Research Article

Fractional-Order *PI* Control of First Order Plants with Guaranteed Time Specifications

F. J. Castillo-Garcia,[1] A. San Millan-Rodriguez,[2] V. Feliu-Batlle,[2] and L. Sanchez-Rodriguez[3]

[1] *Ayssel Mecatronica, St/Pedro Pardo 23, 13195 Poblete (Ciudad Real), Spain*

[2] *Department of Electrical, Electronics, Control Engineering and Communications, Castilla-La Mancha University, Av. Camilo Jose Cela s/n, 13071 Ciudad Real, Spain*

[3] *School of Industrial Engineering, Castilla-La Mancha University, Tecnology Campus 'Antigua Fabrica de Armas' s/n, 45071 Toledo, Spain*

Correspondence should be addressed to F. J. Castillo-Garcia; f.castillo@ayssel.com

Cardiospheres (CSs) are self-assembling multicellular clusters from the cellular outgrowth from cardiac explants cultured in nonadhesive substrates. They contain a core of primitive, proliferating cells, and an outer layer of mesenchymal/stromal cells and differentiating cells that express cardiomyocyte proteins and connexin 43. Because CSs contain both primitive cells and committed progenitors for the three major cell types present in the heart, that is, cardiomyocytes, endothelial cells, and smooth muscle cells, and because they are derived from percutaneous endomyocardial biopsies, they represent an attractive cell source for cardiac regeneration. In preclinical studies, CS-derived cells (CDCs) delivered to infarcted hearts resulted in improved cardiac function. CDCs have been tested safely in an initial phase-1 clinical trial in patients after myocardial infarction. Whether or not CDCs are superior to purified populations, for example, c-kit+ cardiac stem cells, or to gene therapy approaches for cardiac regeneration remains to be evaluated.

1. Introduction

The conventional proportional plus integral plus derivative controller (*PID*) is the most frequently used control strategy in industry because of its simplicity, robustness performance, and the availability of many effective and simple tuning methods based on a minimum knowledge of the plant [1–5]. Some surveys have shown that 90% of the industrial control loops belong to the *PID* controller family: *P*, *PD*, *PI*, or *PID* [6, 7].

The design of controllers based on frequency specifications is a fairly broadly used approach (e.g., [8]). Specifications such as phase margin ϕ, gain margin M_g, gain crossover frequency ω_c, and phase crossover frequency ω_g are commonly used. The use of these specifications is twofold: some of them like phase margin and gain crossover frequency are related to time specifications such as overshoot and speed of response, while others like phase and gain margins are related

to robustness to delay and gain process changes, respectively. An interesting feature of these techniques is therefore that they allow the design of the dynamic response of closed-loop control systems and also permit the design of robustness properties for the controller [9]. Achieving robustness properties in the controller by using time domain or Laplace transform domain based design techniques becomes significantly more complicated than in the frequency domain.

Besides, fractional calculus is a mathematical tool that has found an application in the subject of automatic control in the last three decades [10, 11]. From the generalization of the *PID* controller to the $PI^\alpha D^\beta$ controller [11], several works have demonstrated that the use of fractional-order controllers such as that previously mentioned, or some simplified versions such as D^β, I^α, PI^α, or PD^β controllers, allows the performance of some industrial *PID* controllers to be improved in aspects such as robustness (e.g. [11–13]), output response (e.g.,

in servo systems [14]), disturbance rejection (e.g., [15]), and reducing actuator effort (e.g., reduction of saturation effects [16]). These controllers have been applied to both fractional, and integer-order processes, and their advantages have been reported both in simulated and experimental results, for example, [17, 18].

The application of these controllers is often oriented under the point of view of an optimization problem. In [19] fractional-order-proportional-integral-derivative (FOPID) controllers are synthesized using a single objective optimization process involving a user-specified peak overshoot and rise time. In [20] a fractional-order-proportional-integral-derivative ($PI^\lambda D^\delta$) controller is also tuned by minimizing the integral time absolute error (ITAE) by a particle swarm optimization process. The obtained controller is compared to a PID controller also tuned using the proposed method. Both resulting controllers are highly effective, but the superiority of the fractional-order one is demonstrated. In [21] the optimization of fractional algorithms for the discrete-time control of linear and nonlinear systems is studied. The application of fractional derivatives in control is formulated as an optimization problem in order to minimize the integral squared error (ISE) of the error signal. The optimization problem is solved by means of evolutionary concepts. Some simulations for controlling a second order plant with different gain and time constants are carried out. The results show that the proposed method exhibits a good performance and adaptability to different types of systems.

Linear time invariant processes of integer-order (described by linear integer order differential equations of constant coefficients) are often approximated by first or second order transfer functions. For these particular cases, the scientific literature has established well-known simple relations (which may be approximated or exact in some cases) between time and frequency characteristics, for example, [8]. Time specifications, usually considered in the closed-loop system, are steady state error, e_{ss}, overshoot, M_p, and settling time, t_s. These frequency-time relationships allow the design of controllers that verify closed-loop time specifications using design techniques suited for frequency specifications.

Designing standard (integer-order) controllers in the frequency domain is often easier than in the time domain. Moreover, frequency domain techniques are the most used ones to design fractional-order controllers, because these techniques enjoy the same advantages as the ones used to design integer-order controllers in this domain, while designing fractional-order controllers in the time domain or Laplace transform domain becomes much more complicated than when designing integer-order controllers in these domains.

However, designing controllers (both of fractional or integer order) using frequency methods requires an accurate translation from frequency to time specifications. If integer-order processes were controlled using fractional-order regulators, the overall closed-loop transfer function would become fractional-order too, and the relations between frequency and time domain specifications would be significantly different from the ones stated in integer-order controller cases. Consequently, these fractional-order controllers tuned

using frequency specifications yield closed-loop systems that do not verify the desired time specifications.

Some research has been reported on the relation between frequency and time specifications in closed-loop systems that use fractional-order controllers. In [22], the authors analytically obtained the expressions of time specifications for a fractional integrator using the Mittag-Leffler function [23]. The equations of settling time, t_s, peak time, t_p, and overshoot, M_p, were expressed as functions of the fractional order of the controller, α, and the required gain crossover frequency, ω_c. Nevertheless, when the open-loop transfer function is other than the one stated in this paper, the analytical deduction of these relationships becomes very difficult. A methodology was developed in [24] for obtaining the frequency specifications that yield exact time specifications in the case of controlling a first order plant with a PI^α controller. This is based on an optimization procedure that provided a polynomial rule that defined how the frequency specifications used for the design of an integer-order PI controller had to be modified in order to design a fractional-order PI controller with the same time specifications. This methodology was applied later to design a fractional-order controller combined with a Smith predictor for a main irrigation canal, which was robust to the large variations experienced by the canal parameters [25].

