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Research Article

Existence and Iterative Algorithms of Positive Solutions for a Higher Order Nonlinear Neutral Delay Differential Equation

Zeqing Liu,[1] Ling Guan,[1] Sunhong Lee,[2] and Shin Min Kang[2]

[1] *Department of Mathematics, Liaoning Normal University, Dalian, Liaoning 116029, China*
[2] *Department of Mathematics and RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea*

Correspondence should be addressed to Sunhong Lee; sunhong@gnu.ac.kr and Shin Min Kang, smkang@gnu.ac.kr

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This paper is concerned with the higher order nonlinear neutral delay differential equation $[a(t)(x(t) + b(t)x(t - \tau))^{(m)}]^{(n-m)} + [h(t, x(h_1(t)), \dots, x(h_l(t)))]^{(i)} + f(t, x(f_1(t)), \dots, x(f_l(t))) = g(t)$, for all $t \geq t_0$. Using the Banach fixed point theorem, we establish the existence results of uncountably many positive solutions for the equation, construct Mann iterative sequences for approximating these positive solutions, and discuss error estimates between the approximate solutions and the positive solutions. Nine examples are included to dwell upon the importance and advantages of our results.

1. Introduction and Preliminaries

In recent years, the existence problems of nonoscillatory solutions for neutral delay differential equations of first, second, third, and higher order have been studied intensively by using fixed point theorems; see, for example, [1–12] and the references therein.

Using the Banach, Schauder, and Krasnoselskii fixed point theorems, Zhang et al. [9] and Liu et al. [7] considered the existence of nonoscillatory solutions for the following first order neutral delay differential equations:

$$\begin{gathered}
[x(t) + P(t)x(t-\tau)]' + Q_1(t)x(t-\tau_1) - Q_2(t)x(t-\tau_2) = 0, \quad \forall t \geq t_0, \\
[x(t) + c(t)x(t-\tau)]' + h(t) f(x(t-\sigma_1), x(t-\sigma_2), \dots, x(t-\sigma_k)) = g(t), \quad \forall t \geq t_0,
\end{gathered} \tag{1}$$

where $P \in C([t_0, +\infty), \mathbb{R} \setminus \{\pm 1\})$ and $c \in C([t_0, +\infty), \mathbb{R})$. Making use of the Banach and Krasnoselskii fixed point theorems, Kulenović and Hadžiomerspahić [2] and Zhou [10] studied the existence of a nonoscillatory solution for the following second order neutral differential equations:

$$\begin{gathered}
[x(t) + cx(t-\tau)]'' + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0, \quad \forall t \geq t_0, \\
\left[r(t)(x(t) + P(t)x(t-\tau))'\right]' + \sum_{i=1}^{m} Q_i(t) f_i(x(t-\sigma_i)) = 0, \quad \forall t \geq t_0,
\end{gathered} \tag{2}$$

where $c \in \mathbb{R} \setminus \{\pm 1\}$ and $P \in C([t_0, \infty), \mathbb{R})$. Zhou and Zhang [11], Zhou et al. [12], and Liu et al. [4], respectively, investigated the existence of nonoscillatory solutions for the following higher order neutral delay differential equations:

$$[x(t) + cx(t-\tau)]^{(n)} + (-1)^{n+1}[P(t)x(t-\sigma) - Q(t)x(t-\delta)] = 0, \quad \forall t \geq t_0,$$

$$\begin{aligned}&[x(t)+P(t)x(t-\tau)]^{(n)}\\&\qquad+\sum_{i=1}^{m}Q_i(t)f_i(x(t-\sigma_i))=g(t),\quad\forall t\ge t_0,\\&[x(t)+ax(t-\tau)]^{(n)}\\&\qquad+(-1)^{n+1}f(t,x(t-\sigma_1),x(t-\sigma_2),\dots,x(t-\sigma_k))\\&\quad=g(t),\quad\forall t\ge t_0,\end{aligned}\tag{3}$$

where $c\in\mathbb{R}\setminus\{\pm1\}$, $P\in C([t_0,\infty),\mathbb{R})$ and $a\in\mathbb{R}\setminus\{-1\}$. Candan [1] proved the existence of a bounded nonoscillatory solution for the higher order nonlinear neutral differential equation:

$$\begin{aligned}&\left[r(t)(x(t)+P(t)x(t-\tau))^{(n-1)}\right]'\\&\quad+(-1)^n\left[Q_1(t)g_1(x(t-\sigma_1))\right.\\&\qquad\left.-Q_2(t)g_2(x(t-\sigma_2))-f(t)\right]=0,\quad\forall t\ge t_0,\end{aligned}\tag{4}$$

where $P\in C([t_0,\infty),\mathbb{R}\setminus\{\pm1\})$.

Motivated by the results in [1–12], in this paper we consider the following higher order nonlinear neutral delay differential equation:

$$\begin{aligned}&\left[a(t)(x(t)+b(t)x(t-\tau))^{(m)}\right]^{(n-m)}\\&\quad=+\left[h(t,x(h_1(t)),\dots,x(h_l(t)))\right]^{(i)}\\&\quad=+f(t,x(f_1(t)),\dots,x(f_l(t)))=g(t),\quad\forall t\ge t_0,\end{aligned}\tag{5}$$

where $m,n\in\mathbb{N}$ and $i\in\mathbb{N}_0$ with $i\le n-m-1$, $\tau>0$, $a\in C([t_0,+\infty),\mathbb{R}\setminus\{0\})$, $b,g,f_j,h_j\in C([t_0,+\infty),\mathbb{R})$, $h\in C^i([t_0,+\infty)\times\mathbb{R}^l,\mathbb{R})$ and $f\in C([t_0,+\infty)\times\mathbb{R}^l,\mathbb{R})$ with

$$\lim_{t\to+\infty}h_j(t)=\lim_{t\to+\infty}f_j(t)=+\infty,\quad j\in\{1,2,\dots,l\}.\tag{6}$$

It is clear that (5) includes (1)–(4) as special cases. Utilizing the Banach fixed point theorem, we prove several existence results of uncountably many positive solutions for (5), construct a few Mann iterative schemes, and discuss error estimates between the sequences generated by the Mann iterative schemes and the positive solutions. Nine examples are given to show that the results presented in this paper extend substantially the existing ones in [1, 2, 4, 5, 8, 9, 11].

Throughout this paper, we assume that $\mathbb{R}=(-\infty,+\infty)$, $\mathbb{R}^+=[0,+\infty)$, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$,

$$\begin{aligned}&H_j=\frac{1}{(m-1)!\,(n-m-j-1)!},\quad j\in\{0,i\},\\&\gamma=\min\left\{t_0-\tau,\inf_{t\ge t_0}h_j(t),\inf_{t\ge t_0}f_j(t):j\in\{1,2,\dots,l\}\right\},\end{aligned}\tag{7}$$

$CB([\gamma,+\infty),\mathbb{R})$ stands for the Banach space of all continuous and bounded functions in $[\gamma,+\infty)$ with norm $\|x\|=\sup_{t\ge\gamma}|x(t)|$, and for any $M>N>0$

$$\begin{aligned}\Omega_1(N,M)=&\{x\in CB([\gamma,+\infty),\mathbb{R}):\\&N\le x(t)\le M,\ \forall t\ge\gamma\},\\\Omega_2(N,M)=&\left\{x\in CB([\gamma,+\infty),\mathbb{R}):\frac{N}{b(t+\tau)}\le x(t)\right.\\&\le\frac{M}{b(t+\tau)},\ \forall t\ge T;\frac{N}{b(T+\tau)}\\&\left.\le x(t)\le\frac{M}{b(T+\tau)},\ \forall t\in[\gamma,T)\right\},\\\Omega_3(N,M)=&\left\{x\in CB([\gamma,+\infty),\mathbb{R}):-\frac{N}{b(t+\tau)}\le x(t)\right.\\&\le-\frac{M}{b(t+\tau)},\ \forall t\ge T;-\frac{N}{b(T+\tau)}\\&\left.\le x(t)\le-\frac{M}{b(T+\tau)},\ \forall t\in[\gamma,T)\right\}.\end{aligned}\tag{8}$$

It is easy to check that $\Omega_1(N,M)$, $\Omega_2(N,M)$ and $\Omega_3(N,M)$ are closed subsets of $CB([\gamma,+\infty),\mathbb{R})$.

By a solution of (5), we mean a function $x\in C([\gamma,+\infty),\mathbb{R})$ for some $T>1+|t_0|+\tau+|\gamma|$, such that $a(t)(x(t)+b(t)x(t-\tau))^{(m)}$ are $n-m$ times continuously differentiable in $[T,+\infty)$ and such that (5) is satisfied for $t\ge T$.

Lemma 1. *Let $\tau>0$, $c\ge0$, $F\in C([c,+\infty)^3,\mathbb{R}^+)$ and $G\in C([c,+\infty)^2,\mathbb{R}^+)$. Then*

(a) $\int_c^{+\infty}\int_r^{+\infty}\int_u^{+\infty}rF(s,u,r)\,ds\,du\,dr<+\infty\Leftrightarrow$ $\sum_{j=0}^{\infty}\int_{c+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}F(s,u,r)\,ds\,du\,dr<+\infty$;

(b) $\int_c^{+\infty}\int_u^{+\infty}uG(s,u)\,ds\,du<+\infty\Leftrightarrow$ $\sum_{j=0}^{\infty}\int_{c+j\tau}^{+\infty}\int_u^{+\infty}G(s,u)\,ds\,du<+\infty$;

(c) *if* $\int_c^{+\infty}\int_r^{+\infty}\int_u^{+\infty}rF(s,u,r)\,ds\,du\,dr<+\infty$, *then*

$$\begin{aligned}&\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}F(s,u,r)\,ds\,du\,dr\\&\quad\le\frac{1}{\tau}\int_{t+\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}rF(s,u,r)\,ds\,du\,dr\\&\quad<+\infty,\quad\forall t\ge c;\end{aligned}\tag{9}$$

(d) *if* $\int_c^{+\infty}\int_u^{+\infty} uG(s,u)ds\,du < +\infty$, *then*

$$\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty} G(s,u)\,ds\,du \leq \frac{1}{\tau}\int_{t+\tau}^{+\infty}\int_u^{+\infty} uG(s,u)\,ds\,du < +\infty, \quad \forall t \geq c. \tag{10}$$

Proof. Let $[t]$ denote the largest integral number not exceeding $t \in \mathbb{R}$. Note that

$$\lim_{r\to+\infty}\frac{[(r-c)/\tau]+1}{r} = \frac{1}{\tau}, \tag{11}$$

$$c+n\tau \leq r < c+(n+1)\tau \Longleftrightarrow n \leq \frac{r-c}{\tau} < n+1, \quad \forall n \in \mathbb{N}_0. \tag{12}$$

Clearly (12) means that

$$\begin{aligned}
\sum_{j=0}^{\infty}\int_{c+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr
&= \int_c^{+\infty}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr \\
&\quad + \int_{c+\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr \\
&\quad + \int_{c+2\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr \\
&\quad + \int_{c+3\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr + \cdots \\
&= \int_c^{c+\tau}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr \\
&\quad + 2\int_{c+\tau}^{c+2\tau}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr \\
&\quad + 3\int_{c+2\tau}^{c+3\tau}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr \\
&\quad + 4\int_{c+3\tau}^{c+4\tau}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr + \cdots \\
&= \sum_{n=0}^{\infty}\int_{c+n\tau}^{c+(n+1)\tau}\int_r^{+\infty}\int_u^{+\infty} (n+1)F(s,u,r)\,ds\,du\,dr \\
&= \sum_{n=0}^{\infty}\int_{c+n\tau}^{c+(n+1)\tau}\int_r^{+\infty}\int_u^{+\infty}\left(\left[\frac{r-c}{\tau}\right]+1\right) \times F(s,u,r)\,ds\,du\,dr \\
&= \int_c^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\left(\left[\frac{r-c}{\tau}\right]+1\right)F(s,u,r)\,ds\,du\,dr.
\end{aligned} \tag{13}$$

Thus (a) follows from (11) and (13).

