Accepted 4 April 2013

Academic Editor: Qamrul Hasan Ansari

Let X be a real reflexive Banach space with a weakly continuous duality mapping J_φ. Let C be a nonempty weakly closed star-shaped (with respect to u) subset of X. Let $\mathcal{F} = \{T(t) : t \in [0, +\infty)\}$ be a uniformly continuous semigroup of asymptotically nonexpansive self-mappings of C, which is uniformly continuous at zero. We will show that the implicit iteration scheme: $y_n = \alpha_n u + (1-\alpha_n)T(t_n)y_n$, for all $n \in \mathbb{N}$, converges strongly to a common fixed point of the semigroup $\mathcal{F}$ for some suitably chosen parameters $\{\alpha_n\}$ and $\{t_n\}$. Our results extend and improve corresponding ones of Suzuki (2002), Xu (2005), and Zegeye and Shahzad (2009).

1. Introduction

Let C be a nonempty subset of a (real) Banach space X and $T : C \to C$ a mapping. The *fixed point set* of T is defined by $F(T) = \{x \in C : Tx = x\}$. We say that T is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all x, y in C and *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \le k_n \|x - y\|$ for all x, y in C and n in $\mathbb{N}$.

Set $\mathbb{R}^+ := [0, +\infty)$. We call a one-parameter family $\mathcal{F} := \{T(t) : t \in \mathbb{R}^+\}$ of mappings from C into C a *strongly continuous semigroup of Lipschitzian mappings* if

(1) for each $t > 0$, there exists a bounded function $k(\cdot) : (0, +\infty) \to (0, +\infty)$ such that

$$\|T(t)x - T(t)y\| \le k(t)\|x - y\| \quad \forall x, y \in C, \tag{1}$$

(2) $T(0)x = x$ for all x in C,

(3) $T(s + t) = T(s)T(t)$ for all s, t in $\mathbb{R}^+$,

(4) for each x in C, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into C is continuous.

Note that $\liminf_{t\to 0^+} k(t) \ge 1$. If $k(t) = L$ for all $t > 0$ in (1), then $\mathcal{F}$ is called a *strongly continuous semigroup of uniformly L-Lipschitzian mappings*. If $k(t) = 1$ for all $t > 0$ in (1), then $\mathcal{F}$ is called a *strongly continuous semigroup of nonexpansive mappings*. If $k(t) \ge 1$ for all $t > 0$ and $\lim_{t\to+\infty} k(t) = 1$ in (1), then $\mathcal{F}$ is called a *strongly continuous semigroup of asymptotically nonexpansive mappings*. Moreover, $\mathcal{F}$ is said to be *(right) uniformly continuous* if it also holds:

(5) For any bounded subset B of C, we have

$$\lim_{t\to 0^+} \sup_{x\in B} \|T(t)x - x\| = 0. \tag{2}$$

We denote by $F(\mathcal{F})$ the set of common fixed points of $\mathcal{F}$; that is, $F(\mathcal{F}) := \bigcap_{t\in\mathbb{R}^+} F(T(t))$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972. They proved that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive mapping has a fixed point. Several authors have studied the problem of the existence of fixed points of asymptotically nonexpansive mappings in Banach spaces having rich geometric structure; see [2] and the references therein.

Consider a nonempty closed convex subset C of a (real) Banach space X. A classical method to study nonexpansive mappings is to approximate them by contractions. More precisely, for a fixed element u in C, define for each t in $(0,1)$ a contraction G_t by

$$G_t x = tu + (1-t)\, Tx \quad \forall x \in C. \tag{3}$$

Let x_t be the fixed point of G_t; that is,

$$x_t = tu + (1-t)\, Tx_t. \tag{4}$$

Browder [3] (Reich [4], resp.) proves that as $t \to 0^+$, the point x_t converges strongly to a fixed point of T if X is a Hilbert space (uniformly smooth Banach space, resp.). Many authors (see, e.g., [5–13]) have studied strong convergence of approximates $\{x_t\}$ for asymptotically nonexpansive self-mappings T in Banach spaces under the additional assumption $x_t - Tx_t \to 0$ as $n \to \infty$. This additional assumption can be removed when T is uniformly asymptotically regular.

Suzuki [14] initiated the following implicit iteration process for a semigroup $\mathcal{F} := \{T(t) : t \in \mathbb{R}^+\}$ of nonexpansive mappings in a Hilbert space:

$$y_n = \alpha_n u + (1-\alpha_n)\, T(t_n)\, y_n, \quad \forall n \in \mathbb{N}. \tag{5}$$

Xu [15] extended Suzuki's result to uniformly convex Banach spaces with weakly sequentially continuous duality mappings. Recently, Zegeye and Shahzad [16] extended results of Xu [15] and established the following strong convergence theorem.

Theorem ZS. *Let C be a nonempty closed convex bounded subset of a real uniformly convex Banach space X with a weakly continuous duality mapping J_φ with gauge φ. Let $\mathcal{F} := \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous asymptotically nonexpansive semigroups with net $\{k(t)\} \subset [1,+\infty)$. Assume that $F(\mathcal{F})$ is a sunny nonexpansive retract of C with P as the sunny nonexpansive retraction. Assume that $\{t_n\} \subset (0,+\infty)$ and $\{\alpha_n\} \subset (0,1)$ such that $(1-\alpha_n)k(t_n) \le 1$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} t_n = \lim_{n\to\infty}(\alpha_n/t_n) = 0$, and $\{\alpha_n/(1-(1-\alpha_n)k(t_n))\}$ is bounded. Let u in C be fixed.*

(1) *There exists a sequence $\{y_n\}$ in C such that*

$$y_n = \alpha_n u + (1-\alpha_n)\, T(t_n)\, y_n \quad \forall n \in \mathbb{N}, \tag{6}$$

which converges strongly to an element of $F(\mathcal{F})$.

