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Research Article

An Alternative Method for the Study of Impulsive Differential Equations of Fractional Orders in a Banach Space

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This paper is concerned with the existence, uniqueness, and stability of the solution of some impulsive fractional problem in a Banach space subjected to a nonlocal condition. Meanwhile, we give a new concept of a solution to impulsive fractional equations of multiorders. The derived results are based on Banach's contraction theorem as well as Schaefer's fixed point theorem.

1. Introduction

It is well known that the theory of fractional calculus deals with the concepts of differentiation and integration of arbitrary orders, real and complex. Actually, the real importance of fractional derivatives lies in their nonlocal character which gives rise to a long memory effect and thus to a better insight into the modelled processes. On the other hand, since models using classical derivatives are just a special case of those using fractional derivatives, then most of the investigators in different areas such as electronics, viscoelasticity, satellite guidance, medicine, anomalous diffusion, signal processing, and many other branches of science and technology have revisited some classical dynamic systems in the framework of fractional derivatives to get better results; see the references [1–8]. We point out that most of dynamic systems are naturally governed by fractional differential equations; for further applications of fractional derivatives in other areas and useful backgrounds we refer the reader to the works [1–5, 7–12].

As far as we are concerned with impulsive fractional differential equations, we intend to improve and correct in this paper some existence results established earlier in [4, 13–18] for impulsive fractional differential equations. There have been in the last couple of years several concepts of solutions satisfying some fractional equations subjected to impulsive conditions, see [13, 14, 18, 19], while the authors of [18] claimed that their new concept is the more realistic than the existing ones. Actually, we believe that nobody holds all the truth about this subject and a lot of dark sides of these approaches are not yet well elucidated.

Regarding the concept of a solution for impulsive fractional equations introduced by [18] we point out that Lemma 2.6 which has been used by the authors to obtain the equivalence between an impulsive fractional problem and an integral equation is false as we see in the following counterexample.

In the famous book of Nagy and Riesz [20, page 48], there is an example of monotonic continuous function $F : [0,1] \to \mathbb{R}$ which is not constant in any subinterval of $[0,1]$ and satisfies $F' = 0$, almost everywhere in $[0,1]$. So, in terms of Caputo's derivative we would have formally for any $\alpha \in (0,1)$

$$
{}^{C}D_{0^+}^{\alpha} F(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} F'(s)\, ds = 0, \quad t \in [0,1],
$$

$$
F(a) = F_0, \quad 0 < a < 1, \text{ with } F_0 \text{ being the value of } F \text{ at } a. \tag{1}
$$

However, there is no apparent equivalence between this problem and the fractional integral representation of F defined in Lemma 2.6 [18]; otherwise the function $F(t)$ would be constant and equal to F_0 throughout the interval $[0,1]$

which is a contradiction! Moreover, since in the same work Lemma 2.7 is based on the latter lemma then it is not correct and may lead to apparent contradiction.

Our main contribution in this paper is the study of new fractional problems of several orders in a Banach space subjected to some impulsive conditions of the form

$$\begin{aligned} &{}^{C}D^{\alpha_k}_{t_k^+} u(t) \\ &\quad = A(t,u)\,u(t) + F\Bigg(t, u(t), \\ &\qquad\qquad \int_a^t h(t,s,u(\sigma(s)))\,ds, \\ &\qquad\qquad \int_a^{t_{k+1}} k(t,s,u(\tau(s)))\,ds\Bigg), \\ &\qquad\qquad t \in J_k,\ k = 0,\dots,m, \\ &u(a) = u_0 \in E, \\ &u(t_k^+) = u(t_k^-) + I_k(u(t_k^-)), \quad k = 1,\dots,m. \end{aligned} \tag{2}$$

Let us first give a concrete example of such a problem in $\mathbb{R}$; namely;

$$\begin{aligned} {}^{C}D^{\alpha}_{0^+} u(t) &= t^p - 1, \quad t \in J_0 = [0,1], \\ {}^{C}D^{\beta}_{1^+} u(t) &= (t-1)^q, \quad t \in J_1 = (1,T], \\ u(0) &= 1, \\ u(1^+) &= u(1^-) + 2, \end{aligned} \tag{3}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, $T > 1$, and $p, q \in \mathbb{R}^+$. So, we look for a piecewise continuous function $u : [0,T] \to \mathbb{R}$ satisfying (3). Solving the subproblem

$$\begin{aligned} {}^{C}D^{\alpha}_{0^+} u(t) &= t^p - 1, \quad t \in J_0, \\ u(0) &= 1, \end{aligned} \tag{4}$$

we obtain

$$\begin{aligned} u(t) &= 1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^p\,ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\,ds \\ &= 1 + \frac{\Gamma(p+1)}{\Gamma(\alpha+p+1)} t^{\alpha+p} - \frac{1}{\Gamma(\alpha+1)} t^{\alpha}, \end{aligned} \tag{5}$$

from which we get $u(1) = 1 + (\Gamma(p+1)/\Gamma(\alpha+p+1)) - (1/\Gamma(\alpha+1))$.

On the other hand, the solution of the subproblem

$$\begin{aligned} {}^{C}D^{\beta}_{1^+} u(t) &= (t-1)^q, \quad t \in J_1, \\ u(1^+) &= u(1) + 2 = 3 + \frac{\Gamma(p+1)}{\Gamma(\alpha+p+1)} - \frac{1}{\Gamma(\alpha+1)} \end{aligned} \tag{6}$$

is given by

$$\begin{aligned} u(t) &= u(1^+) + \frac{1}{\Gamma(\beta)} \int_1^t (t-s)^{\beta-1} (s-1)^q\,ds \\ &= 3 + \frac{\Gamma(p+1)}{\Gamma(\alpha+p+1)} - \frac{1}{\Gamma(\alpha+1)} \\ &\quad + \frac{\Gamma(q+1)}{\Gamma(\beta+q+1)} (t-1)^{\beta+q}. \end{aligned} \tag{7}$$

Hence, the piecewise continuous function

$$u(t) = \begin{cases} 1 + \dfrac{\Gamma(p+1)}{\Gamma(\alpha+p+1)} t^{\alpha+p} - \dfrac{1}{\Gamma(\alpha+1)} t^{\alpha}, & t \in J_0, \\ 3 + \dfrac{\Gamma(p+1)}{\Gamma(\alpha+p+1)} - \dfrac{1}{\Gamma(\alpha+1)} & \\ \quad + \dfrac{\Gamma(q+1)}{\Gamma(\beta+q+1)} (t-1)^{\beta+q}, & t \in J_1, \end{cases} \tag{8}$$

is a solution to the impulsive fractional problem (3).

A particular problem of (3) is as follows:

$$\begin{aligned} {}^{C}D^{\gamma}_{0^+} u(t) &= t^p - 1, \quad t \in J_0, \\ {}^{C}D^{\gamma}_{1^+} u(t) &= (t-1)^q, \quad t \in J_1, \\ u(0) &= 1, \\ u(1^+) &= u(1^-) + 2 \end{aligned} \tag{9}$$

corresponding to the case $\gamma = \alpha = \beta$ whose solution is

$$u(t) = \begin{cases} 1 + \dfrac{\Gamma(p+1)}{\Gamma(\gamma+p+1)} t^{\gamma+p} - \dfrac{1}{\Gamma(\gamma+1)} t^{\gamma}, & t \in J_0, \\ 3 + \dfrac{\Gamma(p+1)}{\Gamma(\gamma+p+1)} - \dfrac{1}{\Gamma(\gamma+1)} & \\ \quad + \dfrac{\Gamma(q+1)}{\Gamma(\gamma+q+1)} (t-1)^{\gamma+q}, & t \in J_1. \end{cases} \tag{10}$$

The paper is organized as follows. We present in Section 2 our problem as we establish some equivalence between the the given problem and a nonlinear integral equation. Next, we state a piecewise-continuous type of the Ascoli-Arzela theorem as well as Schaefer's fixed point theorem in order to apply them subsequently in our proofs. In Section 3 we use the Banach contraction theorem to establish an existence and uniqueness theorem of a quasilinear impulsive fractional problem in an abstract Banach space. In Section 4 we apply Schaefer's fixed point theorem to some semilinear impulsive fractional problem in a finite dimensional Banach space to obtain the existence of a piecewise continuous solution; on the other hand we prove the stability of the obtained solution with respect to the initial value. Finally, we conclude the paper by a concrete example illustrating one of our results.