Assume that $\int_c^{+\infty}\int_r^{+\infty}\int_u^{+\infty} rF(s,u,r)ds\,du\,dr < +\infty$. As in the proof of (a), we infer that

$$\begin{aligned}
\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr
&= \int_{t+\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\left[\frac{r-t}{\tau}\right]F(s,u,r)\,ds\,du\,dr \\
&\leq \frac{1}{\tau}\int_{t+\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty} rF(s,u,r)\,ds\,du\,dr \\
&\leq \frac{1}{\tau}\int_c^{+\infty}\int_r^{+\infty}\int_u^{+\infty} rF(s,u,r)\,ds\,du\,dr \\
&< +\infty, \quad \forall t \geq c,
\end{aligned} \tag{14}$$

that is, (c) holds.

Similar to the proofs of (a) and (c), we conclude that (b) and (d) hold. This completes the proof. □

2. Existence of Uncountably Many Positive Solutions and Mann Iterative Schemes

Now we show the existence of uncountably many positive solutions for (5) and discuss the convergence of the Mann iterative sequences to these positive solutions.

Theorem 2. *Assume that there exist three constants M, N, and b_0 and four functions $P, Q, R, W \in C([t_0,+\infty),\mathbb{R}^+)$ satisfying*

$$0 < N < M, \quad b_0 < \frac{M-N}{2M}, \quad |b(t)| \leq b_0 \text{ eventually}; \tag{15}$$

$$\begin{gathered}
|f(t,u_1,\ldots,u_l) - f(t,\bar{u}_1,\ldots,\bar{u}_l)| \leq P(t)\max\{|u_j-\bar{u}_j| : 1 \leq j \leq l\}, \\
|h(t,u_1,\ldots,u_l) - h(t,\bar{u}_1,\ldots,\bar{u}_l)| \leq R(t)\max\{|u_j-\bar{u}_j| : 1 \leq j \leq l\}, \\
\forall (t,u_1,\ldots,u_l,\bar{u}_1,\ldots,\bar{u}_l) \in [t_0,+\infty)\times[N,M]^{2l};
\end{gathered} \tag{16}$$

$$|f(t,u_1,\ldots,u_l)| \leq Q(t), \qquad |h(t,u_1,\ldots,u_l)| \leq W(t), \qquad \forall (t,u_1,\ldots,u_l) \in [t_0,+\infty)\times[N,M]^l; \tag{17}$$

$$\int_{t_0}^{+\infty}\int_u^{+\infty}\frac{|u|^{m-1}}{|a(u)|}\left[|s|^{n-m-1}\max\{P(s),Q(s),|g(s)|\} + |s|^{n-m-i-1}\max\{R(s),W(s)\}\right]ds\,du < +\infty. \tag{18}$$

Then

(a) for any $L \in (b_0M+N,(1-b_0)M)$, there exist $\theta \in (0,1)$ and $T > 1+|t_0|+\tau+|\gamma|$ such that for each $x_0 \in \Omega_1(N,M)$,

the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ *generated by the following scheme*

$$x_{k+1}(t) = \begin{cases} (1-\alpha_k)x_k(t) \\ \quad +\alpha_k\Big\{L - b(t)x_k(t-\tau) + (-1)^n H_0 \\ \quad\quad \times \int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{a(u)} \\ \quad\quad\quad \times [g(s) - f(s, x_k(f_1(s)), \dots, x_k(f_l(s)))]\,ds\,du \\ \quad\quad +(-1)^{n-i-1}H_i \\ \quad\quad \times \int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{a(u)} \\ \quad\quad\quad \times h(x_k(h_1(s)), \dots, x_k(h_l(s)))\,ds\,du\Big\}, \\ \quad\quad\quad t \ge T,\ k \in \mathbb{N}_0, \\ (1-\alpha_k)x_k(T) \\ \quad +\alpha_k\Big\{L - b(T)x_k(T-\tau) \\ \quad\quad +(-1)^n H_0 \\ \quad\quad \times \int_T^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-T)^{m-1}}{a(u)} \\ \quad\quad\quad \times [g(s) - f(s, x_k(f_1(s)), \dots, x_k(f_l(s)))]\,ds\,du \\ \quad\quad +(-1)^{n-i-1}H_i \\ \quad\quad \times \int_T^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-T)^{m-1}}{a(u)} \\ \quad\quad\quad \times h(s, x_k(h_1(s)), \dots, x_k(h_l(s)))\,ds\,du\Big\}, \\ \quad\quad\quad t_0 \le t < T,\ k \in \mathbb{N}_0 \end{cases} \tag{19}$$

converges to a positive solution $x \in \Omega_1(N,M)$ *of* (5) *and has the following error estimate:*

$$\|x_{k+1} - x\| \le e^{-(1-\theta)\sum_{p=0}^{k}\alpha_p}\|x_0 - x\|, \quad \forall k \in \mathbb{N}_0, \tag{20}$$

where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *such that*

$$\sum_{k=0}^{\infty}\alpha_k = +\infty; \tag{21}$$

(*b*) *Equation* (5) *has uncountably many positive solutions in* $\Omega_1(N,M)$.

Proof. Firstly, we prove that (a) holds. Set $L \in (b_0M + N, (1-b_0)M)$. From (15) and (18), we know that there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying

$$|b(t)| \le b_0, \quad \forall t \ge T; \tag{22}$$

$$\theta = b_0 + \int_T^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|}\big[H_0 s^{n-m-1}P(s) + H_i s^{n-m-i-1}R(s)\big]\,ds\,du; \tag{23}$$

$$\int_T^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|}\big[H_0 s^{n-m-1}(|g(s)| + Q(s)) + H_i s^{n-m-i-1}W(s)\big]\,ds\,du < \min\{(1-b_0)M - L, L - b_0M - N\}. \tag{24}$$

Define a mapping $S_L : \Omega_1(N,M) \to \mathrm{CB}([\gamma,+\infty),\mathbb{R})$ by

$$S_Lx(t) = \begin{cases} L - b(t)x(t-\tau) + (-1)^n H_0 \\ \quad \times \int_t^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-t)^{m-1}}{a(u)} \\ \quad \times [g(s) - f(s, x(f_1(s)), \dots, x(f_l(s)))]\,ds\,du \\ \quad +(-1)^{n-i-1}H_i \\ \quad \times \int_t^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{a(u)} \\ \quad \times h(s, x(h_1(s)), \dots, x(h_l(s)))\,ds\,du, \\ \quad\quad t \ge T,\ x \in \Omega_1(N,M), \\ S_Lx(T), \quad \gamma \le t < T,\ x \in \Omega_1(N,M). \end{cases} \tag{25}$$

It is obvious that S_Lx is continuous for each $x \in \Omega_1(N,M)$. By means of (16), (22), (23), and (25), we deduce that for any $x, y \in \Omega_1(N,M)$ and $t \ge T$

$$\begin{aligned} &|S_Lx(t) - S_Ly(t)| \\ &\quad \le |b(t)|\,|x(t-\tau) - y(t-\tau)| \\ &\quad\quad + H_0\int_t^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-t)^{m-1}}{|a(u)|} \\ &\quad\quad \times |f(s, x(f_1(s)), \dots, x(f_l(s))) \\ &\quad\quad\quad - f(s, y(f_1(s)), \dots, y(f_l(s)))|\,ds\,du \\ &\quad\quad + H_i\int_t^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{|a(u)|} \\ &\quad\quad \times |h(s, x(h_1(s)), \dots, x(h_l(s))) \\ &\quad\quad\quad - h(s, y(h_1(s)), \dots, y(h_l(s)))|\,ds\,du \end{aligned}$$

$$\begin{aligned}&\le b_0\|x-y\|+\|x-y\|\\&\quad\times\int_T^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}P(s)\right.\\&\qquad\left.+H_i s^{n-m-i-1}R(s)\right]ds\,du\\&=\theta\|x-y\|,\end{aligned}\tag{26}$$

which yields that

$$\|S_L x-S_L y\|\le\theta\|x-y\|,\quad\forall x,y\in\Omega_1(N,M).\tag{27}$$

On the basis of (17), (22), (24), and (25), we acquire that for any $x\in\Omega_1(N,M)$ and $t\ge T$

$$\begin{aligned}&S_L x(t)\\&\le L+|b(t)|x(t-\tau)\\&\quad+H_0\int_t^{+\infty}\int_u^{+\infty}\frac{s^{n-m-1}u^{m-1}}{|a(u)|}\\&\qquad\times\left[|g(s)|+\left|f(s,x(f_1(s)),\ldots,x(f_l(s)))\right|\right]ds\,du\\&\quad+H_i\int_t^{+\infty}\int_u^{+\infty}\frac{s^{n-m-i-1}u^{m-1}}{|a(u)|}\\&\qquad\times\left|h(s,x(h_1(s)),\ldots,x(h_l(s)))\right|ds\,du\\&\le L+b_0M\\&\quad+\int_T^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|}\\&\qquad\times\left[H_0 s^{n-m-1}(|g(s)|+Q(s))+H_i s^{n-m-i-1}W(s)\right]ds\,du\\&< L+b_0M+\min\{(1-b_0)M-L,L-b_0M-N\}\\&\le M,\\&S_L x(t)\\&\ge L-|b(t)|x(t-\tau)\\&\quad-H_0\int_t^{+\infty}\int_u^{+\infty}\frac{s^{n-m-1}u^{m-1}}{|a(u)|}\\&\qquad\times\left[|g(s)|+\left|f(s,x(f_1(s)),\ldots,x(f_l(s)))\right|\right]ds\,du\\&\quad-H_i\int_t^{+\infty}\int_u^{+\infty}\frac{s^{n-m-i-1}u^{m-1}}{|a(u)|}\\&\qquad\times\left|h(s,x(h_1(s)),\ldots,x(h_l(s)))\right|ds\,du\\&\ge L-b_0M-\int_T^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|}\\&\qquad\times\left[H_0 s^{n-m-1}(|g(s)|+Q(s))+H_i s^{n-m-i-1}W(s)\right]ds\,du\\&> L-b_0M-\min\{(1-b_0)M-L,L-b_0M-N\}\\&\ge N,\end{aligned}\tag{28}$$

which guarantee that $S_L(\Omega_1(N,M))\subseteq\Omega_1(N,M)$. Consequently, (27) gives that S_L is a contraction mapping in $\Omega_1(N,M)$ and it has a unique fixed point $x\in\Omega_1(N,M)$. It is easy to see that $x\in\Omega_1(N,M)$ is a positive solution of (5).