(2) *Every sequence $\{x_n\}$ defined iteratively with any x_1 in C, and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\, T(t_n)\, x_n \quad \forall n \in \mathbb{N}, \tag{7}$$

converges strongly to an element of $F(\mathcal{F})$, provided that $\|x_{n+1} - x_n\| \le D\alpha_n$ for some $D > 0$.

Problem 1. Is it possible to drop the uniform convexity assumption in Theorem ZS?

Motivated by Schu [13], the purpose of this paper is to further analyze strong convergence of (6) and (7) for strongly continuous semigroups of asymptotically nonexpansive mappings defined on a set which is not necessarily convex. It is important and actually quite surprising that we are able to do so for the class of Banach spaces which are not necessarily uniformly convex. It should be noted that, in this generality, Theorem ZS does not apply. Our results are definitive, settle Problem 1, and also improve results of Suzuki [14], Xu [15], and Zegeye and Shahzad [16].

2. Preliminaries

Let C be a nonempty subset of a (real) Banach space X with dual space X^*. We call a mapping $T : C \to C$ *weakly contractive* if

$$\|Tx - Ty\| \le \|x-y\| - \psi(\|x-y\|) \quad \forall x, y \in C, \tag{8}$$

where $\psi : [0,+\infty) \to [0,+\infty)$ is a continuous and nondecreasing function such that $\psi(0) = 0$, $\psi(t) > 0$ for $t > 0$, and $\lim_{t\to+\infty}\psi(t) = +\infty$.

By a *gauge* we mean a continuous strictly increasing function φ defined on $\mathbb{R}^+ := [0,+\infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to+\infty}\varphi(r) = +\infty$. Associated with a gauge φ, the (generally multivalued) *duality mapping* $J_\varphi : X \to X^*$ is defined by

$$J_\varphi(x) := \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\,\varphi(\|x\|),\ \|x^*\| = \varphi(\|x\|)\}. \tag{9}$$

Clearly, the (normalized) duality mapping J corresponds to the gauge $\varphi(t) = t$. In general,

$$J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x), \quad x \ne 0. \tag{10}$$

Recall that X is said to have a *weakly* (resp, *sequentially*) *continuous duality mapping* if there exists a gauge φ such that the duality mapping J_φ is single valued and (resp, sequentially) continuous from X with the weak topology to X^* with the weak* topology. Every ℓ^p $(1 < p < +\infty)$ space has a weakly continuous duality mapping with the gauge $\varphi(t) = t^{p-1}$ (for more details see [17, 18]). We know that if X admits a weakly sequentially continuous duality mapping, then X satisfies *Opial's condition*; that is, if $\{x_n\}$ is a sequence weakly convergent to x in X, then there holds the inequality

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|, \quad y \in X \text{ with } y \ne x. \tag{11}$$

Browder [19] initiated the study of certain classes of nonlinear operators by means of a duality mapping J_φ. Define

$$\Phi(t) := \int_0^t \varphi(s)\, ds \quad \forall t \ge 0. \tag{12}$$

Then, it is known that $J_\varphi(x)$ is the subdifferential of the convex function $\Phi(\|\cdot\|)$ at x; that is,

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad x \in X, \tag{13}$$

where ∂ denotes the subdifferential in the sense of convex analysis. We need the subdifferential inequality

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, j(x+y)\rangle \quad \forall x, y \in X,\ j(x+y) \in J_\varphi(x+y). \tag{14}$$

For a smooth X, we have

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle \quad \forall x, y \in X, \tag{15}$$

or considering the normalized duality mapping J, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle \quad \forall x, y \in X. \tag{16}$$

Assume that a sequence $\{x_n\}$ in X converges weakly to a point x in X. Then the following identity holds;

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|x-y\|) \quad \forall y \in X. \tag{17}$$

Remark 2. For any k with $0 \leq k \leq 1$, we have

$$\begin{gathered}\varphi(kt) \leq \varphi(t) \quad \forall t > 0, \\ \Phi(kt) = \int_0^{kt} \varphi(s)\,ds = k\int_0^t \varphi(ku)\,du \\ \leq k\int_0^t \varphi(u)\,du = k\Phi(t) \quad \forall t > 0.\end{gathered} \tag{18}$$

We need the following demiclosedness principle for asymptotically nonexpansive mappings in a Banach space.

Lemma 3 (see [17, Corollary 5.6.4], [10]). *Let X be a Banach space with a weakly continuous duality mapping $J_\varphi : X \to X^*$ with gauge function φ. Let C be a nonempty closed convex subset of X and $T : C \to C$ an asymptotically nonexpansive mapping. Then, $I - T$ is demiclosed at zero; that is, if $\{x_n\}$ is a sequence in C which converges weakly to x and if the sequence $\{x_n - Tx_n\}$ converges strongly to zero, then $x - Tx = 0$.*

Let C be a convex subset of a Banach space X and D a nonempty subset of C. Then, a continuous mapping P from C onto D is called a *retraction* if $Px = x$ for all x in D; that is, $P^2 = P$. A retraction P is said to be *sunny* if $P(Px+t(x-Px)) = Px$ for each x in C and $t \geq 0$ with $Px + t(x - Px)$ in C. If the sunny retraction P is also nonexpansive, then D is said to be a *sunny nonexpansive retract* of C. The sunny nonexpansive retraction Q from C onto D is unique if X is smooth.