This paper continues the research started with the last two articles. The relationship between frequency and time specifications is studied in the case of the closed-loop control of first order systems, but now several structures of the fractional order controller of PI type are considered: the PI^α, II^β, and $I^\alpha D^{1-\alpha}$ controllers. All of them have been broadly used in the fractional-order control scientific literature, and each one exhibits different features. Moreover this paper develops analytical expressions that define, for a given controller structure, the family of controllers that guarantee desired time specifications t_s and M_p, in the sense of providing controllers that yield a settling time equal to or less than t_s, together with an overshoot equal to or less than M_p. The methodology proposed here allows for choosing in an easy manner, among all the controllers that verify the desired time specifications, the one that optimizes an additional control goal like the minimization of the energy consumption or the maximization of robustness features.

2. Materials and Methods

2.1. Basic Statements

2.1.1. Process Normalization.
As mentioned in Section 1, this work is focused on controlling first-order processes, hereafter FO, whose transfer functions are of the form

$$G(s) = \frac{K}{Ts + 1}, \tag{1}$$

where K and T are the process gain and the time constant, respectively.

In order to obtain general results, the above transfer function is normalized by scaling the time t by T ($t_n = t/T$) and

the process output y by K ($y_n = y/K$). It yields then the normalized transfer function:

$$G_n(s) = \frac{1}{s+1}. \tag{2}$$

2.1.2. Controller Design from Frequency Specifications.
The standard negative unity feedback control scheme is used to control (1) as shown in Figure 1, where $Y_n^*(s)$, $E_n(s)$, and $U_n(s)$ are the Laplace transforms of the normalized signals: reference, $y_n^*(t)$, error, $e_n(t)$, and control, $u_n(t)$. The normalized controller has been denoted by $R_n(s)$.

We seek to design controllers for process (1) that verify certain typical design frequency specifications: (a) a desired phase margin (ϕ), which provides the desired damping and robustness to changes in time delay; (b) a desired gain crossover frequency (ω_c), which provides the desired nominal speed of response, and (c) zero steady state error to a step command, which implies—in the case of *FO* processes—that the controller must include an integral term (of integer or fractional order), according to the Final Value Theorem (see, e.g., [8]).

Since we intend to use the normalized process (2), it is also necessary to normalize the gain crossover frequency by applying the formula $\omega_{cn} = \omega_c T$. The phase margin specification does not change because of the process normalization. The three specifications (ϕ, ω_{cn}) and zero steady state error can therefore be attained by using a normalized controller $R_n(s)$ with a pole at the origin (of integer or fractional order), and at least two parameters to be tuned. We thus propose different versions of the fractional-order *PI* controller to fulfill these three specifications.

The condition of having a given phase margin (ϕ) and a gain crossover frequency (ω_{cn}) can be expressed in a compact form using complex numbers:

$$R_n(j\omega_{cn}) G_n(j\omega_{cn}) = e^{-j(\pi-\phi)}. \tag{3}$$

By denoting $z_r = \Re\{-e^{j\phi}/G_n(j\omega_{cn})\}$ and $z_i = \Im\{-e^{j\phi}/G_n(j\omega_{cn})\}$, where $\Re\{\cdot\}$ and $\Im\{\cdot\}$ represent real and imaginary components of a complex number, respectively, the conditions to tune the controller parameters obtained from (3) are required

$$R_n(j\omega_{cn}) = z_r + j z_i. \tag{4}$$

Operating $-e^{j\phi}/G_n(j\omega_{cn})$ yields

$$\begin{aligned} z_r &= -\cos(\phi) + \omega_{cn}\sin(\phi), \\ z_i &= -\omega_{cn}\cos(\phi) - \sin(\phi). \end{aligned} \tag{5}$$

Expressions (5) are the tuning equations of a controller $R_n(s)$ designed to fulfill the frequency specifications (ϕ, ω_{cn}) with the process $G_n(s)$ of (2).

2.1.3. Controllers

Generic Fractional-Order Controller. In Figure 1, $R_n(s)$ represents the transfer function of a generic normalized controller.
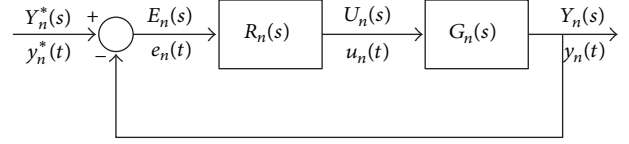

FIGURE 1: Unity feedback control scheme.

This controller must exhibit at least two parameters to be tuned. The controllers that will be considered for $R_n(s)$, and that will be compared in this paper, present the following general structure:

$$R_n(s) = \frac{K_{\alpha n}}{s^\alpha} + \frac{K_{\beta n}}{s^\beta}, \tag{6}$$

where $0 < \alpha \le 1$ and $\alpha - 1 \le \beta < \alpha$. In this controller, the α term provides the main fractional-order integral action, in order to remove steady state errors, and the role of the β term is to improve the transient response, providing with an action that can range from a fractional-order derivative action ($-1 \le \beta < 0$) to a fractional-order integral action ($0 < \beta < \alpha$), including the case of a proportional action ($\beta = 0$). We consider controllers with fractional-order derivatives and integrals not larger than one. The general structure (6) includes as particular cases the controllers to be compared in this paper.

Standard PI Controller. The standard *PI* controller ($\alpha = 1$, $\beta = 0$) can be written as

$$R_{0n}(s) = K_{\beta n} + \frac{K_{\alpha n}}{s}. \tag{7}$$

Note that this controller has two parameters to be tuned, $K_{\beta n}$ and $K_{\alpha n}$.

Fractional-Order I^α Controller. Fractional-order integral controller I^α ($0 < \alpha \le 1$, $K_{\beta n} = 0$):

$$R_{1n}(s) = \frac{K_{\alpha n}}{s^\alpha}. \tag{8}$$

This controller has also two parameters to be tuned, $K_{\alpha n}$ and α.