It follows from (19), (25), and (27) that

$$\begin{aligned}&|x_{k+1}(t)-x(t)|\\&=\Bigg|(1-\alpha_k)x_k(t)\\&\quad+\alpha_k\Bigg\{L-b(t)x_k(t-\tau)+(-1)^nH_0\\&\qquad\times\int_t^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-t)^{m-1}}{a(u)}\\&\qquad\times\left[g(s)-f(s,x_k(f_1(s)),\ldots,x_k(f_l(s)))\right]ds\,du\\&\qquad+(-1)^{n-i-1}H_i\\&\qquad\times\int_t^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{a(u)}\\&\qquad\times h(s,x_k(h_1(s)),\ldots,x_k(h_l(s)))\,ds\,du\Bigg\}-x(t)\Bigg|\\&\le(1-\alpha_k)|x_k(t)-x(t)|+\alpha_k|S_Lx_k(t)-S_Lx(t)|\\&\le(1-\alpha_k)|x_k(t)-x(t)|+\alpha_k\theta|x_k(t)-x(t)|\\&=(1-(1-\theta)\alpha_k)|x_k(t)-x(t)|\\&\le e^{-(1-\theta)\alpha_k}\|x_k-x\|\\&\le e^{-(1-\theta)\sum_{p=0}^{k}\alpha_p}\|x_0-x\|,\quad\forall k\in\mathbb{N}_0,\ t\ge T,\end{aligned}\tag{29}$$

which yields that

$$\|x_{k+1}-x\|\le e^{-(1-\theta)\sum_{p=0}^{k}\alpha_p}\|x_0-x\|,\quad\forall k\in\mathbb{N}_0.\tag{30}$$

That is, (20) holds. Thus (20) and (21) ensure that $\lim_{k\to\infty}x_k=x$.

Secondly, we show that (b) holds. Let $L_1,L_2\in(b_0M+N,(1-b_0)M)$ with $L_1\ne L_2$. In light of (15) and (18), we know that for each $p\in\{1,2\}$, there exist $\theta_p\in(0,1)$, T_p and T^*

with $T_p > 1 + |t_0| + \tau + |\gamma|$ and $T^* > \max\{T_1, T_2\}$ satisfying (22)–(24) and

$$\int_{T^*}^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \left[H_0 s^{n-m-1} P(s) + H_i s^{n-m-i-1} R(s)\right] ds\, du < |L_1 - L_2|, \tag{31}$$

where θ and T are replaced by θ_p and T_p, respectively. Let the mapping S_{L_p} be defined by (25) with L and T replaced by L_p and T_p, respectively. As in the proof of (a), we deduce easily that the mapping S_{L_p} possesses a unique fixed point $z_p \in \Omega_1(N, M)$, that is, z_p is a positive solution of (5) in $\Omega_1(N, M)$. In order to prove (b), we need only to show that $z_1 \neq z_2$. In fact, (25) means that for each $t \geq T^*$ and $p \in \{1, 2\}$

$$\begin{aligned} z_p(t) &= L_p - b(t) z_p(t-\tau) + (-1)^n H_0 \\ &\quad \times \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{a(u)} \\ &\quad \times \left[g(s) - f\left(s, z_p(f_1(s)), \ldots, z_p(f_l(s))\right)\right] ds\, du \\ &\quad + (-1)^{n-i-1} H_i \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{a(u)} \\ &\quad \times h\left(s, z_p(h_1(s)), \ldots, z_p(h_l(s))\right) ds\, du. \end{aligned} \tag{32}$$

It follows from (16), (22), (31), and (32) that for each $t \geq T^*$

$$\begin{aligned} &|z_1(t) - z_2(t)| \\ &\quad \geq |L_1 - L_2| - |b(t)| \, |z_1(t-\tau) - z_2(t-\tau)| \\ &\qquad - H_0 \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{|a(u)|} \\ &\qquad \times |f(s, z_1(f_1(s)), \ldots, z_1(f_l(s))) \\ &\qquad\quad - f(s, z_2(f_1(s)), \ldots, z_2(f_l(s)))| \, ds\, du \\ &\qquad - H_i \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{|a(u)|} \\ &\qquad \times |h(s, z_1(h_1(s)), \ldots, z_1(h_l(s))) \\ &\qquad\quad - h(s, z_2(h_1(s)), \ldots, z_2(h_l(s)))| \, ds\, du \\ &\quad \geq |L_1 - L_2| - b_0 \|z_1 - z_2\| \\ &\qquad - \|z_1 - z_2\| \int_{T^*}^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\ &\qquad \times \left[H_0 s^{n-m-1} P(s) + H_i s^{n-m-i-1} R(s)\right] ds\, du \\ &\quad \geq |L_1 - L_2| - (b_0 + |L_1 - L_2|) \|z_1 - z_2\|, \end{aligned} \tag{33}$$

which implies that

$$\|z_1 - z_2\| \geq \frac{|L_1 - L_2|}{1 + b_0 + |L_1 - L_2|} > 0, \tag{34}$$

that is, $z_1 \neq z_2$. This completes the proof. □

Theorem 3. *Assume that there exist three constants M, N, and b_0 and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (16)–(18) *and*

$$0 < N < M, \quad b_0 < \frac{M - N}{M}, \quad 0 \leq b(t) \leq b_0 \text{ eventually}. \tag{35}$$

Then

(a) *for any $L \in (b_0 M + N, M)$, there exist $\theta \in (0, 1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that for each $x_0 \in \Omega_1(N, M)$, the Mann iterative sequence $\{x_k\}_{k \in \mathbb{N}_0}$ generated by* (19) *converges to a positive solution $x \in \Omega_1(N, M)$ of* (5) *and has the error estimate* (20), *where $\{\alpha_k\}_{k \in \mathbb{N}_0}$ is an arbitrary sequence in $[0, 1]$ satisfying* (21)*;*

(b) *Equation* (5) *has uncountably many positive solutions in $\Omega_1(N, M)$.*

Proof. Let $L \in (b_0 M + N, M)$. Equations (18) and (36) ensure that there exist $\theta \in (0, 1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying (23),

$$0 \leq b(t) \leq b_0, \quad \forall t \geq T; \tag{36}$$

$$\begin{aligned} &\int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \left[H_0 s^{n-m-1} (|g(s)| + Q(s)) \right. \\ &\qquad\qquad \left. + H_i s^{n-m-i-1} W(s)\right] ds\, du \\ &\quad < \min\{M - L, L - b_0 M - N\}. \end{aligned} \tag{37}$$

Define a mapping $S_L : \Omega_1(N, M) \to \mathrm{CB}([\gamma, +\infty), \mathbb{R})$ by (25). Obviously, $S_L x$ is continuous for every $x \in \Omega_1(N, M)$.

Using (16), (23), (25), and (36), we conclude that for any $x, y \in \Omega_1(N, M)$ and $t \geq T$

$$\begin{aligned}
&\left|S_L x(t) - S_L y(t)\right| \\
&\quad \leq b(t)\left|x(t-\tau) - y(t-\tau)\right| \\
&\qquad + H_0 \int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{|a(u)|} \\
&\qquad \times \left|f(s, x(f_1(s)), \dots, x(f_l(s))) \right. \\
&\qquad\quad \left. - f(s, y(f_1(s)), \dots, y(f_l(s)))\right| ds\, du \\
&\qquad + H_i \int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{|a(u)|} \\
&\qquad \times \left|h(s, x(h_1(s)), \dots, x(h_l(s))) \right. \\
&\qquad\quad \left. - h(s, y(h_1(s)), \dots, y(h_l(s)))\right| ds\, du \\
&\quad \leq b_0 \|x-y\| + \|x-y\| \\
&\qquad \times \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1} P(s)\right. \\
&\qquad\qquad \left. + H_i s^{n-m-i-1} R(s)\right] ds\, du \\
&\quad = \theta \|x-y\|,
\end{aligned} \tag{38}$$

which implies that (27) holds. In light of (17), (25), (36), and (37), we know that for any $x \in \Omega_1(N, M)$ and $t \geq T$

$$\begin{aligned}
&S_L x(t) \\
&\quad \leq L + H_0 \int_t^{+\infty}\int_u^{+\infty} \frac{s^{n-m-1}u^{m-1}}{|a(u)|} \\
&\qquad\qquad \times \left[\left|g(s)\right| + \left|f(s, x(f_1(s)), \dots, x(f_l(s)))\right|\right] ds\, du \\
&\qquad + H_i \int_t^{+\infty}\int_u^{+\infty} \frac{s^{n-m-i-1}u^{m-1}}{|a(u)|} \\
&\qquad\qquad \times \left|h(s, x(h_1(s)), \dots, x(h_l(s)))\right| ds\, du \\
&\quad \leq L + \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}\left(\left|g(s)\right| + Q(s)\right)\right. \\
&\qquad\qquad \left. + H_i s^{n-m-i-1} W(s)\right] ds\, du \\
&\quad < L + \min\{M - L, L - b_0 M - N\} \\
&\quad \leq M, \\
&S_L x(t) \\
&\quad \geq L - |b(t)|\, x(t-\tau) \\
&\qquad - H_0 \int_t^{+\infty}\int_u^{+\infty} \frac{s^{n-m-1}u^{m-1}}{|a(u)|} \\
&\qquad\qquad \times \left[\left|g(s)\right| + \left|f(s, x(f_1(s)), \dots, x(f_l(s)))\right|\right] ds\, du \\
&\qquad - H_i \int_t^{+\infty}\int_u^{+\infty} \frac{s^{n-m-i-1}u^{m-1}}{|a(u)|} \\
&\qquad\qquad \times \left|h(s, x(h_1(s)), \dots, x(h_l(s)))\right| ds\, du \\
&\quad \geq L - b_0 M - \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\
&\qquad\qquad \times \left[H_0 s^{n-m-1}\left(\left|g(s)\right| + Q(s)\right)\right. \\
&\qquad\qquad \left. + H_i s^{n-m-i-1} W(s)\right] ds\, du \\
&\quad > L - b_0 M - \min\{M - L, L - b_0 M - N\} \\
&\quad \geq N,
\end{aligned} \tag{39}$$

which mean that $S_L(\Omega_1(N, M)) \subseteq \Omega_1(N, M)$. Equation (27) guarantees that S_L is a contraction mapping in $\Omega_1(N, M)$ and it possesses a unique fixed point $x \in \Omega_1(N, M)$. As in the proof of Theorem 2, we infer that $x \in \Omega_1(N, M)$ is a positive solution of (5). The rest of the proof is similar to that of Theorem 2 and is omitted. This completes the proof. □

Theorem 4. *Assume that there exist three constants M, N, and b_0 and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying (16)–(18) and*

$$0 < N < M, \quad b_0 < \frac{M - N}{M}, \quad -b_0 \leq b(t) \leq 0 \ \text{eventually}. \tag{40}$$

Then

(a) *for any $L \in (N, (1 - b_0)M)$, there exist $\theta \in (0, 1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that for each $x_0 \in \Omega_1(N, M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (19) converges to a positive solution $x \in \Omega_1(N, M)$ of (5) and has the error estimate (20), where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in $[0, 1]$ satisfying (21);*