Lemma 4 (see Goebel and Reich [20, Lemma 13.1]). *Let C be a convex subset of a smooth Banach space X, D a nonempty subset of C, and P a retraction from C onto D. Then, the following are equivalent.*

(a) *P is sunny and nonexpansive.*

(b) *$\langle x - Px, J(z - Px)\rangle \leq 0$ for all $x \in C$, $z \in D$.*

(c) *$\langle x - y, J(Px - Py)\rangle \geq \| Px - Py\|^2$ for all $x, y \in C$.*

Lemma 5 (see [21]). *Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two real sequences such that*

(i) *$\{\alpha_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$,*

(ii) *$\limsup_{n\to\infty}\gamma_n \leq 0$.*

Let $\{\lambda_n\}$ be a sequence of nonnegative numbers which satisfies the inequality

$$\lambda_{n+1} \leq (1-\alpha_n)\lambda_n + \alpha_n\gamma_n, \quad n \in \mathbb{N}. \tag{19}$$

Then, $\lim_{n\to\infty}\lambda_n = 0$.

3. Existence of Common Fixed Points

We begin with the following.

Proposition 6. *Let C be a nonempty closed subset of a Banach space X. Let $\mathcal{F} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly continuous semigroup of asymptotically nonexpansive mappings from C into itself with a net $\{k(t) : t \in (0, +\infty)\}$. Let $\{b_n\}$ be a sequence in $(0, 1)$ and $\{t_n\}$ a sequence in $(0, +\infty)$ with $k(t_n) - 1 < b_n$ for all n in $\mathbb{N}$. Assume that $u \in C$ and $b_n u + (1 - b_n)T(t_n)x \in C$ for all x in C and n in $\mathbb{N}$.*

(a) *There exists a sequence $\{y_n\}$ in C defined by*

$$y_n = b_n u + (1-b_n)T(t_n)y_n \quad \forall n \in \mathbb{N}. \tag{20}$$

(b) *If the sequence $\{y_n\}$ described by (20) is bounded and $\lim_{n\to\infty}t_n = \lim_{n\to\infty}(b_n/t_n) = 0$, then*

$$y_n - T(t)y_n \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty,\ \forall t > 0. \tag{21}$$

Proof. (a) Set $\varrho_n := (k(t_n) - 1)/b_n$. Since $k(t_n) - 1 < b_n$ for all n in $\mathbb{N}$, it follows that $\varrho_n < 1 \leq k(t_n)$, and hence $0 < (1 - b_n)k(t_n) = (1 - b_n)(1 + \varrho_n b_n) < 1$ for all n in $\mathbb{N}$. For each n in $\mathbb{N}$, the mapping $G_n : C \to C$ defined by

$$G_n y := b_n u + (1-b_n)T(t_n)y, \quad y \in C \tag{22}$$

is a contraction with Lipschitz constant $(1 - b_n)k(t_n)$. Therefore, there exists a sequence $\{y_n\}$ in C described by (20).

(b) Suppose that the sequence $\{y_n\}$ in C described by (20) is bounded and $\lim_{n\to\infty}t_n = \lim_{n\to\infty}(b_n/t_n) = 0$. From (20), we have

$$\|T(t_n)y_n\| = \frac{1}{1-b_n}\|y_n - b_n u\| \leq \frac{1}{1-b_n}(\|y_n\| + b_n\|u\|). \tag{23}$$

Without loss of generality, we may assume that $\{b_n\}$ is bounded away from 1. Then, there exists a positive constant δ such that $b_n \leq \delta < 1$ for all n in $\mathbb{N}$. Since $B := \{y_n : n \in \mathbb{N}\}$ is bounded, it follows from (23) that $\{T(t_n)y_n\}$ is bounded.

Set $L := \sup\{k(t) : t \in \mathbb{R}^+\}$. For $t > 0$, we have

$$\begin{aligned} &\|y_n - T(t)y_n\| \\ &\le \sum_{i=0}^{[t/t_n]-1} \|T(it_n)y_n - T((i+1)t_n)y_n\| \\ &\quad + \left\|T\left(\left[\frac{t}{t_n}\right]t_n\right)y_n - T(t)y_n\right\| \\ &\le \left[\frac{t}{t_n}\right] L\|y_n - T(t_n)y_n\| + L\left\|T\left(t - \left[\frac{t}{t_n}\right]t_n\right)y_n - y_n\right\| \\ &= Lb_n\left[\frac{t}{t_n}\right]\|u - T(t_n)y_n\| + L\left\|T\left(t - \left[\frac{t}{t_n}\right]t_n\right)y_n - y_n\right\| \\ &\le tL\left(\frac{b_n}{t_n}\right)\|u - T(t_n)y_n\| + L\sup_{y\in B}\|T(s_n)y - y\|, \end{aligned} \tag{24}$$

for all n in $\mathbb{N}$, where $s_n = t - [t/t_n]t_n$. Note that $s_n = t - [t/t_n]t_n \le t_n \to 0$ as $n \to \infty$. Since $\mathcal{F}$ is uniformly continuous at 0, it follows that $\sup_{y\in B} \| T(s_n)y - y \| \to 0$ as $n \to \infty$. Therefore, $y_n - T(t)y_n \to 0$ as $n \to \infty$. □

Remark 7. (a) If C is star shaped with respect to u in C, then the assumption "$b_n u + (1 - b_n)T(t_n)x \in C$ for all x in C and n in $\mathbb{N}$" in Proposition 6 is automatically satisfied.

(b) If $\mathcal{F} = \{T(t) : t \in \mathbb{R}^+\}$ is a strongly continuous semigroup of nonexpansive mappings with $F(\mathcal{F}) \neq \emptyset$, the sequence $\{y_n\}$ in C described by (20) is bounded (see [15, Theorem 3.3]).