2. Preliminaries

The main purpose of this paper is the investigation of the existence and uniqueness of solution corresponding to the following impulsive fractional integrodifferential equation in a Banach space $(E, \|\cdot\|)$

$$\begin{aligned} {}^{C}D_{t_k^+}^{\alpha_k} u(t) &= A(t,u)\,u(t) + F\Big(t, u(t), \\ &\int_a^t h(t,s,u(\sigma(s)))\,ds, \\ &\int_a^{t_{k+1}} k(t,s,u(\tau(s)))\,ds\Big), \\ &t \in J_k,\ k = 0,\dots,m, \\ u(a) &= u_0 \in E, \\ u(t_k^+) &= u(t_k^-) + I_k(u(t_k^-)), \quad k = 1,\dots,m, \end{aligned} \tag{11}$$

where

(i) $J = [a,T]$ with $0 \le a < T < \infty$ and $J_0 = [a,t_1]$, $J_k = (t_k, t_{k+1}]$; $k = 1,\dots,m$,

(ii) ${}^{C}D_{t_k^+}^{\alpha_k}$ is Caputo's fractional derivative of order $\alpha_k \in (0,1)$, $k = 0,\dots,m$,

(iii) $A : J \times E \to \mathscr{B}(E)$ is a continuous operator, where $\mathscr{B}(E)$ is the Banach space of bounded linear operators on E in itself,

(iv) $I_k : E \to E$, $t_0 = a < t_1 < \cdots < t_m < t_{m+1} = T$; $u(t_k^+) = \lim_{h\to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim_{h\to 0^-} u(t_k + \epsilon) = u(t_k)$ are, respectively, the right and left limits of $u(t)$ at the discontinuity point $t = t_k$.

We set the following hypotheses:

(j) the functions $\sigma, \tau : J \to J$ are continuous with $a \le \sigma(t) \le t$ and $a \le \tau(t) \le t$, for every $t \in J$,

(jj) the nonlinear function $F : J \times E \times E \times E \to E$ is continuous, and

$$\begin{aligned} &h : D \times E \longrightarrow E, \quad D = \left\{(t,s) \in \mathbb{R}^2 : a \le s \le t \le T\right\}, \\ &k : D_0 \times E \longrightarrow E, \\ &\text{where } D_0 = \left\{(t,s) \in \mathbb{R}^2 : t \in J, a \le s \le T\right\}, \end{aligned} \tag{12}$$

are two continuous functions over $D \times E$ and $D_0 \times E$, respectively.

We will use in the sequel the following notation:

$$\begin{aligned} Hu(t) &= \int_a^t h(t,s,u(\sigma(s)))\,ds, \\ K_{t_{k+1}} u(t) &= \int_a^{t_{k+1}} k(t,s,u(\tau(s)))\,ds, \quad k = 0,1,\dots,m, \\ \Phi_{t_{k+1}}(t,u(t)) &= F\left(t, u(t), Hu(t), K_{t_{k+1}} u(t)\right), \\ &\qquad k = 0,1,\dots,m. \end{aligned} \tag{13}$$

We recall that $\mathscr{C} = \mathscr{C}(J;E)$ is the Banach space of continuous functions $u : J \to E$ endowed with the norm

$$\|u\|_{\mathscr{C}} = \sup_{t\in J} \|u(t)\|. \tag{14}$$

Next, we introduce the definition of the fractional derivative in the sense of Caputo. We have the following.

Definition 1. We define the left-sided fractional Riemann-Liouville integral of order $\alpha \in (0,1)$ of a function $f : [c,d] \to E$ as follows:

$$J_{c^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} f(s)\,ds, \quad t > c. \tag{15}$$

We define the left-sided fractional derivative of order $\alpha \in (0,1)$ of a function $f : [c,d] \to E$ in the sense of Caputo by

$${}^{C}D_{c^+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_c^t (t-s)^{-\alpha} f'(s)\,ds, \quad t > c. \tag{16}$$

Remark 2. (1) We point out that the previous integrals are understood in the sense of Bochner.

(2) We assume of course that the function f satisfies the necessary conditions for which those integrals are well defined.

Next, we consider the linear functional space

$$\begin{aligned} \mathscr{PC}(J;E) &= \{u : J \longrightarrow E,\ u \in \mathscr{C}((t_k, t_{k+1}];E), \\ &\quad k = 0,\dots,m \text{ s.t. } u(t_k^-) \text{ and } u(t_k^+) \\ &\quad \text{exist with } u(t_k^-) = u(t_k),\ k = 1,\dots,m\} \end{aligned} \tag{17}$$

equipped with the norm

$$\|u\|_{\mathscr{PC}} = \sup_{t\in J} \|u(t)\|. \tag{18}$$

We obtain a Banach space $(\mathscr{PC}(J;E), \|\cdot\|_{\mathscr{PC}})$.

Now, we recall the definition of the solution of the problem (11).

Definition 3. A function $u \in \mathscr{PC}(J;E)$ is said to be a solution of the problem (11) if ${}^{C}D_{t_k^+}^{\alpha_k} u(t)$ exists in J_k, for $k = 0,\dots,m$, and satisfies

(i) the equation ${}^{C}D_{t_k^+}^{\alpha_k} u(t) = A(t,u)u(t) + \Phi_{t_{k+1}}(t,u(t))$ in J_k, $k = 0,\dots,m$,

(ii) the initial condition $u(a) = u_0$,

(iii) the impulsive conditions $u(t_k^+) = u(t_k^-) + I_k(u(t_k^-))$, $k = 1,\dots,m$.

Lemma 4. *A function $u \in \mathscr{PC}(J;E)$ satisfies the following nonlinear integral equation*

$$u(t)=\begin{cases} u_0+\dfrac{1}{\Gamma(\alpha_0)}\displaystyle\int_a^t (t-s)^{\alpha_0-1}A(s,u)\,u(s)\,ds \\ \quad +\dfrac{1}{\Gamma(\alpha_0)}\displaystyle\int_a^t (t-s)^{\alpha_0-1}\Phi_{t_1}(s,u)\,ds, \\ \qquad\qquad t\in[a,t_1], \\ u_0+\displaystyle\sum_{i=1}^{k} I_i\left(u\left(t_i^-\right)\right) \\ \quad +\dfrac{1}{\Gamma(\alpha_k)}\displaystyle\int_{t_k}^t (t-s)^{\alpha_k-1}A(s,u)\,u(s)\,ds \\ \quad +\displaystyle\sum_{i=1}^{k}\dfrac{1}{\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left(t_i-s\right)^{\alpha_{i-1}-1}A(s,u)\,u(s)\,ds \\ \quad +\dfrac{1}{\Gamma(\alpha_k)}\displaystyle\int_{t_k}^t (t-s)^{\alpha_k-1}\Phi_{t_{k+1}}(s,u)\,ds \\ \quad +\displaystyle\sum_{i=1}^{k}\dfrac{1}{\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left(t_i-s\right)^{\alpha_{i-1}-1}\Phi_{t_i}(s,u)\,ds, \\ \qquad\qquad t\in J_k,\ k=1,\dots,m, \end{cases} \tag{19}$$

if and only if it is a solution to problem (11).