(b) *Equation (5) has uncountably many positive solutions in $\Omega_1(N, M)$.*

Proof. Set $L \in (N, (1 - b_0)M)$. It follows from (18) and (40) that there exist $\theta \in (0, 1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying (23),

$$-b_0 \leq b(t) \leq 0, \quad \forall t \geq T; \tag{41}$$

$$\begin{aligned}
&\int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}\left(\left|g(s)\right| + Q(s)\right)\right. \\
&\qquad \left. + H_i s^{n-m-i-1} W(s)\right] ds\, du \\
&\quad < \min\{L - N, (1 - b_0) M - L\}.
\end{aligned} \tag{42}$$

Define a mapping $S_L : \Omega_1(N,M) \to \mathrm{CB}([\gamma,+\infty),\mathbb{R})$ by (25). Distinctly, $S_L x$ is continuous for each $x \in \Omega_1(N,M)$. In terms of (16), (23), (25), and (41), we reason that for any $x, y \in \Omega_1(N,M)$ and $t \ge T$

$$\begin{aligned}
&\left|S_L x(t) - S_L y(t)\right| \\
&\quad \le |b(t)| \left|x(t-\tau) - y(t-\tau)\right| \\
&\qquad + H_0 \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{|a(u)|} \\
&\qquad \times \left| f(s, x(f_1(s)), \ldots, x(f_l(s))) \right. \\
&\qquad\quad \left. - f(s, y(f_1(s)), \ldots, y(f_l(s))) \right| ds\, du \\
&\qquad + H_i \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{|a(u)|} \\
&\qquad \times \left| h(s, x(h_1(s)), \ldots, x(h_l(s))) \right. \\
&\qquad\quad \left. - h(s, y(h_1(s)), \ldots, y(h_l(s))) \right| ds\, du \\
&\quad \le b_0 \|x-y\| + \|x-y\| \\
&\qquad \times \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \left[H_0 s^{n-m-1} P(s) \right. \\
&\qquad\qquad \left. + H_i s^{n-m-i-1} R(s) \right] ds\, du \\
&\quad = \theta \|x-y\|,
\end{aligned} \tag{43}$$

which means that (27) holds. Owing to (17), (25), (41), and (42), we earn that for any $x \in \Omega_1(N,M)$ and $t \ge T$

$$\begin{aligned}
S_L x(t) &\le L + |b(t)|\, x(t-\tau) \\
&\quad + H_0 \int_t^{+\infty} \int_u^{+\infty} \frac{s^{n-m-1} u^{m-1}}{|a(u)|} \\
&\qquad \times \left[|g(s)| + \left| f(s, x(f_1(s)), \ldots, \right.\right. \\
&\qquad\qquad \left.\left. x(f_l(s))) \right| \right] ds\, du \\
&\quad + H_i \int_t^{+\infty} \int_u^{+\infty} \frac{s^{n-m-i-1} u^{m-1}}{|a(u)|} \\
&\qquad \times \left| h(s, x(h_1(s)), \ldots, \right. \\
&\qquad\qquad \left. x(h_l(s))) \right| ds\, du \\
&\le L + b_0 M + \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\
&\qquad \times \left[H_0 s^{n-m-1} (|g(s)| + Q(s)) \right. \\
&\qquad\qquad \left. + H_i s^{n-m-i-1} W(s) \right] ds\, du \\
&< L + b_0 M + \min\{L-N, (1-b_0)M - L\} \\
&\le M, \\
S_L x(t) &\ge L - H_0 \int_t^{+\infty} \int_u^{+\infty} \frac{s^{n-m-1} u^{m-1}}{|a(u)|} \\
&\qquad \times \left[|g(s)| + \left| f(s, x(f_1(s)), \ldots, \right.\right. \\
&\qquad\qquad \left.\left. x(f_l(s))) \right| \right] ds\, du \\
&\quad - H_i \int_t^{+\infty} \int_u^{+\infty} \frac{s^{n-m-i-1} u^{m-1}}{|a(u)|} \\
&\qquad \times \left| h(s, x(h_1(s)), \ldots, \right. \\
&\qquad\qquad \left. x(h_l(s))) \right| ds\, du \\
&\ge L - \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\
&\qquad \times \left[H_0 s^{n-m-1} (|g(s)| + Q(s)) \right. \\
&\qquad\qquad \left. + H_i s^{n-m-i-1} W(s) \right] ds\, du \\
&> L - \min\{L-N, (1-b_0)M - L\} \\
&\ge N,
\end{aligned} \tag{44}$$

which yield that $S_L(\Omega_1(N,M)) \subseteq \Omega_1(N,M)$. Thus (27) ensures that S_L is a contraction mapping in $\Omega_1(N,M)$ and it owns a unique fixed point $x \in \Omega_1(N,M)$. As in the proof of Theorem 2, we infer that $x \in \Omega_1(N,M)$ is a positive solution of (5). The rest of the proof is parallel to that of Theorem 2, and hence is elided. This completes the proof. □

Theorem 5. *Assume that there exist three constants M, N, and b_0 and four functions $P, Q, R, W \in C([t_0,+\infty), \mathbb{R}^+)$ satisfying* (18) *and*

$$M > N > 0, \quad b_0 > \frac{M}{M-N}, \quad b(t) \ge b_0 \text{ eventually}; \tag{45}$$

$$\begin{aligned}
&\left| f(t, u_1, \ldots, u_l) - f(t, \bar{u}_1, \ldots, \bar{u}_l) \right| \\
&\quad \le P(t) \max\left\{ \left|u_j - \bar{u}_j\right| : 1 \le j \le l \right\}, \\
&\left| h(t, u_1, \ldots, u_l) - h(t, \bar{u}_1, \ldots, \bar{u}_l) \right| \\
&\quad \le R(t) \max\left\{ \left|u_j - \bar{u}_j\right| : 1 \le j \le l \right\}, \\
&\forall (t, u_1, \ldots, u_l, \bar{u}_1, \ldots, \bar{u}_l) \in [t_0, +\infty) \times \left[0, \frac{M}{b_0}\right]^{2l};
\end{aligned} \tag{46}$$

$$\begin{aligned}
&\left| f(t, u_1, \ldots, u_l) \right| \le Q(t), \quad \left| h(t, u_1, \ldots, u_l) \right| \le W(t), \\
&\qquad \forall (t, u_1, \ldots, u_l) \in [t_0, +\infty) \times \left[0, \frac{M}{b_0}\right]^{l}.
\end{aligned} \tag{47}$$

Then

(*a*) *for any* $L \in (N + M/b_0, M)$, *there exist* $\theta \in (0,1)$ *and* $T > 1+|t_0|+\tau+|\gamma|$ *such that for each* $x_0 \in \Omega_2(N,M)$, *the Mann iterative sequence* $\{x_k\}_{k\in\mathbb{N}_0}$ *generated by the following scheme*

$$x_{k+1}(t) = \begin{cases} (1-\alpha_k)\,x_k(t) + \dfrac{\alpha_k}{b(t+\tau)} \\ \quad \times \Big\{ L - x_k(t+\tau) + (-1)^n H_0 \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad \times \left[g(s) - f\left(s, x_k(f_1(s)), \ldots, x_k(f_l(s))\right)\right] ds\,du \\ \quad +(-1)^{n-i-1} H_i \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad \times h\left(s, x_k(h_1(s)), \ldots, x_k(h_l(s))\right) ds\,du \Big\}, \\ \qquad t \ge T,\ k \in \mathbb{N}_0, \\ (1-\alpha_k)\,x_k(T) + \dfrac{\alpha_k}{b(t+\tau)} \\ \quad \times \Big\{ L - x_k(T+\tau) + (-1)^n H_0 \\ \quad \times \displaystyle\int_{T+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-T-\tau)^{m-1}}{a(u)} \\ \quad \times \left[g(s) - f\left(s, x_k(f_1(s)), \ldots, x_k(f_l(s))\right)\right] ds\,du \\ \quad +(-1)^{n-i-1} H_i \\ \quad \times \displaystyle\int_{T+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-T-\tau)^{m-1}}{a(u)} \\ \quad \times h\left(s, x_k(h_1(s)), \ldots, x_k(h_l(s))\right) ds\,du \Big\}, \\ \qquad \gamma \le t < T,\ k \in \mathbb{N}_0 \end{cases} \tag{48}$$

converges to a positive solution $x \in \Omega_2(N,M)$ *of* (5) *and has the error estimate* (20), *where* $\{\alpha_k\}_{k\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *with* (21);

(*b*) *Equation* (5) *has uncountably many positive solutions in* $\Omega_2(N,M)$.

Proof. First of all, we show that (a) holds. Set $L \in (N + M/b_0, M)$. It follows from (18) and (45) that there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that

$$b(t) \ge b_0, \quad \forall t \ge T; \tag{49}$$

$$\theta = \frac{1}{b_0} + \frac{1}{b_0}\int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \times \left[H_0 s^{n-m-1} P(s) + H_i s^{n-m-i-1} R(s)\right] ds\,du; \tag{50}$$

$$\int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right) + H_i s^{n-m-i-1} W(s)\right] ds\,du < \min\left\{M - L, L - \frac{M}{b_0} - N\right\}. \tag{51}$$

Define a mapping $S_L : \Omega_2(N,M) \to CB([\gamma,+\infty),\mathbb{R})$ by

$$S_L x(t) = \begin{cases} \dfrac{L}{b(t+\tau)} - \dfrac{x(t+\tau)}{b(t+\tau)} + \dfrac{(-1)^n H_0}{b(t+\tau)} \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad \times \left[g(s) - f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right)\right] ds\,du \\ \quad + \dfrac{(-1)^{n-i-1} H_i}{b(t+\tau)} \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad \times h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) ds\,du, \\ \qquad t \ge T,\ x \in \Omega_2(N,M), \\ S_L x(T), \quad \gamma \le t < T,\ x \in \Omega_2(N,M). \end{cases} \tag{52}$$

In light of (46), (49), (50), and (52), we conclude that for $x, y \in \Omega_2(N,M)$ and $t \ge T$

$$\begin{aligned} &\left|S_L x(t) - S_L y(t)\right| \\ &\le \frac{1}{b(t+\tau)}\left|x(t+\tau) - y(t+\tau)\right| + \frac{H_0}{b(t+\tau)} \\ &\quad \times \int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{s^{n-m-1}u^{m-1}}{|a(u)|} \\ &\quad \times \left| f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right) - f\left(s, y(f_1(s)), \ldots, y(f_l(s))\right)\right| ds\,du \\ &\quad + \frac{H_i}{b(t+\tau)} \\ &\quad \times \int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{s^{n-m-i-1}u^{m-1}}{a(u)} \\ &\quad \times \left| h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) - h\left(s, y(h_1(s)), \ldots, y(h_l(s))\right)\right| ds\,du \\ &\le \frac{1}{b_0}\|x-y\| + \frac{1}{b_0}\|x-y\| \\ &\quad \times \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1} P(s) + H_i s^{n-m-i-1} R(s)\right] ds\,du \\ &= \theta\|x-y\|, \end{aligned} \tag{53}$$

which yields that

$$\|S_Lx - S_Ly\| \le \theta \|x - y\|, \quad \forall x, y \in \Omega_2(N, M). \tag{54}$$