Theorem 8. *Let X be a real reflexive Banach space with a weakly continuous duality mapping J_φ with gauge function φ, and C a nonempty weakly closed subset of X. Let $\mathcal{F} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly continuous semigroup of asymptotically nonexpansive mappings from C into itself with a net $\{k(t) : t \in (0, +\infty)\}$. Let $\{b_n\}$ be a sequence in $(0, 1)$ and $\{t_n\}$ a sequence in $(0, +\infty)$ with $k(t_n) - 1 < b_n$ for all n in $\mathbb{N}$ satisfying the condition*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{b_n}{t_n} = \lim_{n\to\infty} \frac{k(t_n) - 1}{b_n} = 0. \tag{25}$$

Assume that $u \in C$ such that the sequence $\{y_n\}$ described by (20) is bounded in C. Then,

(a) *$F(\mathcal{F}) \neq \emptyset$.*

(b) *$\{y_n\}$ converges strongly to an element $y^* \in F(\mathcal{F})$ which holds the inequality*

$$\langle y^* - u, J(y^* - v)\rangle \le 0 \quad \forall v \in F(\mathcal{F}). \tag{26}$$

Proof. By Proposition 6 (b), we have $y_n - T(t)y_n \to 0$ as $n \to \infty$ for all $t > 0$. Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup y^* \in C$.

(a) For each $r > 0$, we have

$$\begin{aligned} &\|T(r)^m x - T(r)^m y\| \\ &\quad = \|T(mr)x - T(mr)y\| \\ &\quad \le k(mr)\|x - y\| \quad \forall x, y \in C,\ m \in \mathbb{N}, \end{aligned} \tag{27}$$

that is, each $T(r)$ is asymptotically nonexpansive mapping. Since $\lim_{n\to\infty}\|y_n - T(t)y_n\| = 0$ for all $t > 0$, it follows from Lemma 3 that $T(t)y^* = y^*$ for all $t > 0$. Hence, $F(\mathcal{F}) \neq \emptyset$.

(b) Since $\{y_n\}$ is bounded, there exists a constant $M \ge 0$ such that $\|y_n - p\| \le M$ for all n in $\mathbb{N}$ and p in $F(\mathcal{F})$. For any v in $F(\mathcal{F})$, from (15) and Remark 2, we have

$$\begin{aligned} &\Phi(\|y_n - y^*\|) \\ &\quad = \Phi(\|b_n u + (1 - b_n)T(t_n)y_n - y^*\|) \\ &\quad \le \Phi((1 - b_n)\|T(t_n)y_n - y^*\|) \\ &\qquad + b_n\langle u - y^*, J_\varphi(y_n - y^*)\rangle \\ &\quad \le \Phi((1 - b_n)k(t_n)\|y_n - y^*\|) \\ &\qquad + b_n\langle u - y^*, J_\varphi(y_n - y^*)\rangle \\ &\quad \le (1 - b_n)k(t_n)\Phi(\|y_n - y^*\|) \\ &\qquad + b_n\langle u - y^*, J_\varphi(y_n - y^*)\rangle. \end{aligned} \tag{28}$$

Since $k(t) \ge 1$ for all $t > 0$, we have

$$\begin{aligned} \Phi(\|y_n - y^*\|) &\le \frac{k(t_n) - 1}{b_n}\Phi(\|y_n - y^*\|) \\ &\quad + \langle u - y^*, J_\varphi(y_n - y^*)\rangle \\ &\le \frac{k(t_n) - 1}{b_n}\Phi(M) + \langle u - y^*, J_\varphi(y_n - y^*)\rangle. \end{aligned} \tag{29}$$

Observing that $(k(t_{n_i}) - 1)/b_{n_i} \to 0$, $y_{n_i} \rightharpoonup y^*$, and J_φ is weakly continuous, we conclude from (29) that $y_{n_i} \to y^* \in C$ as $i \to \infty$ because C is closed.

We prove that $\{y_n\}$ converges strongly to y^*. Suppose, for contradiction, that $\{y_{n_j}\}$ is another subsequence of $\{y_n\}$ such that $y_{n_j} \to z^* \in C$ with $z^* \neq y^*$. Since $\lim_{n\to\infty}(y_n - T(t)y_n) = 0$ for all $t > 0$, we have $z^* \in F(\mathcal{F})$. For any v in $F(\mathcal{F})$, we have

$$\begin{aligned} &\langle y_n - T(t_n)y_n, J_\varphi(y_n - v)\rangle \\ &\quad = \langle y_n - v + T(t_n)v - T(t_n)y_n, J_\varphi(y_n - v)\rangle \\ &\quad = \|y_n - v\|\varphi(\|y_n - v\|) \\ &\qquad - \langle T(t_n)y_n - T(t_n)v, J_\varphi(y_n - v)\rangle \\ &\quad \ge -(k(t_n) - 1)\|y_n - v\|\varphi(\|y_n - v\|) \quad \forall n \in \mathbb{N}. \end{aligned} \tag{30}$$

From (20), we have

$$\begin{aligned}\left\langle y_n - u, J_\varphi (y_n - v)\right\rangle &= (1-b_n)\left\langle T(t_n) y_n - u, J_\varphi(y_n - v)\right\rangle \\ &= (1-b_n)\left\langle T(t_n) y_n - y_n + y_n - u, J_\varphi(y_n - v)\right\rangle.\end{aligned} \tag{31}$$

It follows that

$$\begin{aligned}&\left\langle y_n - u, J_\varphi(y_n - v)\right\rangle \\ &\quad\le \frac{1-b_n}{b_n}\left\langle T(t_n) y_n - y_n, J_\varphi(y_n - v)\right\rangle \\ &\quad\le (1-b_n)\frac{k(t_n)-1}{b_n}\|y_n - v\|\varphi(\|y_n - v\|) \\ &\quad\le \frac{k(t_n)-1}{b_n} M\varphi(M)\end{aligned} \tag{32}$$

for all v in $F(\mathcal{F})$ and n in $\mathbb{N}$. Since $\lim_{n\to\infty}(k(t_n)-1)/b_n = 0$, we obtain from (32) that

$$\begin{aligned}&\left\langle y^* - u, J_\varphi(y^* - z^*)\right\rangle \le 0, \\ &\langle z^* - u, J_\varphi(z^* - y^*)\rangle \le 0.\end{aligned} \tag{33}$$

Addition of (33) yields

$$\left\langle y^* - z^*, J_\varphi(y^* - z^*)\right\rangle \le 0. \tag{34}$$

Hence, $y^* = z^*$, a contradiction. Therefore, $\{y_n\}$ converges strongly to $y^* \in C$.