Proof. Since we have $u(t_k^-)=u(t_k)$, then $u(t_k^+)=u(t_k)+I_k(u(t_k^-))$.

Now, for $t\in J_0=[a,t_1]$, the solution of the problem

$$\begin{gathered} {}^{c}D_{a^+}^{\alpha_0}u(t)=A(t,u)\,u(t)+\Phi_{t_1}(t,u),\quad t\in J_0, \\ u(a)=u_0\in E \end{gathered} \tag{20}$$

is given by

$$\begin{aligned} u(t)&=u_0+\frac{1}{\Gamma(\alpha_0)}\int_a^t (t-s)^{\alpha_0-1}A(s,u)\,u(s)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_0)}\int_a^t (t-s)^{\alpha_0-1}\Phi_{t_1}(s,u)\,ds,\quad t\in J_0. \end{aligned} \tag{21}$$

We have for $t=t_1$ the following relation $u(t_1^+)=u(t_1)+I_1(u(t_1^-))$, and so

$$\begin{aligned} u\left(t_1^+\right)&=\frac{1}{\Gamma(\alpha_0)}\int_a^{t_1}\left(t_1-s\right)^{\alpha_0-1}A(s,u)\,u(s)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_0)}\int_a^{t_1}\left(t_1-s\right)^{\alpha_0-1}\Phi_{t_1}(s,u)\,ds \\ &\quad+u_0+I_1\left(u\left(t_1^-\right)\right). \end{aligned} \tag{22}$$

Next, for $t\in J_1=(t_1,t_2]$, we have

$$\begin{aligned} u(t)&=u\left(t_1^+\right)+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t (t-s)^{\alpha_1-1}A(s,u)\,u(s)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t (t-s)^{\alpha_1-1}\Phi_{t_2}(s,u)\,ds \\ &=u_0+\frac{1}{\Gamma(\alpha_0)}\int_a^{t_1}\left(t_1-s\right)^{\alpha_0-1}A(s,u)\,u(s)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_0)}\int_a^{t_1}\left(t_1-s\right)^{\alpha_0-1}\Phi_{t_1}(s,u)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t (t-s)^{\alpha_1-1}A(s,u)\,u(s)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t (t-s)^{\alpha_1-1}\Phi_{t_2}(s,u)\,ds+I_1\left(u\left(t_1^-\right)\right) \end{aligned} \tag{23}$$

from which we infer that

$$\begin{aligned} u\left(t_2^+\right)&=u_0+\frac{1}{\Gamma(\alpha_0)}\int_a^{t_1}\left(t_1-s\right)^{\alpha_0-1}A(s,u)\,u(s)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_0)}\int_a^{t_1}\left(t_1-s\right)^{\alpha_0-1}\Phi_{t_1}(s,u)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2}\left(t_2-s\right)^{\alpha_1-1}A(s,u)\,u(s)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2}\left(t_2-s\right)^{\alpha_1-1}\Phi_{t_2}(s,u)\,ds \\ &\quad+I_1\left(u\left(t_1^-\right)\right)+I_2\left(u\left(t_2^-\right)\right). \end{aligned} \tag{24}$$

Arguing as before we obtain for $t\in J_2$

$$\begin{aligned} u(t)&=u\left(t_2^+\right)+\frac{1}{\Gamma(\alpha_2)}\int_{t_2}^t (t-s)^{\alpha_2-1}A(s,u)\,u(s)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_2)}\int_{t_2}^t (t-s)^{\alpha_2-1}\Phi_{t_3}(s,u)\,ds \\ &=u_0+\frac{1}{\Gamma(\alpha_0)}\int_a^{t_1}\left(t_1-s\right)^{\alpha_0-1}A(s,u)\,u(s)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_0)}\int_a^{t_1}\left(t_1-s\right)^{\alpha_0-1}\Phi_{t_1}(s,u)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2}\left(t_2-s\right)^{\alpha_1-1}A(s,u)\,u(s)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2}\left(t_2-s\right)^{\alpha_1-1}\Phi_{t_2}(s,u)\,ds \\ &\quad+\frac{1}{\Gamma(\alpha_2)}\int_{t_2}^t (t-s)^{\alpha_2-1}A(s,u)\,u(s)\,ds \end{aligned}$$

$$+\frac{1}{\Gamma(\alpha_2)}\int_{t_2}^{t}(t-s)^{\alpha_2-1}\Phi_{t_3}(s,u)\,ds$$
$$+I_1(u(t_1^-))+I_2(u(t_2^-)). \tag{25}$$

Reasoning by induction we get, for any $t\in J_k$, $k=1,\ldots,m$, the general expression

$$\begin{aligned} u(t) &= u_0+\frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}A(s,u)u(s)\,ds \\ &+\sum_{i=1}^{k}\frac{1}{\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha_{i-1}-1}A(s,u)u(s)\,ds \\ &+\frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}\Phi_{t_{k+1}}(s,u)\,ds \\ &+\sum_{i=1}^{k}\frac{1}{\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha_{i-1}-1}\Phi_{t_i}(s,u)\,ds \\ &+\sum_{i=1}^{k}I_i(u(t_i^-)). \end{aligned} \tag{26}$$

Conversely, we assume that u satisfies (19). If $t=a$, then $u(a)=u_0$.

Now, using the fact that Caputo's derivative of a constant is zero, then, for every $t\in J_k$, $k=0,\ldots,m$, we get

$$\begin{aligned} &{}^{C}D_{t_k^+}^{\alpha_k}u(t) \\ &={}^{C}D_{t_k^+}^{\alpha_k}\left[\frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}A(s,u)u(s)\,ds\right] \\ &\quad+{}^{C}D_{t_k}^{\alpha_k}\left[\frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}\Phi_{t_{k+1}}(s,u)\,ds\right] \\ &={}^{C}D_{t_k^+}^{\alpha_k}\left(J_{t_k^+}^{\alpha_k}A(t,u)u(t)\right)+{}^{C}D_{t_k^+}^{\alpha_k}\left(J_{t_k^+}^{\alpha_k}\Phi_{t_{k+1}}(t,u)\right). \end{aligned} \tag{27}$$

So

$${}^{C}D_{t_k^+}^{\alpha_k}u(t)=A(t,u)u(t)+\Phi_{t_{k+1}}(t,u), \tag{28}$$

for every $t\in J_k$, $k=0,\ldots,m$.

Also we can easily show that

$$u(t_k^+)=u(t_k)+I_k(u(t_k^-)),\quad k=1,\ldots,m. \tag{29}$$

□

We conclude this section by introducing some useful theorems which will be used in the sequel.

Theorem 5 ($\mathscr{PC}$-type Ascoli-Arzela theorem [21]). *Let E be a Banach space and $\mathscr{W}\subset\mathscr{PC}(J,E)$. If the following conditions are satisfied*

(i) *$\mathscr{W}$ is a uniformly bounded subset of $\mathscr{PC}(J,E)$;*

(ii) *$\mathscr{W}$ is equicontinuous in (t_k,t_{k+1}), $k=0,1,2,\ldots,m$;*

(iii) *$\mathscr{W}(t)=\{u(t):u\in\mathscr{W},\ t\in J\setminus\{t_k\}\}$, $\mathscr{W}(t_k^+)=\{u(t_k^+):u\in\mathscr{W}\}$, and $\mathscr{W}(t_k^-)\equiv\{u(t_k^-):u\in\mathscr{W}\}$ are relatively compact subsets of E,*

then $\mathscr{W}$ is a relatively compact subset of $\mathscr{PC}(J,E)$.