In view of (47), (49), (51), and (52), we obtain that for any $x \in \Omega_2(N, M)$ and $t \ge T$

$$\begin{aligned}
S_Lx(t) &\le \frac{1}{b(t+\tau)} \times \Bigg\{ L - x(t+\tau) + H_0 \int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{s^{n-m-1}u^{m-1}}{|a(u)|} \times [|g(s)| + |f(s, x(f_1(s)), \dots, x(f_l(s)))|]\, ds\, du \\
&\quad + H_i \int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{s^{n-m-i-1}u^{m-1}}{|a(u)|} \times |h(s, x(h_1(s)), \dots, x(h_l(s)))|\, ds\, du \Bigg\} \\
&\le \frac{1}{b(t+\tau)} \times \Bigg\{ L - \frac{N}{b(t+\tau)} + \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \times \left[H_0 s^{n-m-1} (|g(s)| + Q(s)) + H_i s^{n-m-i-1} W(s) \right] ds\, du \Bigg\} \\
&< \frac{1}{b(t+\tau)} \times \left(L - \frac{N}{b(t+\tau)} + \min\left\{ M - L, L - \frac{M}{b_0} - N \right\} \right) \\
&\le \frac{M}{b(t+\tau)}, \\
S_Lx(t) &\ge \frac{1}{b(t+\tau)} \times \Bigg\{ L - x(t+\tau) - H_0 \int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{s^{n-m-1}u^{m-1}}{|a(u)|} \times [|g(s)| + |f(s, x(f_1(s)), \dots, x(f_l(s)))|]\, ds\, du \\
&\quad - H_i \int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{s^{n-m-i-1}u^{m-1}}{|a(u)|} \times |h(s, x(h_1(s)), \dots, x(h_l(s)))|\, ds\, du \Bigg\} \\
&\ge \frac{1}{b(t+\tau)} \times \Bigg\{ L - \frac{M}{b(t+\tau)} - \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \times \left[H_0 s^{n-m-1} (|g(s)| + Q(s)) + H_i s^{n-m-i-1} W(s) \right] ds\, du \Bigg\} \\
&> \frac{1}{b(t+\tau)} \times \left(L - \frac{M}{b(t+\tau)} - \min\left\{ M - L, L - \frac{M}{b_0} - N \right\} \right) \\
&\ge \frac{N}{b(t+\tau)},
\end{aligned} \tag{55}$$

which imply that $S_L(\Omega_2(N, M)) \subseteq \Omega_2(N, M)$. It follows from (50) and (54) that S_L is a contraction mapping in $\Omega_2(N, M)$ and it has a unique fixed point $x \in \Omega_2(N, M)$. It is clear that $x \in \Omega_2(N, M)$ is a positive solution of (5).

Note that (48), (52), and (54) undertake that

$$\begin{aligned}
&|x_{k+1}(t) - x(t)| \\
&\quad = \Bigg| (1 - \alpha_k) x_k(t) + \frac{\alpha_k}{b(t+\tau)} \times \Bigg\{ L - x_k(t+\tau) + (-1)^n H_0 \\
&\qquad \times \int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{a(u)} \times [g(s) - f(s, x_k(f_1(s)), \dots, sx_k(f_l(s)))]\, ds\, du \\
&\qquad + (-1)^{n-i-1} H_i
\end{aligned}$$

$$\times \int_{t+\tau}^{+\infty} \int_{u}^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{a(u)} \times h(s, x_k(h_1(s)), \ldots, x_k(h_l(s)))\, ds\, du \Big\} - x(t) \Big|$$
$$\le (1-\alpha_k)|x_k(t) - x(t)| + \alpha_k |S_L x_k(t) - S_L x(t)|$$
$$\le (1-\alpha_k)|x_k(t) - x(t)| + \alpha_k\theta |x_k(t) - x(t)|$$
$$= (1-(1-\theta)\alpha_k)|x_k(t) - x(t)|$$
$$\le e^{-(1-\theta)\alpha_k}\|x_k - x\|$$
$$\le e^{-(1-\theta)\sum_{p=0}^{k}\alpha_p}\|x_0 - x\|, \quad \forall k \in \mathbb{N}_0,\ t \ge T, \tag{56}$$

which indicates that (20) holds. Thus (20) and (21) assure that $\lim_{k\to\infty} x_k = x$.

Next we prove that (b) holds. Let $L_1, L_2 \in (N + M/b_0, M)$ with $L_1 \ne L_2$. As in the proof of (a) we infer that for each $p \in \{1,2\}$ there exist $\theta_p \in (0,1)$, $T_p > 1 + |t_0| + \tau + |\gamma|$ and S_{L_p} satisfying (49)–(52), where L, θ, T, and S_L are replaced by L_p, θ_p, T_p, and S_{L_p}, respectively, and S_{L_p} has a unique fixed point $z_p \in \Omega_2(N, M)$, which is a positive solution of (5) in $\Omega_2(N, M)$. It follows that for each $t \ge T_p$ and $p \in \{1,2\}$

$$z_p(t) = \frac{L_p}{b(t+\tau)} - \frac{z_p(t+\tau)}{b(t+\tau)} + \frac{(-1)^n H_0}{b(t+\tau)} \times \int_{t+\tau}^{+\infty}\int_{u}^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{a(u)} \times \left[g(s) - f\left(s, z_p(f_1(s)), \ldots, z_p(f_l(s))\right)\right] ds\, du$$
$$+ \frac{(-1)^{n-i-1}H_i}{b(t+\tau)} \times \int_{t+\tau}^{+\infty}\int_{u}^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{a(u)} \times h\left(s, z_p(h_1(s)), \ldots, z_p(h_l(s))\right) ds\, du. \tag{57}$$

On behalf of proving (b), we need only to show that $z_1 \ne z_2$. Notice that (18) guarantees that there exits $T_3 > \max\{T_1, T_2\}$ satisfying

$$\int_{T_3}^{+\infty}\int_{u}^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}P(s) + H_i s^{n-m-i-1}R(s)\right] ds\, du < \frac{|L_1 - L_2|}{1 + 2\|z_1 - z_2\|}. \tag{58}$$

Due to (46), (51), (57), and (58), we conclude that for each $t \ge T_3$

$$\left| z_1(t) - z_2(t) + \frac{z_1(t+\tau)}{b(t+\tau)} - \frac{z_2(t+\tau)}{b(t+\tau)} \right|$$
$$\ge \frac{1}{b(t+\tau)} \times \Big(|L_1 - L_2| - H_0 \int_{T_3}^{+\infty}\int_{u}^{+\infty} \frac{s^{n-m-1}u^{m-1}}{|a(u)|} \times |f((s, z_1(f_1(s)), \ldots, z_1(f_l(s))) - f(s, z_2(f_1(s)), \ldots, z_2(f_l(s))))|\, ds\, du$$
$$- H_i \int_{T_3}^{+\infty}\int_{u}^{+\infty} \frac{s^{n-m-i-1}u^{m-1}}{|a(u)|} \times |h(s, z_1(f_1(s)), \ldots, z_1(f_l(s))) - h(s, z_2(f_1(s)), \ldots, z_2(f_l(s)))|\, ds\, du \Big)$$
$$\ge \frac{1}{b(t+\tau)} \times \Big(|L_1 - L_2| - \|z_1 - z_2\| \times \int_{T_3}^{+\infty}\int_{u}^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}P(s) + H_i s^{n-m-i-1}R(s)\right] ds\, du \Big)$$
$$> \frac{1}{b(t+\tau)}\left(|L_1 - L_2| - \|z_1 - z_2\| \frac{|L_1 - L_2|}{1 + 2\|z_1 - z_2\|} \right)$$
$$> \frac{|L_1 - L_2|}{2b(t+\tau)}$$
$$> 0, \tag{59}$$

which yields that $z_1 \ne z_2$. This completes the proof. □

Theorem 6. *Assume that there exist three constants M, N, and b_0 and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (18), (46), (47), *and*

$$0 < N < M, \quad \frac{M}{M-N} < b_0, \quad b(t) \le -b_0 \text{ eventually}. \tag{60}$$

Then

(*a*) *for any* $L \in (N, (1-1/b_0)M)$, *there exist* $\theta \in (0,1)$ *and* $T > 1+|t_0|+\tau+|\gamma|$ *such that for each* $x_0 \in \Omega_3(N,M)$, *the Mann iterative sequence* $\{x_k\}_{k\in\mathbb{N}_0}$ *generated by the following scheme*

$$x_{k+1}(t) = \begin{cases} (1-\alpha_k)\,x_k(t) + \dfrac{\alpha_k}{b(t+\tau)} \\ \times \Big\{ -L - x_k(t+\tau) + (-1)^n H_0 \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{a(u)} \\ \qquad \times \left[g(s) - f\left(s, x_k(f_1(s)),\ldots, x_k(f_l(s))\right)\right] ds\,du \\ \quad +(-1)^{n-i-1}H_i \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad \times h\left(s, x_k(h_1(s)),\ldots, x_k(h_l(s))\right) ds\,du \Big\}, \\ \hfill t \geq T,\ k \in \mathbb{N}_0, \\ (1-\alpha_k)\,x_k(T) + \dfrac{\alpha_k}{b(t+\tau)} \\ \times \Big\{ -L - x_k(T+\tau) + (-1)^n H_0 \\ \quad \times \displaystyle\int_{T+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-T-\tau)^{m-1}}{a(u)} \\ \qquad \times \left[g(s) - f\left(s, x_k(f_1(s)),\ldots, x_k(f_l(s))\right)\right] ds\,du \\ \quad +(-1)^{n-i-1}H_i \\ \quad \times \displaystyle\int_{T+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-T-\tau)^{m-1}}{a(u)} \\ \quad \times h\left(s, x_k(h_1(s)),\ldots, x_k(h_l(s))\right) ds\,du \Big\}, \\ \hfill \gamma \leq t < T,\ k \in \mathbb{N}_0 \end{cases} \tag{61}$$

converges to a positive solution $x \in \Omega_3(N,M)$ *of* (5) *and has the error estimate* (20), *where* $\{\alpha_k\}_{k\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *satisfying* (21);

(*b*) *Equation* (5) *has uncountably many positive solutions in* $\Omega_3(N,M)$.