Fractional-Order PI^α Controller. Fractional-order integral controller PI^α ($0 < \alpha \le 1$, $\beta = 0$):

$$R_{2n}(s) = K_{\beta n} + \frac{K_{\alpha n}}{s^\alpha} = \frac{K_{\beta n}s^\alpha + K_{\alpha n}}{s^\alpha}. \tag{9}$$

Fractional-Order II^β Controller. A modification of the standard PI^α controller in the sense of having an integer integral term (which allows the steady state error caused by step commands or step disturbances to be removed more quickly). This is called II^β and has the form ($\alpha = 1$, $0 < \beta < 1$)

$$R_{3n}(s) = \frac{K_{\beta n}}{s^\beta} + \frac{K_{\alpha n}}{s} = \frac{K_{\beta n}s^{1-\beta} + K_{\alpha n}}{s}. \tag{10}$$

Fractional-Order $I^{\alpha}D^{1-\alpha}$ Controller. Another modification of the standard PI^{α} controller is called the $I^{\alpha}D^{1-\alpha}$ controller $(0 < \alpha \leq 1, \beta = \alpha - 1)$:

$$R_{4n}(s) = K_{\beta n}s^{1-\alpha} + \frac{K_{\alpha n}}{s^{\alpha}} = \frac{K_{\beta n}s + K_{\alpha n}}{s^{\alpha}} \qquad (11)$$

which includes as particular cases the PI controller $(\alpha = 1)$ and arbitrarily close approximations to a PD controller $(\alpha \to 0)$.

Note that the three last controllers have three parameters to be tuned, $K_{\beta n}, K_{\alpha n}$, and α. Table 1 resumes all the previous controllers and their parameters to be tuned.

2.1.4. Controllers Tuning Equations. Substituting the general form (6) in condition (4) gives

$$\frac{K_{\beta n}}{(j\omega_{cn})^{\beta}} + \frac{K_{\alpha n}}{(j\omega_{cn})^{\alpha}} = z_r + jz_i. \qquad (12)$$

Taking into account that

$$(j\omega_{cn})^{x} = \omega_{cn}^{x}e^{j(\pi/2)x}$$
$$= \omega_{cn}^{x}\left[\cos\left(\frac{\pi}{2}x\right) + j\sin\left(\frac{\pi}{2}x\right)\right], \quad x \in \Re, \qquad (13)$$

in (12), operating this expression, equating separately the real and imaginary components, and expressing the resulting two equations in a matricial form yield

$$\begin{pmatrix} z_r \\ z_i \end{pmatrix} = \begin{pmatrix} \dfrac{\cos((\pi/2)\beta)}{\omega_{cn}^{\beta}} & \dfrac{\cos((\pi/2)\alpha)}{\omega_{cn}^{\alpha}} \\ -\dfrac{\sin((\pi/2)\beta)}{\omega_{cn}^{\beta}} & -\dfrac{\sin((\pi/2)\alpha)}{\omega_{cn}^{\alpha}} \end{pmatrix} \begin{pmatrix} K_{\beta} \\ K_{\alpha} \end{pmatrix}. \qquad (14)$$

This last expression allows for obtaining the controller gains:

$$K_{\beta n} = \frac{\omega_{cn}^{\beta}}{\sin((\pi/2)(\alpha - \beta))}\left[\sin\left(\frac{\pi}{2}\alpha\right)z_r + \cos\left(\frac{\pi}{2}\alpha\right)z_i\right],$$
$$K_{\alpha n} = -\frac{\omega_{cn}^{\alpha}}{\sin((\pi/2)(\alpha - \beta))}\left[\sin\left(\frac{\pi}{2}\beta\right)z_r + \cos\left(\frac{\pi}{2}\beta\right)z_i\right]$$
$$(15)$$

which express these gains as functions of the orders α and β of the fractional-order operators. Moreover substituting (5) in (15) and operating yield

$$K_{\beta n} = -\frac{\omega_{cn}^{\beta}}{\sin((\pi/2)(\alpha - \beta))}$$
$$\times \left[\sin\left(\phi_n + \frac{\pi}{2}\alpha\right) + \omega_{cn}\cos\left(\phi_n + \frac{\pi}{2}\alpha\right)\right],$$
$$K_{\alpha n} = \frac{\omega_{cn}^{\alpha}}{\sin((\pi/2)(\alpha - \beta))}$$
$$\times \left[\sin\left(\phi_n + \frac{\pi}{2}\beta\right) + \omega_{cn}\cos\left(\phi_n + \frac{\pi}{2}\beta\right)\right].$$
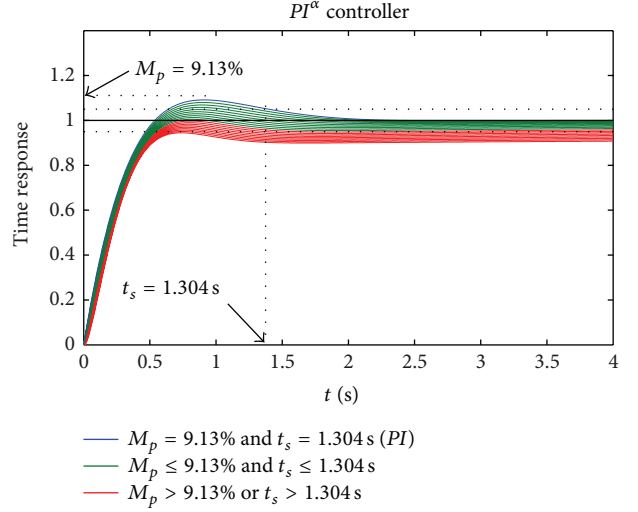$$(16)$$

FIGURE 2: Unity feedback control scheme.

A singular case is the I^{α} controller, which only has one term. In this case combination of (4) and (8) leads to

$$\frac{K_{\alpha n}}{(j\omega_{cn})^{\alpha}} = \frac{K_{\alpha n}}{\omega_{cn}^{\alpha}}e^{-j(\pi/2)\alpha} = z_r + jz_i \qquad (17)$$

which yields

$$\alpha = -\frac{2}{\pi}\arctan\left(\frac{z_i}{z_r}\right),$$
$$K_{\alpha n} = \omega_{cn}^{\alpha}\sqrt{z_r^2 + z_i^2}, \quad \text{if } z_r > 0, \qquad (18)$$
$$K_{\alpha n} = -\omega_{cn}^{\alpha}\sqrt{z_r^2 + z_i^2}, \quad \text{if } z_r < 0.$$

An additional condition is that z_i must have an opposite sign to z_r in order to obtain positive values for α (see the first equation of (18)).