Theorem 6 (Schaefer's fixed point theorem). *Let E be a Banach space and let $\mathscr{T}:E\to E$ be a completely continuous operator. If the set*

$$X=\{u\in E:u=\lambda\mathscr{T}u,\ \lambda\in(0,1)\} \tag{30}$$

is bounded, then $\mathscr{T}$ has at least a fixed point.

3. A Quasilinear Impulsive Fractional Problem

We begin our investigation by the following result which ensures the existence and the uniqueness of the solution of the following impulsive quasilinear problem:

$$\begin{aligned} &{}^{C}D_{t_k^+}^{\alpha_k}u(t) \\ &\quad=A(t,u)u(t)+\Phi_{t_{k+1}}(t,u(t)),\quad t\in J_k,\ k=0,\ldots,m, \\ &u(a)=u_0\in E, \\ &u(t_k^+)=u(t_k)+I_k(u(t_k^-)),\quad k=1,\ldots,m. \end{aligned} \tag{31}$$

We assume that $A:J\times E\to\mathscr{B}(E)$ is continuous and there exists a constant $M>0$ such that

$$\|A(t,u)-A(t,v)\|\le M\|u-v\|,\quad\forall t\in J,\ \forall u,v\in E. \tag{32}$$

We set $M'=\max_{t\in J}\|A(t,0)\|$.

It is not hard to establish the following estimates.

Lemma 7. *Let the functions $h(t,s,u)$ and $k(t,s,u)$ be continuous with respect to the variables s and t, and there are two positive constants C_1 and C_2 such that*

$$\begin{aligned} &\|h(t,s,u)-h(t,s,v)\|\le C_1\|u-v\|, \\ &\qquad\forall t,s\in J,\ \forall u,v\in E, \\ &\|k(t,s,u)-k(t,s,v)\|\le C_2\|u-v\|, \\ &\qquad\forall t,s\in J,\ \forall u,v\in E. \end{aligned} \tag{33}$$

Then, there exist two positive constants C_1' and C_2' so that

$$\begin{aligned} &\|Hu(t)\|\le(T-a)\left(C_1\|u\|_{\mathscr{PC}}+C_1'\right), \\ &\|Hu(t)-Hv(t)\|\le C_1(T-a)\|u-v\|_{\mathscr{PC}}, \end{aligned} \tag{34}$$

and, for $k=0,\ldots,m$, one has

$$\begin{aligned} &\left\|K_{t_{k+1}}u(t)\right\|\le(T-a)\left(C_2\|u\|_{\mathscr{PC}}+C_2'\right), \\ &\left\|K_{t_{k+1}}u(t)-K_{t_{k+1}}v(t)\right\|\le C_2(T-a)\|u-v\|_{\mathscr{PC}}, \end{aligned} \tag{35}$$

for every $u,v\in\mathscr{PC}(J,E)$ and $t\in J$.

We assume the following hypotheses:

(H1) $\alpha_0, \ldots, \alpha_m \in (0,1)$. We set $T' = \max_{0\le i\le m}\{(T-a)^{\alpha_i}\}$ and $\Gamma' = \min_{0\le i\le m}\{\Gamma(\alpha_i+1)\}$.

(H2) There is a positive constant L_1 such that

$$\begin{aligned}&\left\|\Phi_{t_{k+1}}(t,u) - \Phi_{t_{k+1}}(t,v)\right\| \\ &\quad \le \{L_1 + (C_1 + C_2)(T-a)\}\|u-v\|_{\mathscr{PC}}, \\ &\quad \forall t \in J,\ \forall u, v \in \mathscr{PC},\ k = 0,\ldots,m.\end{aligned} \tag{36}$$

We set $L = L_1 + (C_1 + C_2)(T-a)$ and $L_2 = \sup_{t\in J}\|F(t,0,0,0)\|$.

(H3) There is a positive constant $\mu > 0$ such that

$$\begin{aligned}&\left\|I_k(u) - I_k(v)\right\| \le \mu\|u-v\|, \\ &\forall u, v \in E,\ k = 1,\ldots,m.\end{aligned} \tag{37}$$

(H4) The positive real number

$$\gamma = m\mu + (m+1)\left(L + 2rM + M'\right)\frac{T'}{\Gamma'} \tag{38}$$

satisfies $0 < \gamma < 1$.

Next, we state and prove the existence and uniqueness result for the quasilinear integrodifferential problem (31); we have the following.

Theorem 8. *If the assumptions (H1)–(H4) are satisfied, then problem* (31) *has one and only one solution* $u \in \mathscr{PC}(J,E)$.

Proof. Since we are concerned with the existence and uniqueness of the solution of (31) then, it is wise to use the Banach contraction principle in order to establish such results.

Let $\mathscr{B}_r = \{u \in \mathscr{PC}(J,E) : \|u\|_{\mathscr{PC}} \le r\}$ be the closed ball of $\mathscr{PC}(J,E)$ centered at 0 with radius r satisfying the following inequality:

$$\begin{aligned}\varphi(r) :=\ &\left\|u_0\right\| + \frac{(m+1)T'}{\Gamma'}L' \\ &+ \left[\frac{(m+1)T'}{\Gamma'}\left(rM + M' + L\right) + m\mu\right] r \le r,\end{aligned} \tag{39}$$

where

$$L' = \left(C_1' + C_2'\right)(T-a) + L_2. \tag{40}$$

Endowing $\mathscr{B}_r$ with the metric $d(u,v) = \|u-v\|_{\mathscr{PC}}$, for every $u, v \in \mathscr{B}_r$, we obtain a complete metric space $(\mathscr{B}_r, d)$. Next, we define the operator $\Psi : \mathscr{B}_r \to \mathscr{B}_r$ by

$$\begin{aligned}\Psi u(t) = u_0 &+ \frac{1}{\Gamma(\alpha_k)} \\ &\times \int_{t_k}^{t}(t-s)^{\alpha_k-1}A(s,u)u(s)\,ds \\ &+ \sum_{a<t_i<t}\frac{1}{\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha_{i-1}-1}A(s,u)u(s)\,ds \\ &+ \frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}\Phi_{t_{k+1}}(s,u)\,ds \\ &+ \sum_{a<t_i<t}\frac{1}{\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha_{i-1}-1}\Phi_{t_i}(s,u)\,ds \\ &+ \sum_{a<t_i<t}I_i\left(u\left(t_i^-\right)\right), \quad t \in J_k,\ k = 0,\ldots,m.\end{aligned} \tag{41}$$

It is understood that the sum $\sum_{a<t_i<t}$ is zero if $t \in J_0$.

First, we prove that if $u \in \mathscr{PC}(J;E)$, then $\Psi u \in \mathscr{PC}(J;E)$.