Proof. Put $L \in (N, (1-1/b_0)M)$. It follows from (18) and (60) that there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying (50) and

$$\begin{gathered} b(t) \leq -b_0, \quad \forall t \geq T; \\ \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right) + H_i s^{n-m-i-1} W(s)\right] ds\,du \\ < \min\left\{M\left(1-\frac{1}{b_0}\right) - L, L - N\right\}. \end{gathered} \tag{62}$$

Define a mapping $S_L : \Omega_3(N,M) \to \mathrm{CB}([\gamma,+\infty),\mathbb{R})$ by

$$S_L x(t) = \begin{cases} \dfrac{-L}{b(t+\tau)} - \dfrac{x(t+\tau)}{b(t+\tau)} + \dfrac{(-1)^n H_0}{b(t+\tau)} \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad \times \left[g(s) - f\left(s, x(f_1(s)),\ldots, x(f_l(s))\right)\right] ds\,du \\ \quad + \dfrac{(-1)^{n-i-1}H_i}{b(t+\tau)} \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad \times h\left(s, x(h_1(s)),\ldots, x(h_l(s))\right) ds\,du, \\ \hfill t \geq T,\ x \in \Omega_3(N,M), \\ S_L x(T), \quad \gamma \leq t < T,\ x \in \Omega_3(N,M). \end{cases} \tag{63}$$

By virtue of (47), (62), and (63), we know that for any $x \in \Omega_3(N,M)$ and $t \geq T$

$$\begin{aligned} S_L x(t) &\leq \frac{1}{b(t+\tau)} \times \Bigg(-L - x(t+\tau) - H_0 \\ &\quad \times \int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{|a(u)|} \\ &\quad \times \left[|g(s)| + \left|f\left(s, x(f_1(s)),\ldots, x(f_l(s))\right)\right|\right] ds\,du \\ &\quad - H_i \int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{|a(u)|} \\ &\quad \times \left|h\left(s, x(h_1(s)),\ldots, x(h_l(s))\right)\right| ds\,du \Bigg) \\ &\leq \frac{1}{b(t+\tau)} \times \Bigg(-L + \frac{M}{b(t+\tau)} \\ &\quad - \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\ &\quad \times \left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right) + H_i s^{n-m-i-1} W(s)\right] ds\,du \Bigg) \end{aligned}$$

$$< \frac{1}{b(t+\tau)}\left(-L+\frac{M}{b(t+\tau)} - \min\left\{M\left(1-\frac{1}{b_0}\right)-L, L-N\right\}\right) \le \frac{-M}{b(t+\tau)},$$

$$\begin{aligned}
S_Lx(t) &\ge \frac{1}{b(t+\tau)} \times \Bigg(-L - x(t+\tau) + H_0 \times \int_{t+\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{|a(u)|} \\
&\quad \times \left[\left|g(s)\right| + \left|f(s, x(f_1(s)), \ldots, x(f_l(s)))\right|\right] ds\, du \\
&\quad + H_i \int_{t+\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{|a(u)|} \times \left|h(s, x(h_1(s)), \ldots, x(h_l(s)))\right| ds\, du\Bigg) \\
&\ge \frac{1}{b(t+\tau)} \times \Bigg(-L+\frac{N}{b(t+\tau)} + \int_T^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|} \times \left[H_0 s^{n-m-1}\left(\left|g(s)\right| + Q(s)\right) + H_i s^{n-m-i-1}W(s)\right] ds\, du\Bigg) \\
&> \frac{1}{b(t+\tau)}\left(-L+\frac{N}{b(t+\tau)} + \min\left\{M\left(1-\frac{1}{b_0}\right)-L, L-N\right\}\right) \\
&\ge \frac{-N}{b(t+\tau)},
\end{aligned} \tag{64}$$

which imply that $S_L(\Omega_3(N, M)) \subseteq \Omega_3(N, M)$. The rest of the proof is identical with the proof of Theorem 5 and hence is omitted. This completes the proof. □

Theorem 7. *Let $m \ge 2$. Assume that there exist two constants M, N with $M > N > 0$ and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (16)–(18) *and*

$$b(t) = 1 \quad \text{eventually.} \tag{65}$$

Then

(*a*) *for any $L \in (N, M)$, there exist $\theta \in (0, 1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that for each $x_0 \in \Omega_1(N, M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by the following scheme*

$$x_{k+1}(t) = \begin{cases}
(1-\alpha_k)x_k(t) + \alpha_k \times \Bigg\{L + (-1)^n(m-1)H_0 \times \displaystyle\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_r^{+\infty}\int_u^{+\infty}\left(\left((s-u)^{n-m-1}(u-r)^{m-2}\right) \times (a(u))^{-1}\right) \\
\quad \times \left[g(s) - f(s, x_k(f_1(s)), \ldots, x_k(f_l(s)))\right] ds\, du\, dr \\
\quad + (-1)^{n-i-1}(m-1)H_i \times \displaystyle\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_r^{+\infty}\int_u^{+\infty}\left(\left((s-u)^{n-m-i-1}(u-r)^{m-2}\right) \times (a(u))^{-1}\right) \\
\quad \times h(s, x_k(h_1(s)), \ldots, x_k(h_l(s)))\, ds\, du\, dr\Bigg\}, \quad t \ge T,\ k \in \mathbb{N}_0, \\
(1-\alpha_k)x_k(T) + \alpha_k \times \Bigg\{L + (-1)^n(m-1)H_0 \times \displaystyle\sum_{j=1}^{\infty}\int_{T+(2j-1)\tau}^{T+2j\tau}\int_r^{+\infty}\int_u^{+\infty}\left(\left((s-u)^{n-m-1}(u-r)^{m-2}\right) \times (a(u))^{-1}\right) \\
\quad \times \left[g(s) - f(s, x_k(f_1(s)), \ldots, x_k(f_l(s)))\right] ds\, du\, dr \\
\quad + (-1)^{n-i-1}(m-1)H_i \times \displaystyle\sum_{j=1}^{\infty}\int_{T+(2j-1)\tau}^{T+2j\tau}\int_r^{+\infty}\int_u^{+\infty}\left(\left((s-u)^{n-m-i-1}(u-r)^{m-2}\right) \times (a(u))^{-1}\right) \\
\quad \times h(s, x_k(h_1(s)), \ldots, x_k(h_l(s)))\, ds\, du\, dr\Bigg\}, \quad \gamma \le t < T,\ k \in \mathbb{N}_0
\end{cases} \tag{66}$$

converges to a positive solution $x \in \Omega_1(N, M)$ of (5) *and has the error estimate* (20), *where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in* $[0, 1]$ *with* (21);

(*b*) *Equation* (5) *has uncountably many positive solutions in $\Omega_1(N, M)$.*

Proof. Let $L \in (N, M)$. It follows from (18) and (65) that there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying

$$b(t) = 1, \quad \forall t \geq T; \tag{67}$$

$$\theta = \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}P(s) + H_i s^{n-m-i-1}R(s)\right] ds\, du; \tag{68}$$

$$\int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right) + H_i s^{n-m-i-1}W(s)\right] ds\, du < \min\{M - L, L - N\}. \tag{69}$$

Define a mapping $S_L : \Omega_1(N, M) \to \mathrm{CB}([\gamma, +\infty), \mathbb{R})$ by

$$S_L x(t) = \begin{cases} L + (-1)^n (m-1) H_0 \\ \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \left(\left((s-u)^{n-m-1}(u-r)^{m-2}\right) \times (a(u))^{-1}\right) \\ \times \left[g(s) - f\left(s, x\left(f_1(s)\right), \dots, x\left(f_l(s)\right)\right)\right] ds\, du\, dr \\ +(-1)^{n-i-1}(m-1) H_i \\ \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \left(\left((s-u)^{n-m-i-1}(u-r)^{m-2}\right) \times (a(u))^{-1}\right) \\ \times h\left(s, x\left(h_1(s)\right), \dots, x\left(h_l(s)\right)\right) ds\, du\, dr, \\ \qquad t \geq T,\ x \in \Omega_1(N, M), \\ S_L x(T), \qquad \gamma \leq t < T,\ x \in \Omega_1(N, M). \end{cases} \tag{70}$$

With a view to (16), (68), and (70), we derive that for any $x, y \in \Omega_1(N, M)$ and $t \geq T$

$$\begin{aligned} &\left|S_L x(t) - S_L y(t)\right| \\ &\leq (m-1) H_0 \\ &\quad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{|a(u)|} \\ &\quad \times \left|f\left(s, x\left(f_1(s)\right), \dots, x\left(f_l(s)\right)\right) - f\left(s, y\left(f_1(s)\right), \dots, y\left(f_l(s)\right)\right)\right| ds\, du\, dr \\ &\quad + (m-1) H_i \\ &\quad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{|a(u)|} \\ &\quad \times \left|h\left(s, x\left(h_1(s)\right), \dots, x\left(h_l(s)\right)\right) - h\left(s, y\left(h_1(s)\right), \dots, y\left(h_l(s)\right)\right)\right| ds\, du\, dr \\ &\leq (m-1) H_0 \|x - y\| \\ &\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{|a(u)|} P(s)\, ds\, du\, dr \\ &\quad + (m-1) H_i \|x - y\| \\ &\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{|a(u)|} R(s)\, ds\, du\, dr \\ &= H_0 \|x - y\| \\ &\quad \times \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{|a(u)|} P(s)\, ds\, du \\ &\quad + H_i \|x - y\| \\ &\quad \times \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{|a(u)|} R(s)\, ds\, du \\ &\leq \|x - y\| \\ &\quad \times \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1} P(s) + H_i s^{n-m-i-1} R(s)\right] ds\, du \\ &= \theta \|x - y\|, \end{aligned} \tag{71}$$

which gives (27). By virtue of (17), (69), and (70), we deduce that for any $x \in \Omega_1(N, M)$ and $t \geq T$

$$\begin{aligned} &S_L x(t) \\ &\leq L + (m-1) H_0 \\ &\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{|a(u)|} \\ &\quad \times \left[|g(s)| + \left|f\left(s, x\left(f_1(s)\right), \dots, x\left(f_l(s)\right)\right)\right|\right] ds\, du\, dr \\ &\quad + (m-1) H_i \\ &\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{|a(u)|} \\ &\quad \times \left|h\left(s, x\left(h_1(s)\right), \dots, x\left(h_l(s)\right)\right)\right| ds\, du\, dr \\ &\leq L + (m-1) H_0 \\ &\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{|a(u)|} \\ &\quad \times \left[|g(s)| + Q(s)\right] ds\, du\, dr \\ &\quad + (m-1) H_i \\ &\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{|a(u)|} W(s)\, ds\, du\, dr \end{aligned}$$

$$\begin{aligned}
&= L + H_0 \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{|a(u)|} \\
&\qquad \times \left[|g(s)| + Q(s)\right] ds\, du \\
&\quad + H_i \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{|a(u)|} \\
&\qquad \times W(s)\, ds\, du \\
&\le L + \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\
&\qquad \times \left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right) \right. \\
&\qquad \left. + H_i s^{n-m-i-1} W(s)\right] ds\, du \\
&< L + \min\{M-L, L-N\} \\
&\le M, \\
&S_L x(t) \\
&\ge L - (m-1) H_0 \\
&\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{|a(u)|} \\
&\quad \times \left[\, |g(s)| + \left| f\left(s, x\left(f_1(s)\right), \ldots, \right.\right.\right. \\
&\qquad \left.\left.\left. x\left(f_l(s)\right)\right)\right|\right] ds\, du\, dr \\
&\quad - (m-1) H_i \\
&\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{|a(u)|} \\
&\quad \times \left|h\left(s, x\left(h_1(s)\right), \ldots, x\left(h_l(s)\right)\right)\right| ds\, du\, dr \\
&\ge L - \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\
&\quad \times \left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right) + H_i s^{n-m-i-1} W(s)\right] ds\, du \\
&> L - \min\{M-L, L-N\} \\
&\ge N,
\end{aligned} \tag{72}$$

which mean that $S_L(\Omega_1(N, M)) \subseteq \Omega_1(N, M)$. Coupled with (27) and (68), we get that S_L is a contraction mapping in $\Omega_1(N, M)$ and it possesses a unique fixed point $x \in \Omega_1(N, M)$. Clearly, $x \in \Omega_1(N, M)$ is a positive solution of (5).