Finally, from (32), we conclude that y^* satisfies (26). □

Let $\mathcal{F} = \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings from C into itself and $u \in C$. Motivated by Morales and Jung [22, Theorem 1] and Morales [23, Theorem 2], we define

$$E^u_{\mathcal{F}}(C) = \{\lambda u + (1-\lambda) T(t) x : x \in C, \lambda \in [0,1], t > 0\}. \tag{35}$$

Corollary 9. *Let X be a real reflexive Banach space with a weakly continuous duality mapping J_φ with gauge function φ and C a nonempty weakly closed subset of X. Let $\mathcal{F} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly continuous semigroup of asymptotically nonexpansive mappings from C into itself with a net $\{k(t) : t \in (0,+\infty)\}$. Let $\{b_n\}$ be a sequence in $(0,1)$ and $\{t_n\}$ a sequence in $(0,+\infty)$ with $k(t_n) - 1 < b_n$ for all n in $\mathbb{N}$ satisfying condition* (25). *Assume that $u \in C$ such that C is star shaped with respect to u, and set $E^u_{\mathcal{F}}(C)$ defined by* (35) *is bounded.*

(a) *For each n in $\mathbb{N}$, there is exactly one point y_n in C described by* (20).

(b) *$F(\mathcal{F})$ is nonempty. Moreover, $\{y_n\}$ converges strongly to an element y^* in $F(\mathcal{F})$ which holds* (26).

Corollary 9 improves and generalizes several recent results of this nature. Indeed, it extends [13, Theorem 1.7] from the class of asymptotically nonexpansive mappings to a uniformly continuous semigroup of asymptotically nonexpansive mappings without uniformly asymptotic regularity assumption. In particular, Corollary 9 improves Theorem ZS in the following ways.

(1) Convexity of C is not required.

(2) The assumption "uniform convexity" of the underlying space is not required.

(3) For convergence of $\{y_n\}$, condition "$F(\mathcal{F})$ is a sunny nonexpansive retract of C" is not assumed.

Next, we show that $F(\mathcal{F})$ is a nonempty sunny nonexpansive retract of C.

Theorem 10. *Let X be a real reflexive Banach space with a weakly continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex-bounded subset of X. Let $\mathcal{F} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly continuous semigroup of asymptotically nonexpansive mappings from C into itself with net $\{k(t) : t \in (0,+\infty)\}$. Then, $F(\mathcal{F})$ is a nonempty sunny nonexpansive retract of C.*

Proof. By Theorem 8 (a), $F(\mathcal{F}) \neq \emptyset$. As in Theorem 8 (b), for each u in C the sequence $\{y_n\}$ converges strongly to an element y^* in $F(\mathcal{F})$. Define a mapping $Q : C \to F(\mathcal{F})$ by

$$Qu = \lim_{n\to\infty} y_n. \tag{36}$$

By (32), we have

$$\left\langle y_n - u, J_\varphi(y_n - v)\right\rangle \le \frac{k(t_n)-1}{b_n}\operatorname{diam}(C)\,\varphi(\operatorname{diam}(C)) \tag{37}$$

for all u in C, v in $F(\mathcal{F})$, and n in $\mathbb{N}$, where $\operatorname{diam}(B)$ is the diameter of set B. Letting $n \to \infty$, we obtain that

$$\left\langle Qu - u, J_\varphi(Qu - v)\right\rangle \le 0 \quad \forall v \in F(\mathcal{F}). \tag{38}$$

Therefore, by Lemma 4, we conclude that Q is sunny nonexpansive. □

Remark 11. In view of Remark 7 (b), the nonempt*h*ness of the common fixed point set of a uniformly continuous semigroup of nonexpansive mappings implies that the sequence $\{y_n\}$ in C described by (20) is bounded. So we can drop the boundedness assumption of domain C in Theorem 10, provided that $F(\mathcal{F}) \neq \emptyset$.

Corollary 12. *Let X be a real reflexive Banach space with a weakly continuous duality mapping J_φ with gauge function φ and C a nonempty weakly closed subset of X. Let $\mathcal{F} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly continuous semigroup of nonexpansive mappings from C into itself with $F(\mathcal{F}) \neq \emptyset$. Let $\{b_n\}$ be a sequence in $(0,1)$ and $\{t_n\}$ a sequence in $(0,+\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty}(b_n/t_n) = 0$. Assume that $u \in C$ and that C is star shaped with respect to u.*

(a) *For each n in $\mathbb{N}$, there is exactly one point y_n in C described by* (20).

(b) $\{y_n\}$ *converges strongly to an element y^* in $F(\mathcal{F})$ which holds* (26).

Corollary 12 is an improvement of [14, Theorem 3] and [15, Theorem 3.3], where strong convergence theorems were established in Hilbert and uniformly convex Banach spaces, respectively.

4. Approximation of Common Fixed Points

Theorem 13. *Let X be a real reflexive Banach space with a weakly continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex bounded subset of X and $\mathcal{F} = \{T(t) : t \in \mathbb{R}^+\}$ a uniformly continuous semigroup of asymptotically nonexpansive mappings from C into itself with a net $\{k(t) : t \in (0, +\infty)\}$. For given u, x_1 in C, let $\{x_n\}$ be a sequence in C generated by* (7), *where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{t_n\}$ is a decreasing sequence in $(0, +\infty)$ satisfying the following conditions.*

(C1) $\lim_{n\to\infty} t_n = \lim_{n\to\infty}(\alpha_n/t_n) = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\lim_{n\to\infty}(|\alpha_n - \alpha_{n+1}|/\alpha_n) = 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < +\infty$,

(C3) $\lim_{n\to\infty}(\|T(t_n - t_{n+1})x_n - x_n\|/\alpha_{n+1}) = 0$,

(C4) $(1 - \alpha_n)k(t_n) \le 1$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty}((k(t_n) - 1)/\alpha_n) = 0$.