Table 1 shows the tuning equations resulting from particularization of (15) to the controllers proposed in this paper.

Gains of the real controller $R(s)$ are obtained from $R(s) = (1/K)R_n(s')|_{s'=sT}$ being $K_{\alpha} = K_{\alpha n}/(KT^{\alpha})$ and $K_{\beta} = K_{\beta n}/(KT^{\beta})$. Fractional orders of $R(s)$ remain as α and β.

3. Results and Discussion

3.1. Problem Description. We will explain the problem with the following illustrative example: assume a normalized first order plant, $G_n(s)$, and the following normalized frequency specifications: $\omega_{cn} = 4$ rad/s and $\phi = 1.3$ rad. The standard PI controller provides a time response to a unity step command characterized by an overshoot of 9.13%, a settling time of 1.304 s, and a zero steady state error. The simulation of a PI^{α} controller, tuned by means of the same frequency requirements of the PI controller, provides the time responses showed in Figure 2, where we have colored in blue the time response of the PI controller, in green the time responses of the PI^{α} controller with overshoot and settling time less than

TABLE 1: Tuning equation resume.

Controller	Transfer function	Controller parameters	Tuning equations
PI	$R_{0n}(s) = K_{\beta n} + \dfrac{K_{\alpha n}}{s}$	$K_{\beta n}$ and $K_{\alpha n}$	$K_{\beta n} = z_r$ $K_{\alpha n} = -\omega_{cn} z_i$
I^{α}	$R_{1n}(s) = \dfrac{K_{\alpha n}}{s^{\alpha}}$	$K_{\alpha n}$ and α	$K_{\alpha n} = \text{sign}(z_r)\omega_{cn}^{\alpha}(z_r^2 + z_i^2)^{1/2}$ $\alpha = \dfrac{2}{\pi}\tan^{-1}\left(\dfrac{-z_i}{z_r}\right)$
PI^{α}	$R_{2n}(s) = \dfrac{K_{\beta n} s^{\alpha} + K_{\alpha n}}{s^{\alpha}}$	$K_{\beta n}$, $K_{\alpha n}$ and α	$K_{\beta n}(\alpha) = z_r + \dfrac{z_i}{\tan\left((\pi/2)\,\alpha\right)}$ $K_{\alpha n}(\alpha) = -\dfrac{\omega_{cn}^{\alpha}}{\sin\left((\pi/2)\,\alpha\right)} z_i$
II^{β}	$R_{3n}(s) = \dfrac{K_{\beta n} s^{1-\beta} + K_{\alpha n}}{s}$	$K_{\beta n}$, $K_{\alpha n}$ and β	$K_{\beta n}(\beta) = \dfrac{\omega_{cn}^{\beta}}{\cos\left((\pi/2)\,\beta\right)} z_r$ $K_{\alpha n}(\beta) = -\omega_{cn}\left(z_i + z_r \tan\left(\dfrac{\pi}{2}\beta\right)\right)$
$I^{\alpha}D^{1-\alpha}$	$R_{4n}(s) = \dfrac{K_{\beta n} s + K_{\alpha n}}{s^{\alpha}}$	$K_{\beta n}$, $K_{\alpha n}$ and α	$K_{\beta n}(\alpha) = \dfrac{1}{\omega_{cn}^{1-\alpha}}\left(z_r \sin\left(\dfrac{\pi}{2}\alpha\right) + z_i \cos\left(\dfrac{\pi}{2}\alpha\right)\right)$ $K_{\alpha n}(\alpha) = \omega_{cn}^{\alpha}\left(z_r \cos\left(\dfrac{\pi}{2}\alpha\right) - z_i \sin\left(\dfrac{\pi}{2}\alpha\right)\right)$

or equal to the PI ones, and in red the time responses of the PI^{α} controller with overshoot or settling time greater than the PI ones.

At this frequency design point, ω_{cn} = 4 rad/s and ϕ = 1.3 rad, the PI^{α} controller only guarantees the overshoot and the settling time of the PI controller (green colored time responses) when α is in the range $[0.52, 1]$. The other values of alpha, $[0, 0.52)$, make the time response of the controlled system exhibit $t_s > 1.304$ or $M_p > 9.13\%$.

The main objective of this work is to know, for the fractional-order structures presented in the previous section, the range of α that guarantees the time response requirements, overshoot and settling time, of the standard PI controller for each pair of frequency requirements $[\omega_{cn}, \phi]$.

3.2. Obtention of Frequency Specifications Region and Volumes. In this section the integer and fractional-order controllers, $R_{kn}(s)$, with $k = 0, \ldots, 4$, presented in the previous section (see Table 1), are simulated to control the generalized FO plant, $G_n(s)$. These controllers are designed taking into account the pairs of frequency specifications: normalized gain crossover frequency, ω_{cn}, and phase margin, ϕ.

In order to sweep up a wide range of normalized frequency specifications, which include the most of the realistic cases, the following range of variation has been simulated:

$$\omega_{cn} \in [0.1, 10] \text{ rad/s},$$
$$\phi \in [45, 90]° = \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \text{ rad.} \tag{19}$$

For each pair of these normalized frequency specifications, (ω_{cn}, ϕ), the PI controller is tuned by means of its tuning equation (see Table 1) resulting in a time response whose settling time, t_s, and overshoot, M_p, are stored as reference: $t_s^*(\omega_{cn}, \phi)$ and $M_p^*(\omega_{cn}, \phi)$.

Once the PI controller has been simulated, we proceed to simulate the fractional-order controllers I^{α}, PI^{α}, II^{β}, and $I^{\alpha}D^{1-\alpha}$ for each pair (ω_{cn}, ϕ), obtaining for each design point the time response parameters: $t_s(\omega_{cn}, \phi)$ and $M_p(\omega_{cn}, \phi)$.

Define the following functional:

$$\Delta_k(\omega_{cn}, \phi, \alpha) = \begin{cases} 1 & \text{if}\left(t_s(\omega_{cn}, \phi) \le t_s^*(\omega_{cn}, \phi)\right), \\ & \left(M_p(\omega_{cn}, \phi) \le M_p^*(\omega_{cn}, \phi)\right), \\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

where $k = 0, \ldots, 4$ and indicates the structure of the controller (see Table 1).