From (27), (66), and (70), we gain that

$$\begin{aligned}
&\left|x_{k+1}(t) - x(t)\right| \\
&= \Bigg| \left(1-\alpha_k\right) x_k(t) \\
&\quad + \alpha_k \left\{ L + (-1)^n (m-1) H_0 \right. \\
&\qquad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\
&\qquad \times \left[g(s) - f\left(s, x_k\left(f_1(s)\right), \ldots, x_k\left(f_l(s)\right)\right)\right] ds\, du\, dr \\
&\qquad + (-1)^{n-i-1} (m-1) H_i \\
&\qquad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\
&\qquad \left. \times h\left(s, x_k\left(h_1(s)\right), \ldots, x_k\left(h_l(s)\right)\right) ds\, du\, dr \right\} - x(t) \Bigg| \\
&\le \left(1-\alpha_k\right)\left|x_k(t) - x(t)\right| + \alpha_k \left|S_L x_k(t) - S_L x(t)\right| \\
&\le \left(1-\alpha_k\right)\left|x_k(t) - x(t)\right| + \alpha_k \theta \left|x_k(t) - x(t)\right| \\
&= \left(1 - (1-\theta)\alpha_k\right)\left|x_k(t) - x(t)\right| \\
&\le e^{-(1-\theta)\alpha_k} \left\|x_k - x\right\| \\
&\le e^{-(1-\theta)\sum_{p=0}^{k} \alpha_p} \left\|x_0 - x\right\|, \quad \forall k \in \mathbb{N}_0,\ t \ge T,
\end{aligned} \tag{73}$$

which yields (20). It follows from (20) and (21) that $\lim_{k\to\infty} x_k = x$.

Now we prove that (b) holds. Let $L_1, L_2 \in (N, M)$ and $L_1 \neq L_2$. As in the proof of (a), we conclude that for each $p \in \{1, 2\}$, there exist $\theta_p \in (0, 1)$, $T_p > 1 + |t_0| + \tau + |\gamma|$ and $S_{L_p} : \Omega_1(N, M) \to \Omega_1(N, M)$ satisfying (69)–(77), where L, θ, T, and S_L are replaced by L_p, θ_p, T_p, and S_{L_p}, respectively, and S_{L_p} has a unique fixed point $z_p \in \Omega_1(N, M)$, which is a positive solution of (5) in $\Omega_1(N, M)$, that is,

$$\begin{aligned}
z_p(t) &= L_k + (-1)^n (m-1) H_0 \\
&\quad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\
&\quad \times \left[g(s) - f\left(s, z_p\left(f_1(s)\right), \ldots, z_p\left(f_l(s)\right)\right)\right] ds\, du\, dr \\
&\quad + (-1)^{n-i-1} (m-1) H_i \\
&\quad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\
&\quad \times h\left(s, z_p\left(h_1(s)\right), \ldots, z_p\left(h_l(s)\right)\right) ds\, du\, dr, \\
&\qquad \forall t \ge T_p,\ p \in \{1, 2\}.
\end{aligned} \tag{74}$$

For purpose of proving (b), we just need to show that $z_1 \neq z_2$. It follows from (16), (27), (68), and (74) that

$$\begin{aligned} &\left|z_1(t)-z_2(t)\right| \\ &\quad\geq \left|L_1-L_2\right|-(m-1)H_0\left\|z_1-z_2\right\| \\ &\qquad\times\int_t^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\left(\left((s-u)^{n-m-1}(u-r)^{m-2}\right)\right. \\ &\qquad\qquad\left.\times(|a(u)|)^{-1}\right)P(s)\,ds\,du\,dr \\ &\qquad-(m-1)H_i\left\|z_1-z_2\right\| \\ &\qquad\times\int_t^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\left(\left((s-u)^{n-m-i-1}(u-r)^{m-2}\right)\right. \\ &\qquad\qquad\left.\times(|a(u)|)^{-1}\right)R(s)\,ds\,du\,dr \\ &\quad\geq \left|L_1-L_2\right|-\left\|z_1-z_2\right\| \\ &\qquad\times\int_{\max\{T_1,T_2\}}^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|} \\ &\qquad\qquad\times\left[H_0 s^{n-m-1}P(s)\right. \\ &\qquad\qquad\left.+H_i s^{n-m-i-1}R(s)\right]ds\,du \\ &\quad> \left|L_1-L_2\right|-\max\{\theta_1,\theta_2\}\left\|z_1-z_2\right\|, \\ &\qquad\qquad\forall t\geq\max\{T_1,T_2\}, \end{aligned} \tag{75}$$

which yields that

$$\left\|z_1-z_2\right\| \geq \frac{\left|L_1-L_2\right|}{1+\max\{\theta_1,\theta_2\}}>0, \tag{76}$$

that is, $z_1 \neq z_2$. This completes the proof. □

Theorem 8. *Let $m=1$. Assume that there exist two constants M, N with $M>N>0$ and four functions $P, Q, R, W \in C([t_0,+\infty),\mathbb{R}^+)$ satisfying (16), (17), (65), and*

$$\begin{aligned} &\int_{t_0}^{+\infty}\int_u^{+\infty}\frac{1}{|a(u)|} \\ &\qquad\times\left[|s|^{n-2}\max\{P(s),Q(s),|g(s)|\}\right. \\ &\qquad\left.+|s|^{n-i-2}\max\{R(s),W(s)\}\right]ds\,du \\ &\quad<+\infty. \end{aligned} \tag{77}$$

Then

(a) for any $L \in (N,M)$, there exist $\theta \in (0,1)$ and $T > 1+|t_0|+\tau+|\gamma|$ such that for each $x_0 \in \Omega_1(N,M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by the following scheme

$$\begin{aligned} &x_{k+1}(t) \\ &=\begin{cases} (1-\alpha_k)x_k(t) \\ +\alpha_k\left\{L+\dfrac{(-1)^n}{(n-2)!}\right. \\ \quad\times\displaystyle\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_u^{+\infty}\frac{(s-u)^{n-2}}{a(u)} \\ \quad\times\left[g(s)-f\left(s,x_k\left(f_1(s)\right),\ldots,x_k\left(f_l(s)\right)\right)\right]ds\,du \\ \quad+\dfrac{(-1)^{n-i-1}}{(n-i-2)!} \\ \quad\times\displaystyle\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)} \\ \quad\left.\times h\left(s,x_k\left(h_1(s)\right),\ldots,x_k\left(h_l(s)\right)\right)ds\,du\right\}, \\ \qquad\qquad t\geq T,\ k\in\mathbb{N}_0, \\ (1-\alpha_k)x_k(T) \\ +\alpha_k\left\{L+\dfrac{(-1)^n}{(n-2)!}\right. \\ \quad\times\displaystyle\sum_{j=1}^{\infty}\int_{T+(2j-1)\tau}^{T+2j\tau}\int_u^{+\infty}\frac{(s-u)^{n-2}}{a(u)} \\ \quad\times\left[g(s)-f\left(s,x_k\left(f_1(s)\right),\ldots,x_k\left(f_l(s)\right)\right)\right]ds\,du \\ \quad+\dfrac{(-1)^{n-i-1}}{(n-i-2)!} \\ \quad\times\displaystyle\sum_{j=1}^{\infty}\int_{T+(2j-1)\tau}^{T+2j\tau}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)} \\ \quad\left.\times h\left(s,x_k\left(h_1(s)\right),\ldots,x_k\left(h_l(s)\right)\right)ds\,du\right\}, \\ \qquad\qquad \gamma\leq t<T,\ k\in\mathbb{N}_0 \end{cases} \end{aligned} \tag{78}$$

converges to a positive solution $x \in \Omega_1(N,M)$ of (5) and has the error estimate (20), where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ with (21);

(b) Equation (5) has uncountably many positive solutions in $\Omega_1(N,M)$.

Proof. Let $L \in (N, M)$. It follows from (65) and (77) that there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying (67),

$$\theta = \int_T^{+\infty} \int_u^{+\infty} \frac{1}{|a(u)|} \times \left[\frac{s^{n-2}}{(n-2)!} P(s) + \frac{s^{n-i-2}}{(n-i-2)!} R(s) \right] ds\, du, \tag{79}$$

$$\begin{aligned} &\int_T^{+\infty} \int_u^{+\infty} \frac{1}{|a(u)|} \left[\frac{s^{n-2}}{(n-2)!} \left(|g(s)| + Q(s) \right) + \frac{s^{n-i-2}}{(n-i-2)!} W(s) \right] ds\, du \\ &\quad < \min \{ M - L, L - N \}. \end{aligned} \tag{80}$$

Define a mapping $S_L : \Omega_1(N, M) \to \mathrm{CB}([\gamma, +\infty), \mathbb{R})$ by

$$S_L x(t) = \begin{cases} L + \dfrac{(-1)^n}{(n-2)!} \displaystyle\sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{a(u)} \\ \quad \times \left[g(s) - f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right) \right] ds\, du \\ \quad + \dfrac{(-1)^{n-i-1}}{(n-i-2)!} \displaystyle\sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{a(u)} \\ \quad \times h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) ds\, du, \\ \qquad\qquad t \geq T,\ x \in \Omega_1(N, M), \\ S_L x(T), \qquad \gamma \leq t < T,\ x \in \Omega_1(N, M). \end{cases} \tag{81}$$

By virtue of (16), (79), and (81), we derive that for any $x, y \in \Omega_1(N, M)$ and $t \geq T$

$$\begin{aligned} &\left| S_L x(t) - S_L y(t) \right| \\ &\quad \leq \frac{1}{(n-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{|a(u)|} \\ &\qquad \times \left| f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right) - f\left(s, y(f_1(s)), \ldots, y(f_l(s))\right) \right| ds\, du \\ &\qquad + \frac{1}{(n-i-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{|a(u)|} \\ &\qquad \times \left| h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) - h\left(s, y(h_1(s)), \ldots, y(h_l(s))\right) \right| ds\, du \\ &\quad \leq \frac{1}{(n-2)!} \|x - y\| \int_T^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-2}}{|a(u)|} P(s)\, ds\, du \\ &\qquad + \frac{1}{(n-i-2)!} \|x - y\| \times \int_T^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{|a(u)|} R(s)\, ds\, du \\ &\quad \leq \|x - y\| \int_T^{+\infty} \int_u^{+\infty} \frac{1}{|a(u)|} \left[\frac{s^{n-2}}{(n-2)!} P(s) + \frac{s^{n-i-2}}{(n-i-2)!} R(s) \right] ds\, du \\ &\quad = \theta \|x - y\|, \end{aligned} \tag{82}$$

which gives (27). It follows from (17), (80), and (81) that for any $x \in \Omega_1(N, M)$ and $t \geq T$

$$\begin{aligned} &S_L x(t) \\ &\quad \leq L + \frac{1}{(n-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{|a(u)|} \\ &\qquad \times \left[|g(s)| + \left| f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right) \right| \right] ds\, du \\ &\qquad + \frac{1}{(n-i-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{|a(u)|} \\ &\qquad \times \left| h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) \right| ds\, du \\ &\quad \leq L + \int_T^{+\infty} \int_u^{+\infty} \frac{1}{|a(u)|} \\ &\qquad \times \left[\frac{s^{n-2}}{(n-2)!} \left(|g(s)| + Q(s) \right) + \frac{s^{n-i-2}}{(n-i-2)!} W(s) \right] ds\, du \\ &\quad < L + \min \{ M - L, L - N \} \\ &\quad \leq M, \\ &S_L x(t) \\ &\quad \geq L - \frac{1}{(n-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{|a(u)|} \\ &\qquad \times \left[|g(s)| + \left| f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right) \right| \right] ds\, du \\ &\qquad - \frac{1}{(n-i-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{|a(u)|} \\ &\qquad \times \left| h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) \right| ds\, du \\ &\quad \geq L - \int_T^{+\infty} \int_u^{+\infty} \frac{1}{|a(u)|} \end{aligned}$$

$$\times\left[\frac{s^{n-2}}{(n-2)!}\left(|g(s)|+Q(s)\right)+\frac{s^{n-i-2}}{(n-i-2)!}W(s)\right]ds\,du$$
$$> L-\min\{M-L,L-N\}$$
$$\geq N, \tag{83}$$

which mean that $S_L(\Omega_1(N,M)) \subseteq \Omega_1(N,M)$. Combined with (27) and (79), we know that S_L is a contraction mapping in $\Omega_1(N,M)$ and it possesses a unique fixed point $x \in \Omega_1(N,M)$. Obviously, $x \in \Omega_1(N,M)$ is a positive solution of (5).