Then,

(a) $F(\mathcal{F}) \neq \emptyset$.

(b) $\{x_n\}$ *converges strongly to* $Q_{F(\mathcal{F})}(u)$, *where* $Q_{F(\mathcal{F})}$ *is a sunny nonexpansive retraction of C onto $F(\mathcal{F})$.*

Proof. It follows from Theorem 10 that $F(\mathcal{F}) \neq \emptyset$, and there is a sunny nonexpansive retraction $Q_{F(\mathcal{F})}$ of C onto $F(\mathcal{F})$. Set $y^* := Q_{F(\mathcal{F})}(u)$, $B := \{x_n\}$, $L := \sup\{k(t) : t > 0\}$, and $M := \sup\{\|x_n - y^*\| : n \in \mathbb{N}\}$. Since C is bounded, there exists a constant $K > 0$ such that

$$\|x_{n+1} - x_n\| \le K, \qquad \|T(t_n)x_n - u\| \le K \quad \forall n \in \mathbb{N}. \tag{39}$$

Hence, for all $n > 1$, we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad\le \|(\alpha_n - \alpha_{n-1})(u - T(t_{n-1})x_{n-1}) \\
&\qquad + (1 - \alpha_n)(T(t_n)x_n - T(t_{n-1})x_{n-1})\| \\
&\quad\le |\alpha_n - \alpha_{n-1}| K \\
&\qquad + (1 - \alpha_n)\|T(t_n)x_n - T(t_{n-1})x_{n-1}\| \\
&\quad\le |\alpha_n - \alpha_{n-1}| K \\
&\qquad + (1 - \alpha_n)(\|T(t_n)x_n - T(t_n)x_{n-1}\| \\
&\qquad\qquad + \|T(t_n)x_{n-1} - T(t_{n-1})x_{n-1}\|) \\
&\quad\le |\alpha_n - \alpha_{n-1}| K + (1 - \alpha_n)k(t_n)\|x_n - x_{n-1}\| \\
&\qquad + \|T(t_n)x_{n-1} - T(t_{n-1})x_{n-1}\| \\
&\quad\le (1 - \alpha_n)k(t_n)\|x_n - x_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| K + k(t_n)\|T(t_{n-1} - t_n)x_{n-1} - x_{n-1}\| \\
&\quad\le (1 - \alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K \\
&\qquad + L\|T(t_{n-1} - t_n)x_{n-1} - x_{n-1}\| \\
&\qquad + (1 - \alpha_n)(k(t_n) - 1)\|x_n - x_{n-1}\| \\
&\quad\le (1 - \alpha_n)\|x_n - x_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| K + L\|T(t_{n-1} - t_n)x_{n-1} - x_{n-1}\| \\
&\qquad + (k(t_n) - 1)K.
\end{aligned} \tag{40}$$

Hence, by Lemma 5 and assumptions (C1)~(C4), we conclude that $x_{n+1} - x_n \to 0$ as $n \to \infty$. Fix $t > 0$, we have

$$\begin{aligned}
&\|x_n - T(t)x_n\| \\
&\quad\le \sum_{i=0}^{[t/t_n]-1} \|T(it_n)x_n - T((i+1)t_n)x_n\| \\
&\qquad + \left\|T\left(\left[\frac{t}{t_n}\right]t_n\right)x_n - T(t)x_n\right\| \\
&\quad\le \left[\frac{t}{t_n}\right] L\|x_n - T(t_n)x_n\| \\
&\qquad + L\left\|T\left(t - \left[\frac{t}{t_n}\right]t_n\right)x_n - x_n\right\| \\
&\quad\le L\left[\frac{t}{t_n}\right](\|x_n - x_{n+1}\| + \|x_{n+1} - T(t_n)x_n\|) \\
&\qquad + L\left\|T\left(t - \left[\frac{t}{t_n}\right]t_n\right)x_n - x_n\right\| \\
&\quad\le L\left(\frac{t}{t_n}\right)\|x_n - x_{n+1}\| \\
&\qquad + tL\left(\frac{\alpha_n}{t_n}\right)\|u - T(t_n)x_n\| \\
&\qquad + L\max\{\|T(s)x_n - x_n\| : 0 \le s \le t_n\}
\end{aligned} \tag{41}$$

for all n in $\mathbb{N}$, which gives that $x_n - T(t)x_n \to 0$ as $n \to \infty$. We may assume that $x_{n_i} \rightharpoonup p \in C$ as $i \to \infty$. In view of the assumption that the duality mapping J_φ is weakly sequentially continuous, it follows from Lemma 3 that $p \in F(\mathcal{F})$. Thus, by the weak continuity of J_φ and Lemma 4, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \langle u - Q_{F(\mathcal{F})}(u), J_\varphi(x_n - Q_{F(\mathcal{F})}(u))\rangle \\
&\quad= \lim_{i\to\infty} \langle u - Q_{F(\mathcal{F})}(u), J_\varphi(x_{n_i} - Q_{F(\mathcal{F})}(u))\rangle
\end{aligned}$$

$$= \left\langle u - Q_{F(\mathcal{F})}(u), J_\varphi\left(p - Q_{F(\mathcal{F})}(u)\right)\right\rangle \leq 0. \tag{42}$$