In the case of the fractional-order controller I^{α}, which only has two parameters to be tuned (K_{α} and α), Δ functional only depends on ω_{cn} and ϕ. Then we can define a *frequency specifications region* in (17) composed of all the points in which $\Delta(\omega_{cn}, \phi) = 1$. If the I^{α} controller is tuned by means of a pair of frequency specifications, ω_{cn} and ϕ, placed inside the $\Delta(\omega_{cn}, \phi)$ region, this controller will provide a time response with equal or better time specifications, t_s and M_p, than the one provided by the PI controller.

On the other hand, for fractional-order controllers PI^{α}, II^{β} and $I^{\alpha}D^{1-\alpha}$, which have three parameters to be tuned (K_{α}, K_{β}, and α), Δ functional depends on ω_{cn}, ϕ and α. Then we can define a *frequency specifications volume* in (17) composed of all the points in which $\Delta(\omega_{cn}, \phi, \alpha) = 1$. If these fractional-order controllers are tuned by means of a pair of frequency specifications, ω_{cn} and ϕ, and a value of α placed inside the $\Delta(\omega_{cn}, \phi, \alpha)$ volume, these controllers will provide time responses with equal or better time specifications, t_s and M_p, than the ones of the response provided by the PI controller.

3.2.1. Simulation Setup. This section shows the procedure to obtain the *frequency specifications region* $\Delta(\omega_{cn}, \phi)$ for the I^{α}
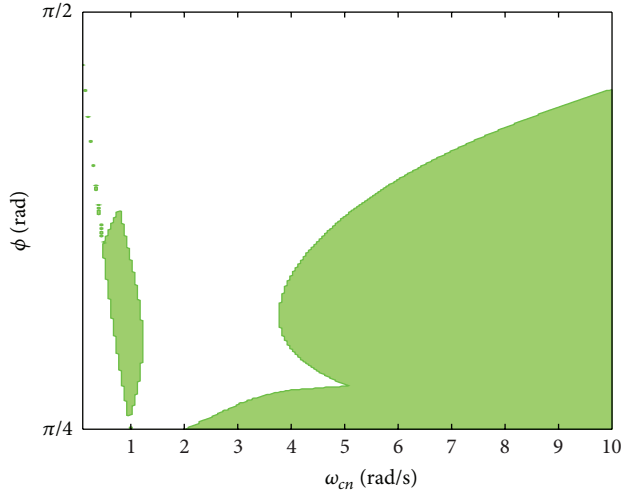
FIGURE 3: Frequency specifications region for I^α controller.

controller and the *frequency specifications volume* $\Delta(\omega_{cn}, \phi, \alpha)$ for the PI^α, II^β, and $I^\alpha D^{1-\alpha}$ controllers.

All the fractional operators of the controllers have been implemented using the Grunwald-Letnikov approximation without truncation (see, e.g., [11]). The structure of each controller has been modified, as shown in Table 2, in order to avoid that the implementation of the fractional-order integral action could provide a nonzero steady state error.

Simulations have been carried out by means of *MATLAB* with a simulation time $t_{sim} = 20/\omega_{cn}$ in order to ensure that the steady state regime has been reached. Each simulation has been carried out with the same number of samples, $n = 2000$, in order to guarantee a reasonable approximation of the fractional-order operators. We have also checked, for each controller, that its implemented structure (see Figure 2) provides the same time response with its original transfer function.

The resolution used to cover ranges (17) in the obtention of the *frequency specifications region* $\Delta(\omega_{cn}, \phi)$ and the *frequency specifications volumes* $\Delta(\omega_{cn}, \phi, \alpha)$ has been $\Delta\omega_{cn} = 0.1$, $\Delta\phi_m = 0.1°$, and the increment used in the sweepup of the α operators has been $\Delta\alpha = 0.05$.

3.2.2. Frequency Specifications Region for I^α Controller. Figure 3 represents the *frequency specifications region* obtained for the I^α controller.

All the I^α controllers which were tuned by means of any pair of frequency specifications, ω_{cn} and ϕ, contained in the filled zone provide the same or better time domain requirements, t_s and M_p, than the PI controller.

3.2.3. Frequency Specifications Volume for PI^α, II^β, and $I^\alpha D^{1-\alpha}$ Controllers. Figure 4 represents the *frequency specifications volume* obtained for the PI^α, II^β, and $I^\alpha D^{1-\alpha}$ controllers.

All the fractional-order controllers (PI^α, II^β, and $I^\alpha D^{1-\alpha}$) which were tuned by means of any pair of frequency specifications, ω_{cn} and ϕ, and a value of α contained in their respective volumes provide the same or better time response requirements, t_s and M_p, than the PI controller.

3.3. Frequency Specifications Region and Volumes Parametrization. In this section we propose simple parametrizations of the 2D region obtained for the I^α controller and the 3D volumes obtained for the PI^α, II^β, and $I^\alpha D^{1-\alpha}$ controllers.

The objective is to provide simple conditions with which one can check if the time response parameters achieved by the fractional-order controllers designed in the frequency domain fulfill the time requirements.

3.3.1. I^α: Frequency Specifications Region Parametrization. Figure 5 compares between the frequency specifications region of the I^α obtained from simulations in the previous section and the simple parametrization found.

The parametrization found for this region is

$$
\left(36.549\phi^2 + 27.248\phi\omega_{cn} - 98.160\phi + 26.150\omega_{cn}^2 \right.
$$
$$
\left. - 74.897\omega_{cn} + 83.317 \leq 1 \right)
$$
$$
\cup \left(18.494\phi^2 - 1.2716\phi\omega_{cn} - 31.371\phi + 0.074272\omega_{cn}^2 \right.
$$
$$
\left. + 0.17648\omega_{cn} + 17.185 \leq 1 \right)
$$
$$
\cup \left(24.608\phi - 16\phi^2 + 0.029160\omega_{cn}^2 \right.
$$
$$
\left. - 0.47876\omega_{cn} - 7.4967 \leq 1 \right)
$$
$$
\cup \left(\phi \leq 0.12526\omega_{cn} + 0.11539 \right).
$$
(21)

If the evaluation of (21) results '*true*', then $\Delta(\omega_{cn}, \phi) = 1$. This parametrization condenses the 96.41% of the points of the real region and provides an error less than 5% in the settling time and overshoot.