Indeed, for each $t \in (t_k, t_{k+1})$, $u \in \mathscr{C}((t_k,t_{k+1}),E)$, and any sufficiently small $\delta > 0$, we have

$$\begin{aligned}&\|\Psi u(t+\delta) - \Psi u(t)\| \\ &\quad \le \frac{\left(M\|u\|_{\mathscr{PC}} + M' + L\right)\|u\|_{\mathscr{PC}} + L'}{\Gamma(\alpha_k)} \\ &\qquad \times \int_{t_k}^{t}\left[(t-s)^{\alpha_k-1} - (t+\delta-s)^{\alpha_k-1}\right]ds \\ &\qquad + \frac{\left(M\|u\|_{\mathscr{PC}} + M' + L\right)\|u\|_{\mathscr{PC}} + L'}{\Gamma(\alpha_k)} \\ &\qquad \times \int_{t}^{t+\delta}(t+\delta-s)^{\alpha_k-1}ds.\end{aligned} \tag{42}$$

Calculating the integrals we find that

$$\|\Psi u(t+\delta) - \Psi u(t)\| \le 3\frac{\left[\left(Mr + M' + L\right)r + L'\right]}{\Gamma'}\delta^{\alpha_k}. \tag{43}$$

Thus, the right-hand side tends to zero as $\delta \to 0$. Likewise one gets $\lim_{\delta\to 0}\|\Psi u(t) - \Psi u(t-\delta)\| = 0$; this shows that Ψu is continuous at t. Hence $\Psi u \in \mathscr{C}((t_k,t_{k+1}),E)$.

Next, for the right endpoint $t = t_{k+1}$ we get for any sufficiently small $\delta > 0$

$$\left\|\Psi u\left(t_{k+1}\right) - \Psi u\left(t_{k+1}-\delta\right)\right\| \le 3\frac{\left[\left(Mr + M' + L\right)r + L'\right]}{\Gamma'}\delta^{\alpha_k}, \tag{44}$$

which shows that the right-hand side tends to zero as $\delta \to 0$, and accordingly, Ψu is continuous at t_{k+1}. Therefore $\Psi u \in \mathscr{PC}(J,E)$.

To prove that $\Psi\mathscr{B}_r \subset \mathscr{B}_r$ we see that, for any $u \in \mathscr{B}_r$ and $t \in J_k$, $k = 0, \ldots, m$, we have

$$\begin{aligned}\|\Psi u(t)\| \leq{} & \|u_0\| + \frac{1}{\Gamma(\alpha_k)} \\ & \times \int_{t_k}^{t} (t-s)^{\alpha_k-1} \|A(s,u)\| \cdot \|u(s)\| \, ds \\ & + \sum_{a<t_i<t} \frac{1}{\Gamma(\alpha_{i-1})} \\ & \times \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha_{i-1}-1} \|A(s,u)\| \cdot \|u(s)\| \, ds \\ & + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k-1} \left\|\Phi_{t_{k+1}}(s,u)\right\| ds \\ & + \sum_{a<t_i<t} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha_{i-1}-1} \left\|\Phi_{t_i}(s,u)\right\| ds \\ & + \sum_{a<t_i<t} \left\|I_i\left(u\left(t_i^-\right)\right)\right\|. \end{aligned} \tag{45}$$

Estimating the right-hand side we find

$$\begin{aligned}&\|\Psi u(t)\| \\ &\leq \|u_0\| + \frac{(m+1)T'}{\Gamma'} \\ &\quad \times \left(M\|u\|_{\mathscr{PC}} + M'\right) \cdot \|u\|_{\mathscr{PC}} + \frac{1}{\Gamma(\alpha_k)} \\ &\quad \times \int_{t_k}^{t} (t-s)^{\alpha_k-1} \left[L\|u\|_{\mathscr{PC}} + (T-a)\left(C_1' + C_2'\right) + L_2\right] ds \\ &\quad + \sum_{a<t_i<t} \frac{1}{\Gamma(\alpha_{i-1})} \\ &\quad \times \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha_{i-1}-1} \left[L\|u\|_{\mathscr{PC}} + (T-a)\left(C_1' + C_2'\right) + L_2\right] ds \\ &\quad + m\mu\|u\|_{\mathscr{PC}}. \end{aligned} \tag{46}$$

So

$$\begin{aligned}&\|\Psi u(t)\| \\ &\quad \leq \|u_0\| + \frac{(m+1)T'}{\Gamma'} L' \\ &\qquad + \left[\frac{(m+1)T'}{\Gamma'}\left(rM + M' + L\right) + m\mu\right] r \\ &\quad \leq \varphi(r) \leq r. \end{aligned} \tag{47}$$

Therefore, $\|\Psi u\|_{\mathscr{PC}} \leq r$, and consequently $\Psi\mathscr{B}_r \subset \mathscr{B}_r$.

Next, we prove that Ψ is a contraction mapping; indeed, for any $u, v \in \mathscr{B}_r$ and $t \in J_k$, $k = 0, \ldots, m$, we have

$$\begin{aligned}&\|\Psi u(t) - \Psi v(t)\| \\ &\quad \leq \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k-1} \|A(s,u)u(s) - A(s,v)v(s)\| \, ds \\ &\qquad + \sum_{a<t_i<t} \frac{1}{\Gamma(\alpha_{i-1})} \\ &\qquad \times \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha_{i-1}-1} \|A(s,u)u(s) - A(s,v)v(s)\| \, ds \\ &\qquad + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k-1} \left\|\Phi_{t_{k+1}}(s,u) - \Phi_{t_{k+1}}(s,v)\right\| ds \\ &\qquad + \sum_{a<t_i<t} \frac{1}{\Gamma(\alpha_{i-1})} \\ &\qquad \times \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha_{i-1}-1} \left\|\Phi_{t_i}(t,u) - \Phi_{t_i}(s,v)\right\| ds \\ &\qquad + \sum_{a<t_i<t} \left\|I_i\left(u\left(t_i^-\right)\right) - I_i\left(v\left(t_i^-\right)\right)\right\|. \end{aligned} \tag{48}$$

Taking into account the previous assumptions we get the following estimate:

$$\begin{aligned}&\|\Psi u(t) - \Psi v(t)\| \\ &\quad \leq m\mu\|u-v\|_{\mathscr{PC}} \\ &\qquad + \frac{\|u-v\|_{\mathscr{PC}}}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k-1} \left[Mr + M' + Mr\right] ds \\ &\qquad + \sum_{a<t_i<t} \frac{\|u-v\|_{\mathscr{PC}}}{\Gamma(\alpha_{i-1})} \\ &\qquad \times \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha_{i-1}-1} \left[Mr + M' + Mr\right] ds \\ &\qquad + \frac{L\|u-v\|_{\mathscr{PC}}}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k-1} ds \\ &\qquad + \sum_{a<t_i<t} \frac{L\|u-v\|_{\mathscr{PC}}}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha_{i-1}-1} ds. \end{aligned} \tag{49}$$

Thus,

$$\begin{aligned}&\|\Psi u(t) - \Psi v(t)\| \\ &\quad \leq \left[m\mu + (m+1)\left(L + 2rM + M'\right)\frac{T'}{\Gamma'}\right] \|u-v\|_{\mathscr{PC}} \\ &\quad \leq \gamma\|u-v\|_{\mathscr{PC}}. \end{aligned} \tag{50}$$

Accordingly, the mapping Ψ has a unique fixed point $u = \Psi u \in \mathscr{B}_r$, which completes the proof. □

4. A Semilinear Impulsive Fractional Problem

In this section we consider a semilinear impulsive fractional integrodifferential problem subjected to a nonlocal condition in a finite dimensional normed space $(E, \|\cdot\|)$. Actually, the finite dimension requirement is due to some technical difficulties in order to prove some compactness properties. The problem is as follows:

$$\begin{gathered} {}^{C}D_{t_k^+}^{\alpha_k} u(t) \\ = A(t) u(t) + \Phi_{t_{k+1}}(t, u(t)), \quad t \in J_k,\ k = 0, \ldots, m, \\ u(a) + g(u) = u_0 \in E, \\ u(t_k^+) = u(t_k) + I_k(u(t_k^-)), \quad k = 1, \ldots, m. \end{gathered} \tag{51}$$

We assume that the mapping $A : J \to \mathscr{B}(E)$ is continuous and we put

$$M'' = \max_{t \in J} \|A(t)\|. \tag{52}$$

We need the following hypothesis:

(H5) there exists a constant $G > 0$ such that the mapping $g : \mathscr{PC}(J, E) \to E$ satisfies

$$\|g(u) - g(v)\| \le G \|u - v\|_{\mathscr{PC}}, \quad \forall u, v \in \mathscr{PC}(J, E). \tag{53}$$

Now, we are ready to state and prove the following result.