Define $\gamma_n := \max\{0, \langle u - y^*, J_\varphi(x_{n+1} - y^*)\rangle\}$. From (7), we obtain

$$\begin{aligned}\Phi(\|x_{n+1} - y^*\|) &= \Phi(\|\alpha_n(u - y^*) + (1-\alpha_n)(T(t_n)x_n - y^*)\|) \\ &\leq \Phi((1-\alpha_n)\|T^n x_n - y^*\|) \\ &\quad + \alpha_n\left\langle u - y^*, J_\varphi(x_{n+1} - y^*)\right\rangle \\ &\leq \Phi((1-\alpha_n)k(t_n)\|x_n - y^*\|) + \alpha_n\gamma_n.\end{aligned} \tag{43}$$

Since $(1-\alpha_n)k(t_n) \leq 1$ for all n in $\mathbb{N}$, it follows from Remark 2 that

$$\begin{aligned}\Phi(\|x_{n+1} - y^*\|) &\leq (1-\alpha_n)k(t_n)\Phi(\|x_n - y^*\|) + \alpha_n\gamma_n \\ &\leq (1-\alpha_n)\Phi(\|x_n - y^*\|) \\ &\quad + \alpha_n\gamma_n + (k(t_n) - 1)\Phi(M).\end{aligned} \tag{44}$$

Note that $\lim_{n\to\infty}\gamma_n = 0$ and $\lim_{n\to\infty}((k(t_n) - 1)/\alpha_n) = 0$. Using Lemma 5, we obtain that $\{x_n\}$ converges strongly to y^*. □

One can carry over Theorem 13 to the so-called viscosity approximation technique (see Xu [21]). We derive a more general result in this direction which is an improvement upon several convergence results in the context of viscosity approximation technique.

Theorem 14. *Let X be a real reflexive Banach space with a weakly continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex-bounded subset of X, $f : C \to C$ a weakly contraction with function ψ, and $\mathcal{F} = \{T(t) : t \in \mathbb{R}^+\}$ a uniformly continuous semigroup of asymptotically nonexpansive mappings from C into itself with a net $\{k(t) : t \in (0, +\infty)\}$. For an arbitrary initial value x_1 in C, let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = \alpha_n f x_n + (1-\alpha_n)T(t_n)x_n \quad \forall n \in \mathbb{N}. \tag{45}$$

Here, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{t_n\}$ is a decreasing sequence in $(0, +\infty)$ satisfying conditions (C1)~(C4). Then,

(a) *$F(\mathcal{F}) \neq \emptyset$.*

(b) *$\{x_n\}$ converges strongly to x^* in $F(\mathcal{F})$, where $x^* = Q_{F(\mathcal{F})}(fx^*)$ and $Q_{F(\mathcal{F})}$ is a sunny nonexpansive retraction of C onto $F(\mathcal{F})$.*

Proof. It follows from Theorem 10 that $F(\mathcal{F}) \neq \emptyset$, and there is a sunny nonexpansive retraction $Q_{F(\mathcal{F})}$ of C onto $F(\mathcal{F})$. Since $Q_{F(\mathcal{F})}f$ is a weakly contractive mapping from C into itself, it follows from [24, Theorem 1] that there exists a unique element x^* in C such that $x^* = Q_{F(\mathcal{F})}(fx^*)$. Such x^* in C is an element of $F(\mathcal{F})$. Now, we define a sequence $\{z_n\}$ in C by

$$z_{n+1} = \alpha_n f x^* + (1-\alpha_n)T(t_n)z_n \quad \forall n \in \mathbb{N}. \tag{46}$$

By Theorem 13, we have that $z_n \to x^* = Q_{F(\mathcal{F})}(fx^*)$. By boundedness of $\{x_n\}$ and $\{z_n\}$, there exists a constant $M \geq 0$ such that $\|x_n - z_n\| \leq M$ for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned}\|x_{n+1} - z_{n+1}\| &\leq \alpha_n\|fx_n - fx^*\| \\ &\quad + (1-\alpha_n)\|T(t_n)x_n - T(t_n)z_n\| \\ &\leq \alpha_n(\|fx_n - fz_n\| + \|fz_n - fx^*\|) \\ &\quad + (1-\alpha_n)\|T(t_n)x_n - T(t_n)z_n\| \\ &\leq \alpha_n(\|x_n - z_n\| - \psi(\|x_n - z_n\|) + \|z_n - x^*\|) \\ &\quad + (1-\alpha_n)k(t_n)\|x_n - z_n\| \\ &\leq \|x_n - z_n\| - \alpha_n\psi(\|x_n - z_n\|) \\ &\quad + \alpha_n\|z_n - x^*\| + (1-\alpha_n)(k(t_n) - 1)\|x_n - z_n\| \\ &\leq \|x_n - z_n\| - \alpha_n\psi(\|x_n - z_n\|) \\ &\quad + \alpha_n\left[\|z_n - x^*\| + \frac{(k(t_n) - 1)}{\alpha_n}M\right].\end{aligned} \tag{47}$$

By [25, Lemma 3.2], we obtain $\|x_n - z_n\| \to 0$. Therefore, $x_n \to x^* = Q_{F(\mathcal{F})}(fx^*)$. □

Theorem 15. *Let X be a real reflexive Banach space with a weakly continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of X and $f : C \to C$ a weakly contraction. Let $\mathcal{F} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly continuous semigroup of nonexpansive mappings from C into itself with $F(\mathcal{F}) \neq \emptyset$. For given x_1 in C, let $\{x_n\}$ be a sequence in C generated by (45), where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{t_n\}$ is a decreasing sequence in $(0, +\infty)$ satisfying conditions (C1)~(C3). Then, $\{x_n\}$ converges strongly to $x^* \in F(\mathcal{F})$, where $x^* = Q_{F(\mathcal{F})}(fx^*)$ and $Q_{F(\mathcal{F})}$ is a sunny nonexpansive retraction of C onto $F(\mathcal{F})$.*

Remark 16. (a) For a related result concerning the strong convergence of the explicit iteration procedure $z_{n+1} := (\lambda/k_n)T^n(z_n)$ to some fixed point of an asymptotically nonexpansive mapping T on star-shaped domain in a reflexive Banach space with a weakly continuous duality mapping, we refer the reader to Schu [13].