3.3.2. PI^α, II^β, and $I^\alpha D^{1-\alpha}$: Frequency Specifications Volume Parametrization. Figure 6 compares between the frequency specifications volume of the PI^α obtained from simulations in the previous Section and the simple parametrization found.
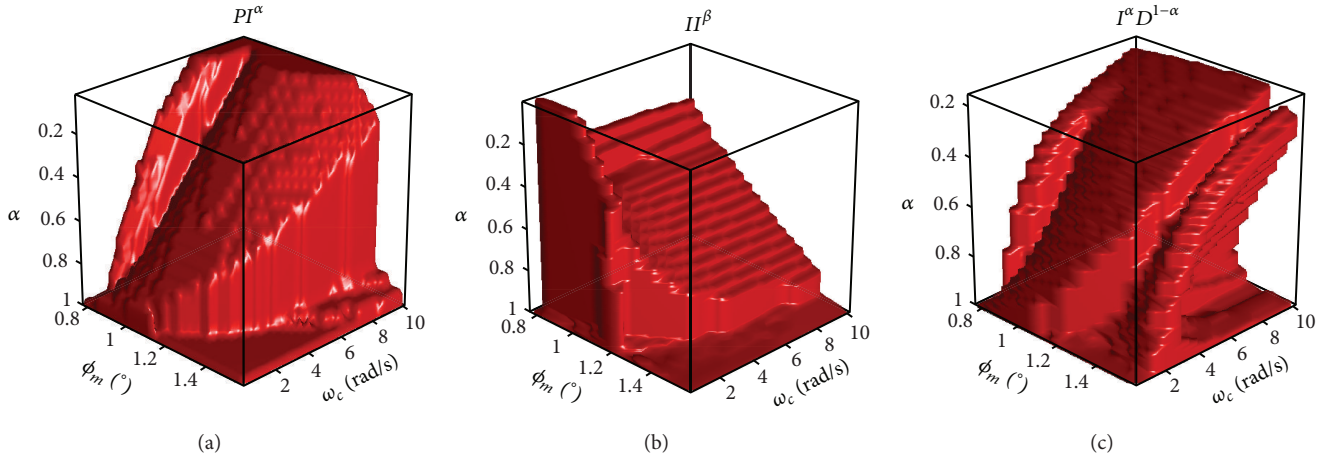
The parametrization of the PI^α volume yields

$$
\left(\frac{\omega_{cn} - 10.071}{9.1055} \right)^2 + \left(\frac{\phi - 0.92}{0.1635} \right)^2 + \left(\frac{\alpha - 1.640}{1.7125} \right)^2
$$
$$
\leq 1 \cup \left(\frac{\omega_{cn} - 9.555}{9.22\alpha + 1.2855} \right)^2 + \left(\frac{\phi - 1.1415 + 0.03\alpha}{0.04944\alpha + 0.2910} \right)^2
$$
$$
\leq 1.
$$
(22)

If the evaluation of (22) results *true*, then $\Delta(\omega_{cn}, \phi, \alpha) = 1$. Parametrization (22) condenses the 82.21% of the points of the real volume and provides an error less than 15% in the settling time and overshoot.

TABLE 2: Implementation of the controllers.

Controller	Original transfer function	Implemented structure	Operator to implement
I^α	$\dfrac{K_{\alpha n}}{s^\alpha}$	$\dfrac{K_{\alpha n}}{s} s^{1-\alpha}$	$D^{1-\alpha}$
PI^α	$\dfrac{K_{\beta n} s^\alpha + K_{\alpha n}}{s^\alpha}$	$K_{\beta n} + \dfrac{K_{\alpha n}}{s} s^\alpha$	$D^{1-\alpha}$
II^β	$\dfrac{K_{\beta n} s^{1-\beta} + K_{\alpha n}}{s}$	$\dfrac{K_{\beta n} s^{1-\beta} + K_{\alpha n}}{s}$	$D^{1-\beta}$
$I^\alpha D^{1-\alpha}$	$\dfrac{K_{\beta n} s + K_{\alpha n}}{s^\alpha}$	$\left(K_{\beta n} + \dfrac{K_{\alpha n}}{s} \right) s^{1-\alpha}$	$D^{1-\alpha}$

FIGURE 4: Frequency specifications volumes for PI^α, II^β, and $I^\alpha D^{1-\alpha}$ controllers.

The parametrization of the II^β volume yields

$$
\left(\frac{\omega_{cn} - 0.771}{0.5055} \right)^2 + \left(\frac{\phi - 0.792}{0.5435} \right)^2 + \left(\frac{\alpha - 1.64}{1.6725} \right)^2
$$

$$
\leq 1 \cup \left\{ \left(\frac{\omega_{cn} - 10.782}{12.5123} \right)^2 + \left(\frac{\phi - 0.759}{0.7132} \right)^2 \right. \tag{23}
$$

$$
\left. \leq 1 \cap \alpha - 0.8\phi \geq -0.3 \right\},
$$

where $\alpha = 1 - \beta$.

If the evaluation of (23) results *true* then $\Delta(\omega_{cn}, \phi, \alpha) = 1$. Parametrization (23) condenses the 87.98% of the points of the real volume and provides an error less than 15% in the settling time and overshoot.

The parametrization of the $I^\alpha D^{1-\alpha}$ volume yields

$$
\left(\frac{\omega_{cn} - 12.071}{12.1055} \right)^2 + \left(\frac{\phi - 0.792}{0.1135} \right)^2 + \left(\frac{\alpha - 1.64}{1.4725} \right)^2
$$

$$
\leq 1 \cup \left(\frac{\omega_{cn} - 11.989}{15.0911} \right)^2 + \left(\frac{\phi - 1.193}{0.5534} \right)^2 + \left(\frac{\alpha - 1.61}{1.4421} \right)^2
$$

$$
\leq 1 \cup \left(\frac{\omega_{cn} - 11.568}{12.1213} \right)^2 + \left(\frac{\phi - 1.521}{0.1132} \right)^2 + \left(\frac{\alpha - 1.63}{1.4426} \right)^2
$$

$$
\leq 1. \tag{24}
$$

If the evaluation of (24) results *true*, then $\Delta(\omega_{cn}, \phi, \alpha) = 1$. Parametrization (24) condenses the 84.73% of the points of the real volume and provides an error less than 15% in the settling time and overshoot.