Theorem 9. *If the assumptions (H1)–(H3) and (H5) are satisfied, then problem* (51) *has at least one solution* $u \in \mathscr{PC}(J, E)$.

Proof. Let us define the operator $Q : \mathscr{PC}(J, E) \to \mathscr{PC}(J, E)$ by

$$\begin{aligned} Qu(t) = {} & u_0 - g(u) + \frac{1}{\Gamma(\alpha_k)} \\ & \times \int_{t_k}^{t} (t-s)^{\alpha_k - 1} A(s) u(s)\, ds \\ & + \sum_{a < t_i < t} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1} - 1} A(s) u(s)\, ds \\ & + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} \Phi_{t_{k+1}}(s, u)\, ds \\ & + \sum_{a < t_i < t} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1} - 1} \Phi_{t_i}(s, u)\, ds \\ & + \sum_{a < t_i < t} I_i(u(t_i^-)), \quad t \in J_k,\ k = 0, \ldots, m. \end{aligned} \tag{54}$$

First, we notice that by using the same technique as that in the proof of the Theorem 8 we can establish that if $u \in \mathscr{PC}(J, E)$, then $Qu \in \mathscr{PC}(J, E)$; that is, the operator Q maps the space $\mathscr{PC}(J, E)$ into itself.

To prove that Q has a fixed point we use Schaefer's fixed point theorem. We proceed in four steps.

Step 1 (Q is continuous). Let $\{u_n\}_{n \ge 1} \subset \mathscr{PC}(J, E)$ such that $u_n \to u$ in $\mathscr{PC}(J, E)$; then

$$\begin{aligned} & \|Qu_n(t) - Qu(t)\| \\ & \quad \le \|g(u_n) - g(u)\| + \frac{1}{\Gamma(\alpha_k)} \\ & \qquad \times \int_{t_k}^{t} (t-s)^{\alpha_k - 1} \|A(s)(u_n(s) - u(s))\|\, ds \\ & \qquad + \sum_{a < t_i < t} \frac{1}{\Gamma(\alpha_{i-1})} \\ & \qquad \times \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1} - 1} \|A(s)(u_n(s) - u(s))\|\, ds \\ & \qquad + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} \left\|\Phi_{t_{k+1}}(s, u_n) - \Phi_{t_{k+1}}(s, u)\right\| ds \\ & \qquad + \sum_{a < t_i < t} \frac{1}{\Gamma(\alpha_{i-1})} \\ & \qquad \times \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1} - 1} \left\|\Phi_{t_i}(s, u_n) - \Phi_{t_i}(s, u)\right\| ds \\ & \qquad + \sum_{a < t_i < t} \|I_i(u_n(t_i^-)) - I_i(u(t_i^-))\|, \\ & \qquad\qquad t \in J_k,\ k = 0, \ldots, m. \end{aligned} \tag{55}$$

Taking into account the assumptions (H2)-(H3) and (H5) and using Lemma 7 we get

$$\begin{aligned} & \|Qu_n(t) - Qu(t)\| \\ & \quad \le G \|u_n - u\|_{\mathscr{PC}} + \frac{M''}{\Gamma(\alpha_k)} \|u_n - u\|_{\mathscr{PC}} \\ & \qquad \times \int_{t_k}^{t} (t-s)^{\alpha_k - 1} ds + M'' \|u_n - u\|_{\mathscr{PC}} \\ & \qquad \times \sum_{a < t_i < t} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1} - 1} ds \\ & \qquad + \frac{L}{\Gamma(\alpha_k)} \|u_n - u\|_{\mathscr{PC}} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} ds \\ & \qquad + L \|u_n - u\|_{\mathscr{PC}} \sum_{a < t_i < t} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1} - 1} ds \\ & \qquad + m\mu \|u_n - u\|_{\mathscr{PC}}, \quad t \in J_k,\ k = 0, \ldots, m. \end{aligned} \tag{56}$$

Calculating the integrals in the right-hand side we obtain

$$\|Qu_n - Qu\|_{\mathscr{PC}} \le \left[G + m\mu + \frac{(m+1)T'}{\Gamma'}\left(M'' + L\right)\right]\|u_n - u\|_{\mathscr{PC}}. \quad (57)$$

So

$$\|Qu_n - Qu\|_{\mathscr{PC}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \quad (58)$$

and accordingly, Q is continuous.

Step 2. Let $\varepsilon > 0$ and $B_\varepsilon = \{u \in \mathscr{PC}(J, E) : \|u\|_{\mathscr{PC}} \le \varepsilon\}$. Define $\mathscr{W} = \{Qu : u \in B_\varepsilon\}$; then for any $u \in B_\varepsilon$ we have

$$\begin{aligned}\|Qu(t)\| \le{}& \|u_0 - g(u)\| + \frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{t}(t-s)^{\alpha_k - 1}\|A(s)\|\,\|u(s)\|\,ds \\ &+ \sum_{a<t_i<t}\frac{1}{\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha_{i-1}-1}\|A(s)\|\,\|u(s)\|\,ds \\ &+ \frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}\left\|\Phi_{t_{k+1}}(s,u)\right\|ds \\ &+ \sum_{a<t_i<t}\frac{1}{\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha_{i-1}-1}\left\|\Phi_{t_i}(s,u)\right\|ds \\ &+ \sum_{a<t_i<t}\left\|I_i\left(u\left(t_i^-\right)\right)\right\|, \quad t \in J_k,\ k = 0,\ldots,m.\end{aligned} \quad (59)$$

Estimating the right-hand side we obtain

$$\begin{aligned}\|Qu(t)\| \le{}& \|u_0\| + G\|u\|_{\mathscr{PC}} + \|g(0)\| + \frac{(m+1)T'}{\Gamma'}M''\cdot\|u\|_{\mathscr{PC}} \\ &+ \frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}\left[L\|u\|_{\mathscr{PC}} + L'\right]ds \\ &+ \sum_{a<t_i<t}\frac{1}{\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha_k-1}\left[L\|u\|_{\mathscr{PC}} + L'\right]ds \\ &+ m\mu\|u\|_{\mathscr{PC}},\end{aligned} \quad (60)$$

implying that

$$\|Qu\|_{\mathscr{PC}} \le \|u_0\| + \|g(0)\| + \frac{(m+1)T'}{\Gamma'}\left(M''\varepsilon + L\varepsilon + L'\right) + G\varepsilon + m\mu\varepsilon := \rho. \quad (61)$$

Hence $\mathscr{W}$ is uniformly bounded.