(b) We remark that condition "$\|x_{n+1} - x_n\| \leq D\alpha_n$ for some $D > 0$" implies that $\|x_{n+1} - x_n\| \to 0$ as $\alpha_n \to 0$. Thus, the assumption "$\|x_{n+1} - x_n\| \leq D\alpha_n$ for some $D > 0$" imposed in Theorem ZS is very strong. In our results, such assumption is avoided. Under a mild assumption, Theorem 13 shows that the sequence $\{x_n\}$ generated by (7) converges strongly to a common fixed point of a uniformly continuous semigroup of asymptotically nonexpansive mappings in a real Banach space without uniform convexity. Therefore, Theorem 13 is a significant improvement of a number of known results (e.g., Theorem ZS and [12, Theorem 4.7]) for semigroups of asymptotically nonexpansive mappings. Corollary 9 and Theorem 13 provide an affirmative answer to Problem 1.

Acknowledgments

This research is supported partially by Taiwan NSC Grants 99-2115-M-037-002-MY3, 98-2923-E-037-001-MY3, 99-2115-M-110-007-MY3. The authors would like to thank the referees for their careful reading and helpful comments.

References

[1] K. Goebel and W. A. Kirk, "A fixed point theorem for asymptotically nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 35, pp. 171–174, 1972.

[2] T. D. Benavides and P. L. Ramírez, "Structure of the fixed point set and common fixed points of asymptotically nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 129, no. 12, pp. 3549–3557, 2001.

[3] F. E. Browder, "Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces," *Archive for Rational Mechanics and Analysis*, vol. 24, pp. 82–90, 1967.

[4] S. Reich, "Strong convergence theorems for resolvents of accretive operators in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 75, no. 1, pp. 287–292, 1980.

[5] L. C. Ceng, A. R. Khan, Q. H. Ansari, and J. C. Yao, "Viscosity approximation methods for strongly positive and monotone operators," *Fixed Point Theory*, vol. 10, no. 1, pp. 35–71, 2009.

[6] L. C. Ceng, H. K. Xu, and J. C. Yao, "The viscosity approximation method for asymptotically nonexpansive mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 4, pp. 1402–1412, 2008.

[7] L. C. Ceng, N. C. Wong, and J. C. Yao, "Fixed point solutions of variational inequalities for a finite family of asymptotically nonexpansive mappings without common fixed point assumption," *Computers & Mathematics with Applications*, vol. 56, no. 9, pp. 2312–2322, 2008.

[8] H. Fukhar-ud-din and A. R. Khan, "Approximating common fixed points of asymptotically nonexpansive maps in uniformly convex Banach spaces," *Computers & Mathematics with Applications*, vol. 53, no. 9, pp. 1349–1360, 2007.

[9] H. Fukhar-ud-din, A. R. Khan, D. O. Regan, and R. P. Agarwal, "An implicit iteration scheme with errors for a finite family of uniformly continuous mappings," *Functional Differential Equations*, vol. 14, no. 2–4, pp. 245–256, 2007.

[10] T. C. Lim and H. K. Xu, "Fixed point theorems for asymptotically nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 22, no. 11, pp. 1345–1355, 1994.

[11] D. R. Sahu, R. P. Agarwal, and D. O'Regan, "Structure of the fixed point set of asymptotically nonexpansive mappings in Banach spaces with weak uniformly normal structure," *Journal of Applied Analysis*, vol. 17, no. 1, pp. 51–68, 2011.

[12] D. R. Sahu, Z. Liu, and S. M. Kang, "Iterative approaches to common fixed points of asymptotically nonexpansive mappings," *The Rocky Mountain Journal of Mathematics*, vol. 39, no. 1, pp. 281–304, 2009.

[13] J. Schu, "Approximation of fixed points of asymptotically nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 112, no. 1, pp. 143–151, 1991.

[14] T. Suzuki, "On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces," *Proceedings of the American Mathematical Society*, vol. 131, no. 7, pp. 2133–2136, 2003.

[15] H. K. Xu, "A strong convergence theorem for contraction semigroups in Banach spaces," *Bulletin of the Australian Mathematical Society*, vol. 72, no. 3, pp. 371–379, 2005.

[16] H. Zegeye and N. Shahzad, "Strong convergence theorems for continuous semigroups of asymptotically nonexpansive mappings," *Numerical Functional Analysis and Optimization*, vol. 30, no. 7-8, pp. 833–848, 2009.

[17] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*, vol. 6 of *Topological Fixed Point Theory and Its Applications*, Springer, New York, NY, USA, 2009.

[18] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[19] F. E. Browder, "Convergence theorems for sequences of nonlinear operators in Banach spaces," *Mathematische Zeitschrift*, vol. 100, pp. 201–225, 1967.

[20] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, 1984.

[21] H. K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[22] C. H. Morales and J. S. Jung, "Convergence of paths for pseudo-contractive mappings in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 128, no. 11, pp. 3411–3419, 2000.

[23] C. H. Morales, "Strong convergence of path for continuous pseudo-contractive mappings," *Proceedings of the American Mathematical Society*, vol. 135, no. 9, pp. 2831–2838, 2007.

[24] B. E. Rhoades, "Some theorems on weakly contractive maps," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 4, pp. 2683–2693, 2001.

[25] Y. I. Alber, C. E. Chidume, and H. Zegeye, "Approximating fixed points of total asymptotically nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2006, Article ID 10673, 20 pages, 2006.