3.4. Application Examples

3.4.1. Example 1. Assume the following *FO* plant:

$$
G(s) = \frac{2.65}{4.21\,s + 1} \tag{25}
$$

which closed-loop control has to verify the following time specifications when a unity step command is applied:

(i) steady-state error: $e_{ss} = 0$,

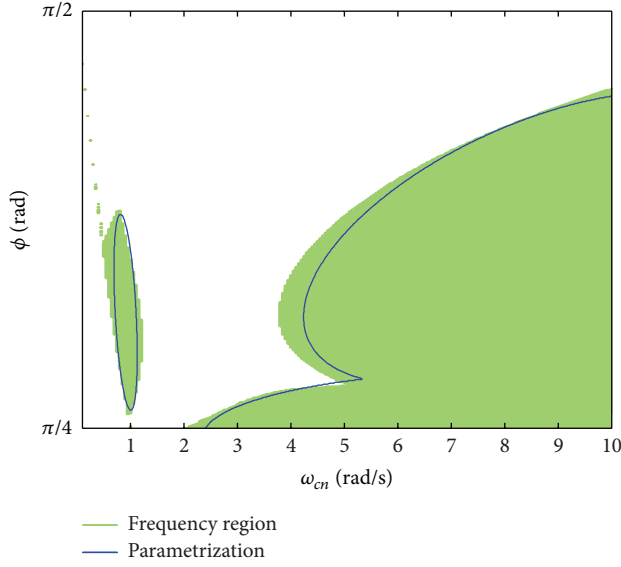(ii) overshoot: $M_p < 10\%$,

(iii) settling time: $t_s < 6$ s.

FIGURE 5: Parametrization of I^α region.

TABLE 3: Example 1 controllers.

Controller	Normalized transfer function
PI	$R_{0n}(s) = \dfrac{3.4636s + 8.2888}{s}$
PI^α	$R_{2n}(s) = \dfrac{1.3545s^{0.5} + 9.7104}{s^{0.5}}$
II^β	$R_{3n}(s) = \dfrac{9.7104s^{0.5} - 5.3232}{s}$
$I^\alpha D^{1-\alpha}$	$R_{4n}(s) = \dfrac{7.8117s + 0.4831}{s^{0.5}}$

The normalization of (25) is $y_n = y/2.65$ and $t_n = t/4.21$ yielding $G_n(s)$ (2). The settling time, t_s, must be also normalized resulting in $t_{sn} = 6/4.21 = 1.42$.

The first step consists of obtaining the equivalent frequency specifications (ω_{cn} and ϕ) from time specifications for the PI controller. The resulting frequency specifications are $\omega_{cn} = 3.93$ rad/s and $\phi = 1.273$ rad.

Using these frequency specifications in the tuning equation of Table 1, we obtain the PI controller and the fractional-order controllers PI^α, II^β, and $I^\alpha D^{1-\alpha}$. The evaluation of $\Delta(\omega_{cn}, \phi, \alpha)$ by means of ((22)–(24)) states that the minimum value of *alpha* for guaranteeing the time domain requirements, $t_{sn} < 6/4.21 = 1.42$ s and $M_p < 10\%$, is $\alpha = 0.55$ for the PI^α controller, $\alpha = 0.72$ for the II^β controller, and $\alpha = 0.45$ for the $I^\alpha D^{1-\alpha}$. In this way if we choose, for example, a value of $\alpha = 0.5$, only the $I^\alpha D^{1-\alpha}$ controller guarantees that $t_{sn} < 1.42$ and $M_p < 10\%$. Figure 7 represents the time response of the resulting controllers. Note that only $I^\alpha D^{1-\alpha}$ controller guarantees the time requirements provided by the PI controller. In the case of the II^β controller the system becomes unstable. All the obtained controllers are summarized in Table 3.

TABLE 4: Example 2 controllers.

Controller	Normalized transfer function
PI	$R_{0n}(s) = \dfrac{6.4875s + 37.7482}{s}$
PI^α	$R_{2n}(s) = \dfrac{18.9095s^{0.5} + 1.7512}{s^{0.5}}$
II^β	$R_{3n}(s) = \dfrac{25.9015s^{0.5} - 13.9576}{s}$
$I^\alpha D^{1-\alpha}$	$R_{4n}(s) = \dfrac{22.4055s + 0.43864}{s^{0.5}}$

3.4.2. Example 2. Assume that the same plant (25) is controlled with the following time requirements to a unity step command:

  (i) steady-state error: $e_{ss} = 0$,

 (ii) overshoot: $M_p < 20\%$,

(iii) settling time: $t_s < 3$ s.

Obtaining the equivalent frequency normalized specifications (ω_{cn} and ϕ) from time specifications for the PI controller, the resulting frequency specifications are $\omega_{cn} = 7.97$ rad/s and $\phi = 1.065$ rad.

Using these frequency specifications in the tuning equation of Table 1, we obtain the PI controller and the fractional-order controllers PI^α, II^β, and $I^\alpha D^{1-\alpha}$. The evaluation of $\Delta(\omega_{cn}, \phi, \alpha)$ by means of ((22)–(24)) states that the minimum value of *alpha* for guaranteeing the time domain requirements, $t_{sn} < 3/4.21 = 0.712$ s and $M_p < 20\%$, is $\alpha = 0.20$ for the PI^α controller, $\alpha = 0.56$ for the II^β controller, and $\alpha = 0.3$ for the $I^\alpha D^{1-\alpha}$. In this way if we choose, for example, a value of $\alpha = 0.5$, all the tuned controllers will guarantee $t_{sn} < 0.712$ and $M_p < 20\%$, except the II^β. Figure 8 represents the time response of the resulting controllers. Note that only II^β does not fulfill the required time specifications. In the same way as the previous example this controller makes the system unstable. All the obtained controllers are summarized in Table 4.

4. Conclusions

This work presents a simulation based analysis of the time specification fulfillment of fractional-order controllers with integral action for controlling a first order plant.

The comparative analysis is applied over several fractional-order controllers: I^α, PI^α, II^β, and $I^\alpha D^{1-\alpha}$. The first one has only two parameters to be tuned while the others have three.