Step 3 (we prove that $\mathscr{W}$ is equicontinuous). Let $u \in B_\varepsilon$; then, for any $t_k < \tau_1 < \tau_2 \le t_{k+1}$, we have

$$\begin{aligned}\|Qu(\tau_2) - Qu(\tau_1)\| \le{}& \frac{1}{\Gamma(\alpha_k)}\times\int_{t_k}^{\tau_1}\left[(\tau_1 - s)^{\alpha_k-1} - (\tau_2 - s)^{\alpha_k-1}\right]\|A(s)\|\,\|u(s)\|\,ds \\ &+ \frac{1}{\Gamma(\alpha_k)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha_k-1}\|A(s)\|\,\|u(s)\|\,ds \\ &+ \frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{\tau_1}\left[(\tau_1 - s)^{\alpha_k-1} - (\tau_2 - s)^{\alpha_k-1}\right]\left\|\Phi_{t_{k+1}}(s,u)\right\|ds \\ &+ \frac{1}{\Gamma(\alpha_k)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha_k-1}\left\|\Phi_{t_{k+1}}(s,u)\right\|ds.\end{aligned} \quad (62)$$

So

$$\begin{aligned}\|Qu(\tau_2) - Qu(\tau_1)\| \le{}& \frac{\left(M''+L\right)\varepsilon + L'}{\Gamma(\alpha_k)}\int_{t_k}^{\tau_1}\left[(\tau_1 - s)^{\alpha_k-1} - (\tau_2 - s)^{\alpha_k-1}\right]ds \\ &+ \frac{\left(M''+L\right)\varepsilon + L'}{\Gamma(\alpha_k)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha_k-1}ds \\ \le{}& 3\frac{\left[\left(M''+L\right)\varepsilon + L'\right]}{\Gamma'}(\tau_2 - \tau_1)^{\alpha_k}.\end{aligned} \quad (63)$$

As $\tau_1 \to \tau_2$, the right-hand side of the previous inequality tends to zero, which means that $\mathscr{W}$ is equicontinuous.

We point out that the closures of the subsets $\mathscr{W}(t) := \{Qu(t) : u \in B_\varepsilon,\ t \in J\backslash\{t_k\},\ k = 1,\ldots,m\}$, $\mathscr{W}(t_k^-) := \{Qu(t_k^-) : u \in B_\varepsilon\}$, and $\mathscr{W}(t_k^+) := \{Qu(t_k^+) : u \in B_\varepsilon\}$, $k = 1,\ldots,m$, are bounded in E ($\dim E < \infty$); hence they are compact.

As a consequence of the previous steps and the $\mathscr{PC}$-type Arzela-Ascoli theorem we conclude that Q is completely continuous.

Step 4. Now, we show that the set

$$X = \{u \in \mathscr{PC}(J,E) : u = \lambda Qu,\ \lambda \in (0,1)\} \quad (64)$$

is bounded.

Let $u \in X$; then $u = \lambda Q u$, for some $\lambda \in (0,1)$. Thus, for each $t \in J$,

$$\begin{aligned}\|u(t)\| &\leq \lambda \|u_0\| + \lambda \|g(u)\| \\ &\quad + \frac{\lambda}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} \|A(s)\| \cdot \|u(s)\| \, ds \\ &\quad + \lambda \sum_{a<t_i<t} \frac{1}{\Gamma(\alpha_{i-1})} \\ &\quad \times \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} \|A(s)\| \cdot \|u(s)\| \, ds \\ &\quad + \frac{\lambda}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} \left\|\Phi_{t_{k+1}}(s,u)\right\| ds \\ &\quad + \lambda \sum_{a<t_i<t} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} \left\|\Phi_{t_i}(s,u)\right\| ds \\ &\quad + \lambda \sum_{a<t_i<t} \left\|I_i(u(t_i^-))\right\| \\ &\leq \lambda \left[\|u_0\| + \|g(0)\| + \frac{(m+1)T'}{\Gamma'} \right. \\ &\quad \left. \times \left(M''\varepsilon + L\varepsilon + L'\right) + G\varepsilon + m\mu\varepsilon \right] \\ &< \infty. \end{aligned} \tag{65}$$

This shows that the set X is bounded.

We conclude by Schaefer's fixed point theorem that the operator Q has a fixed point $u \in \mathscr{PC}(J,E)$ such that $Qu = u$, which means that u is a solution to problem (51). □

Next, we establish the continuous dependence of the solution upon the initial value. We have the following.

Proposition 10. *Under the hypotheses (H1)–(H3) and (H5) the solution of problem* (51) *depends continuously upon its initial value if*

$$G + m\mu + (m+1)\left(L + M''\right)\frac{T'}{\Gamma'} < 1. \tag{66}$$

Proof. Since u is a solution to (51), then it satisfies the integral equation (19). Let v be a solution to problem (51) with initial value $v(a) = v_0 - g(v)$. Then $v(t)$ satisfies the integral equation

$$\begin{aligned} v(t) &= v_0 - g(v) \\ &\quad + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k-1} A(s) v(s) \, ds \\ &\quad + \sum_{a<t_i<t} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} A(s) v(s) \, ds \\ &\quad + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k-1} \Phi_{t_{k+1}}(s,v) \, ds \\ &\quad + \sum_{a<t_i<t} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} \Phi_{t_i}(s,v) \, ds \\ &\quad + \sum_{a<t_i<t} I_i\left(v(t_i^-)\right), \quad t \in J_k, \ k = 0, \dots, m. \end{aligned} \tag{67}$$

Estimating the difference between solutions $u(t)$ and $v(t)$ to (19) and (67), respectively, we get

$$\begin{aligned} \|u(t) - v(t)\| &\leq \|u_0 - v_0\| + G\|u - v\|_{\mathscr{PC}} \\ &\quad + \frac{M''(m+1)T'}{\Gamma'} \|u - v\|_{\mathscr{PC}} \\ &\quad + \frac{L(m+1)T'}{\Gamma'} \|u - v\|_{\mathscr{PC}} \\ &\quad + m\mu \|u - v\|_{\mathscr{PC}}, \quad t \in J_k, \ k = 0, \dots, m. \end{aligned} \tag{68}$$

Taking the supremum over the interval J we find that

$$\|u - v\|_{\mathscr{PC}} \leq \frac{1}{\rho} \|u_0 - v_0\|, \tag{69}$$

where

$$\rho = 1 - G - m\mu - (m+1)\left(L + M''\right)\frac{T'}{\Gamma'}, \tag{70}$$

which proves that the mapping $u_0 \mapsto u$ is continuous from $E \to \mathscr{PC}(J,E)$. □

5. Example

Consider the following impulsive fractional integrodifferential problem

$$\begin{aligned} &D_{0^+}^{1/2} u(t) \\ &\quad = \frac{t}{24} u(t) \cos u(t) + \frac{|u(t)|}{(|u(t)| + 2)(t^2 + 12)} \\ &\quad + \int_0^t \frac{e^{-|u(s)|/6}}{(t+2)^2} ds + \int_0^{1/2} \frac{\sin(u(s)/4)}{e^t + 6} ds, \quad t \in J_0, \end{aligned}$$

$$
\begin{aligned}
&D^{3/4}_{(1/2)^+}u(t)\\
&\quad=\frac{t}{24}u(t)\cos u(t)+\frac{|u(t)|}{(|u(t)|+2)(t^2+12)}\\
&\qquad+\int_0^t\frac{e^{-|u(s)|/6}}{(t+2)^2}ds+\int_0^1\frac{\sin(u(s)/4)}{e^t+6}ds,\quad t\in J_1,\\
&\qquad u(0)=1,\\
&u\left(\frac{1}{2}^+\right)=u\left(\frac{1}{2}^-\right)+\frac{|u((1/2)^-)|}{|u((1/2)^-)|+6}.
\end{aligned}
\tag{71}
$$