The first step consists on using the conventional PI controller as a pattern to obtain the frequency specifications, ω_c and ϕ, with which the time specifications are guaranteed. Assuming a tuning method based on the previous frequency specifications for all the compared controllers, the set of frequency specifications points (ω_{cn} and ϕ) which allows the controllers to reach the time specifications have been obtained by means of simulations.
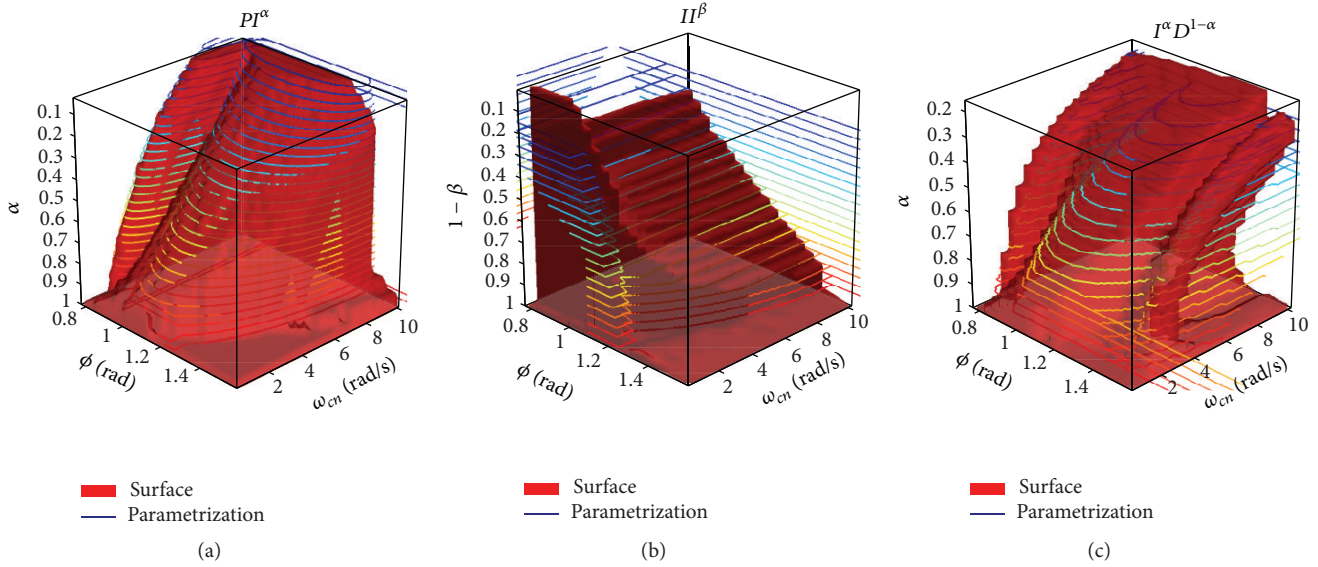
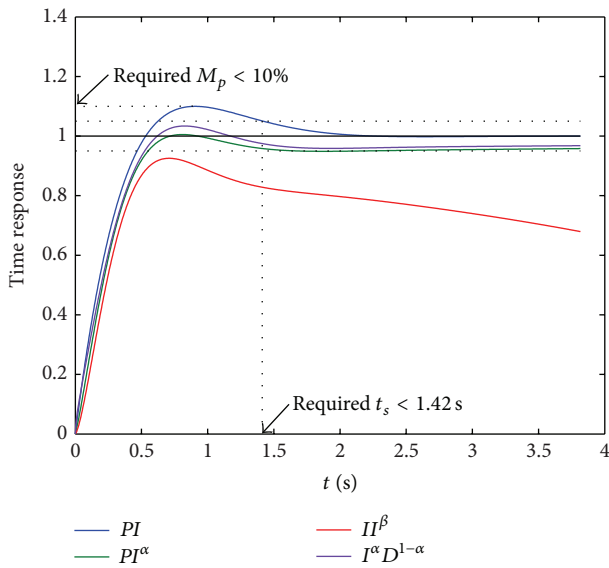FIGURE 6: Parametrization of PI^{α} volume.
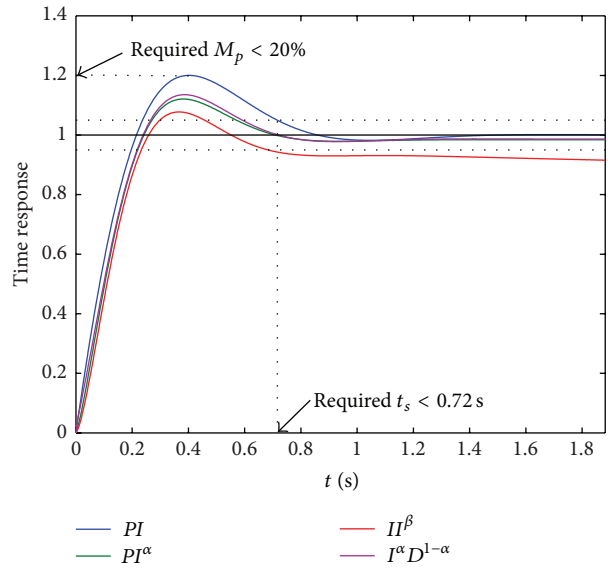

FIGURE 7: Example 1: nominal time responses.

FIGURE 8: Example 2: nominal time responses.

For the I^{α} controller, the result is a set of points, denominating the *frequency specifications region*, which allows for knowing if this fractional-order controller preserves the time specifications of the PI controller.

In the cases of the PI^{α}, II^{β}, and $I^{\alpha}D^{1-\alpha}$ controllers, the result depends on the fractional-order of the integral and/or derivative action of the controller. The set of points provides a *frequency specification volume* which again allows for knowing if these fractional-order controllers preserve the time specification of the PI controller.

The *frequency specifications region* of the I^{α} controller and the *volume region* of the PI^{α}, II^{β}, and $I^{\alpha}D^{1-\alpha}$ controllers have been parameterized by means of simple equations

obtaining a reasonable error between 5% and 15% in the time requirements.

Finally, two application examples have been detailed in order to illustrate the application of the *frequency specifications region* and *frequency specifications volume*. In the first one only the $I^{\alpha}D^{1-\alpha}$ fractional-order controller preserves the time specifications, while in the second one all of them, except the II^{β} controller, preserve them.

The work presented here provides a tool to optimize fractional-order controllers, for example, to maximize the gain margin or minimize energy consumption, and simultaneously to preserve the required nominal time response.

In further work, we will extend this methodology to plants with time delay terms.

Acknowledgments

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