We set $J_0 = [0,1/2]$, $J_1 = (1/2,1]$, and $J = [0,1]$. We take $E = \mathbb{R}$, $a = t_0 = 0$, $t_1 = 1/2$, $T = 1$, $\alpha_0 = 1/2$, $\alpha_1 = 3/4$, and $\mathscr{B}_r = \{u \in \mathscr{PC}(J,E), \|u\|_{\mathscr{PC}} \le r\}$. Define

$$
\begin{aligned}
A(t,u)&=\frac{t}{24}(\cos u)I,\\
Hu(t)&=\int_0^t\frac{e^{-|u(s)|/6}}{(t+2)^2}ds,\\
K_{t_{k+1}}u(t)&=\int_0^{t_{k+1}}\frac{\sin(u(s)/4)}{e^t+6}ds,\quad k=0,1,\\
\Phi_{t_{k+1}}(t,u(t))&=\frac{|u(t)|}{(|u(t)|+2)(t^2+12)}+\int_0^t\frac{e^{-|u(s)|/6}}{(t+2)^2}ds\\
&\quad+\int_0^{t_{k+1}}\frac{\sin(u(s)/4)}{e^t+6}ds,\quad k=0,1,\\
I_1u\left(\frac{1}{2}^-\right)&=\frac{|u((1/2)^-)|}{|u((1/2)^-)|+6}.
\end{aligned}
\tag{72}
$$

For any $u, v \in \mathscr{B}_r$ and $t \in J$, we have

$$|Hu(t)-Hv(t)|\le\frac{1}{24}\|u-v\|_{\mathscr{PC}}.\tag{73}$$

Hence $C_1 = 1/24$. Likewise, one has, for $k = 0, 1$,

$$\left|K_{t_{k+1}}u(t)-K_{t_{k+1}}v(t)\right|\le\frac{1}{24}\|u-v\|_{\mathscr{PC}};\tag{74}$$

then $C_2 = 1/24$.

By (H1), we have

$$
\begin{aligned}
T'&=\max\left\{(T-a)^{\alpha_0},(T-a)^{\alpha_1}\right\}=1,\\
\Gamma'&=\min\left\{\Gamma(\alpha_0+1),\Gamma(\alpha_1+1)\right\}\\
&=\min\left\{\Gamma\left(\frac{3}{2}\right),\Gamma\left(\frac{7}{4}\right)\right\}=\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}.
\end{aligned}
\tag{75}
$$

On the other hand, using (H2) we obtain

$$
\begin{aligned}
&\left|\Phi_{t_{k+1}}(t,u)-\Phi_{t_{k+1}}(t,v)\right|\\
&\quad\le\left|\frac{|u(t)|}{(|u(t)|+2)(t^2+12)}-\frac{|v(t)|}{(|v(t)|+2)(t^2+12)}\right|\\
&\qquad+|Hu(t)-Hv(t)|+\left|K_{t_{k+1}}u(t)-K_{t_{k+1}}v(t)\right|\\
&\quad\le\frac{3}{24}\|u-v\|_{\mathscr{PC}}.
\end{aligned}
\tag{76}
$$

Thus

$$L_1=\frac{1}{24},\qquad L=\frac{3}{24}.\tag{77}$$

Assumption (H3) gives

$$\left|I_1(u)-I_1(v)\right|\le\frac{1}{6}\|u-v\|_{\mathscr{PC}},\tag{78}$$

so $\mu = 1/6$.

Due to the definition of $A(t,u)$ we have $M = M' = 1/24$. Let us now find a threshold for the value of r for which condition (H4) is satisfied. We should have

$$0<\gamma=\frac{1}{6}+\frac{2}{3\sqrt{\pi}}+\frac{r}{3\sqrt{\pi}}<1,\tag{79}$$

so r is any positive number such that $r < (5\sqrt{\pi}-4)/2 = 2.4311$. We conclude by Theorem 8 that problem (71) has a unique solution $u \in \mathscr{PC}([0,1],\mathbb{R})$ such that $\|u\|_{\mathscr{PC}} \le r$.

6. Concluding Remarks

In this work we have first noticed that most of the published papers dealing with impulsive differential equations of fractional orders are not mathematically correct, so we have proved through a concrete counterexample that the concept of solution proposed recently by some authors is not realistic. On the other hand, we introduced a new class of impulsive fractional problems with several fractional orders and we established an equivalence with some integral equation. Moreover, we derived two existence results by using two different fixed point theorems as we proved the stability of the solution of the given problem with respect to the initial value. Finally, we illustrated our first theorem of existence and uniqueness by a concrete example in $\mathbb{R}$.

References

[1] W. M. Ahmad and R. El-Khazali, "Fractional-order dynamical models of love," *Chaos, Solitons and Fractals*, vol. 33, no. 4, pp. 1367–1375, 2007.

[2] B. Bonilla, M. Rivero, L. Rodríguez-Germá, and J. J. Trujillo, "Fractional differential equations as alternative models to nonlinear differential equations," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 79–88, 2007.

[3] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer, Berlin, Germany, 2010.

[4] E. Hernández, D. O'Regan, and K. Balachandran, "On recent developments in the theory of abstract differential equations with fractional derivatives," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 73, no. 10, pp. 3462–3471, 2010.

[5] V. Lakshmikantham, "Theory of fractional functional differential equations," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 69, no. 10, pp. 3337–3343, 2008.

[6] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[7] V. Lakshmikantham, S. Leela, and D. J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Scientific, Cambridge, UK, 2009.

[8] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg, Germany, 2011.

[9] M. Benchohra and B. A. Slimani, "Existence and uniqueness of solutions to impulsive fractional differential equations," *Electronic Journal of Differential Equations*, vol. 2009, article 10, 11 pages, 2009.

[10] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.

[11] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[12] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science, Longhorne, Pa, USA, 1993.

[13] K. Balachandran, S. Kiruthika, and J. J. Trujillo, "Remark on the existence results for fractional impulsive integrodifferential equations in Banach spaces," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 6, pp. 2244–2247, 2012.

[14] K. Balachandran, S. Kiruthika, and J. J. Trujillo, "Existence results for fractional impulsive integrodifferential equations in Banach spaces," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 4, pp. 1970–1977, 2011.

[15] O. K. Jaradat, A. Al-Omari, and S. Momani, "Existence of the mild solution for fractional semilinear initial value problems," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 69, no. 9, pp. 3153–3159, 2008.

[16] G. M. Mophou and G. M. N'Guérékata, "Existence of the mild solution for some fractional differential equations with nonlocal conditions," *Semigroup Forum*, vol. 79, no. 2, pp. 315–322, 2009.

[17] G. M. N'Guérékata, "A Cauchy problem for some fractional abstract differential equation with non local conditions," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 70, no. 5, pp. 1873–1876, 2009.

[18] M. Fečkan, Y. Zhou, and J. Wang, "On the concept and existence of solution for impulsive fractional differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 7, pp. 3050–3060, 2012.

[19] L. Mahto, S. Abbas, and A. Favini, "Analysis of Caputo impulsive fractional order differential equations," *International Journal of Differential Equations*, vol. 2013, Article ID 704547, 11 pages, 2013.

[20] B. Nagy and F. Riesz, *Functional Analysis*, Blackie & Son, London, UK, 1956.

[21] W. Wei, X. Xiang, and Y. Peng, "Nonlinear impulsive integro-differential equations of mixed type and optimal controls," *Optimization*, vol. 55, no. 1-2, pp. 141–156, 2006.