In light of (27), (78), and (81), we gain that

$$\begin{aligned}
&|x_{k+1}(t)-x(t)| \\
&=\Bigg|(1-\alpha_k)x_k(t) \\
&\quad+\alpha_k\Bigg\{L+\frac{(-1)^n}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_u^{+\infty}\frac{(s-u)^{n-2}}{a(u)} \\
&\qquad\times\left[g(s)-f\left(s,x_k(f_1(s)),\dots,x_k(f_l(s))\right)\right]ds\,du \\
&\qquad+\frac{(-1)^{n-i-1}}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)} \\
&\qquad\times h\left(s,x_k(h_1(s)),\dots,x_k(h_l(s))\right)ds\,du\Bigg\}-x(t)\Bigg| \\
&\leq(1-\alpha_k)|x_k(t)-x(t)|+\alpha_k|S_Lx_k(t)-S_Lx(t)| \\
&\leq(1-\alpha_k)|x_k(t)-x(t)|+\alpha_k\theta|x_k(t)-x(t)| \\
&=(1-(1-\theta)\alpha_k)|x_k(t)-x(t)| \\
&\leq e^{-(1-\theta)\alpha_k}\|x_k-x\| \\
&\leq e^{-(1-\theta)\sum_{p=0}^{k}\alpha_p}\|x_0-x\|,\quad \forall k\in\mathbb{N}_0,\ t\geq T,
\end{aligned} \tag{84}$$

which yields (20). It follows from (20) and (21) that $\lim_{k\to\infty}x_k = x$.

Now we prove that (b) holds. Let $L_1, L_2 \in (N,M)$ and $L_1 \neq L_2$. As in the proof of (a), we conclude that for each $p \in \{1,2\}$, there exist $\theta_p \in (0,1)$, $T_p > 1+|t_0|+\tau+|\gamma|$ and $S_{L_p} : \Omega_1(N,M) \to \Omega_1(N,M)$ satisfying (67) and (79)–(81), where L, θ, T, and S_L are replaced by L_p, θ_p, T_p, and S_{L_p}, respectively, and S_{L_p} has a unique fixed point $z_p \in \Omega_1(N,M)$, which is a positive solution of (5) in $\Omega_1(N,M)$, that is,

$$\begin{aligned}
z_p(t)&=L_p+\frac{(-1)^n}{(n-2)!} \\
&\quad\times\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_u^{+\infty}\frac{(s-u)^{n-2}}{a(u)} \\
&\quad\times\left[g(s)-f\left(s,z_p(f_1(s)),\dots,z_p(f_l(s))\right)\right]ds\,du \\
&\quad+\frac{(-1)^{n-i-1}}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)} \\
&\quad\times h\left(s,z_p(h_1(s)),\dots,z_p(h_l(s))\right)ds\,du, \\
&\qquad\qquad \forall t\geq T_p,\ p\in\{1,2\}.
\end{aligned} \tag{85}$$

In order to prove (b), we just need to show that $z_1 \neq z_2$. In view of (16), (27), (79), and (85), we get that

$$\begin{aligned}
&|z_1(t)-z_2(t)| \\
&\geq|L_1-L_2|-\frac{1}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_u^{+\infty}\frac{(s-u)^{n-2}}{|a(u)|} \\
&\quad\times\left|f\left(s,z_2(f_1(s)),\dots,z_2(f_l(s))\right)\right. \\
&\quad\left.-f\left(s,z_1(f_1(s)),\dots,z_1(f_l(s))\right)\right|ds\,du \\
&\quad-\frac{1}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{|a(u)|} \\
&\quad\times\left|h\left(s,z_1(h_1(s)),\dots,z_1(h_l(s))\right)\right. \\
&\quad\left.-h\left(s,z_2(h_1(s)),\dots,z_2(h_l(s))\right)\right|ds\,du \\
&\geq|L_1-L_2|-\frac{\|z_1-z_2\|}{(n-2)!}\int_t^{+\infty}\int_u^{+\infty}\frac{s^{n-2}}{|a(u)|}P(s)\,ds\,du \\
&\quad-\frac{\|z_1-z_2\|}{(n-i-2)!}\int_t^{+\infty}\int_u^{+\infty}\frac{s^{n-i-2}}{|a(u)|}R(s)\,ds\,du \\
&\geq|L_1-L_2|-\|z_1-z_2\| \\
&\quad\times\int_{\max\{T_1,T_2\}}^{+\infty}\int_u^{+\infty}\frac{1}{|a(u)|} \\
&\quad\times\left[\frac{s^{n-2}}{(n-2)!}P(s)+\frac{s^{n-i-2}}{(n-i-2)!}R(s)\right]ds\,du \\
&>|L_1-L_2|-\max\{\theta_1,\theta_2\}\|z_1-z_2\|, \\
&\qquad\qquad \forall t\geq\max\{T_1,T_2\},
\end{aligned} \tag{86}$$

which implies that

$$\|z_1 - z_2\| \geq \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0, \quad (87)$$

that is, $z_1 \neq z_2$. This completes the proof. □

Theorem 9. *Let $m \geq 2$. Assume that there exist two constants M, N with $M > N > 0$ and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (16), (17),

$$\int_{t_0}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty} \frac{|r|\,|u|^m}{|a(u)|} \times \left[|s|^{n-m-1} \max\{P(s), Q(s), |g(s)|\} + |s|^{n-m-i-1} \max\{R(s), W(s)\}\right] ds\,du\,dr < +\infty, \quad (88)$$

$$b(t) = -1 \text{ eventually}. \quad (89)$$

Then

(*a*) *for any $L \in (N, M)$, there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that for each $x_0 \in \Omega_1(N, M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by the following scheme*

$$x_{k+1}(t) = \begin{cases} (1-\alpha_k)\, x_k(t) + \alpha_k \Bigg\{ L + (-1)^{n-1}(m-1) H_0 \\ \quad \times \displaystyle\sum_{j=1}^{\infty} \int_{t+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\ \quad \times \left[g(s) - f\left(s, x_k(f_1(s)), \ldots, x_k(f_l(s))\right)\right] ds\,du\,dr \\ \quad + (-1)^{n-i}(m-1) H_i \\ \quad \times \displaystyle\sum_{j=1}^{\infty} \int_{t+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\ \quad \times h\left(s, x_k(h_1(s)), \ldots, x_k(h_l(s))\right) ds\,du\,dr \Bigg\}, \\ \qquad\qquad t \geq T,\ k \in \mathbb{N}_0, \\ (1-\alpha_k)\, x_k(T) + \alpha_k \Bigg\{ L + (-1)^{n-1}(m-1) H_0 \\ \quad \times \displaystyle\sum_{j=1}^{\infty} \int_{T+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\ \quad \times \left[g(s) - f\left(s, x_k(f_1(s)), \ldots, x_k(f_l(s))\right)\right] ds\,du\,dr \\ \quad + (-1)^{n-i}(m-1) H_i \\ \quad \times \displaystyle\sum_{j=1}^{\infty} \int_{T+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\ \quad \times h\left(s, x_k(h_1(s)), \ldots, x_k(h_l(s))\right) ds\,du\,dr \Bigg\}, \\ \qquad\qquad \gamma \leq t < T,\ k \in \mathbb{N}_0 \end{cases} \quad (90)$$

converges to a positive solution $x \in \Omega_1(N, M)$ of (5) *and has the error estimate* (20), *where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ with* (21);

(*b*) *Equation* (5) *has uncountably many positive solutions in $\Omega_1(N, M)$.*

Proof. Set $L \in (N, M)$. In view of (88) and (89), there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that

$$b(t) = -1, \quad \forall t \geq T; \quad (91)$$

$$\theta = \frac{m-1}{\tau} \int_{T}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty} \frac{r u^{m-2}}{|a(u)|} \times \left[H_0 s^{n-m-1} P(s) + H_i s^{n-m-i-1} R(s)\right] ds\,du\,dr; \quad (92)$$

$$\frac{m-1}{\tau} \int_{T}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty} \frac{r u^{m-2}}{|a(u)|} \times \left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right) + H_i s^{n-m-i-1} W(s)\right] ds\,du\,dr < \min\{M - L, L - N\}. \quad (93)$$

Define a mapping $S_L : \Omega_1(N, M) \to CB([\gamma, +\infty), \mathbb{R})$ by

$$S_L x(t) = \begin{cases} L + (-1)^{n-1}(m-1) H_0 \\ \quad \times \displaystyle\sum_{j=1}^{\infty} \int_{t+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\ \quad \times \left[g(s) - f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right)\right] ds\,du\,dr \\ \quad + (-1)^{n-i}(m-1) H_i \\ \quad \times \displaystyle\sum_{j=1}^{\infty} \int_{t+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\ \quad \times h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) ds\,du\,dr, \\ \qquad\qquad t \geq T,\ x \in \Omega_1(N, M), \\ S_L x(T), \qquad \gamma \leq t < T,\ x \in \Omega_1(N, M). \end{cases} \quad (94)$$

By virtue of (16), (92), (94), and Lemma 1, we acquire that for any $x, y \in \Omega_1(N, M)$ and $t \geq T$

$$|S_L x(t) - S_L y(t)| \leq (m-1) H_0 \sum_{j=1}^{\infty} \int_{t+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{|a(u)|} \times \left| f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right) - f\left(s, y(f_1(s)), \ldots, y(f_l(s))\right)\right| ds\,du\,dr$$

$$
\begin{aligned}
&+ (m-1) H_i \sum_{j=1}^{\infty} \int_{t+j\tau}^{+\infty} \int_{r}^{+\infty} \int_{u}^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{|a(u)|} \\
&\quad \times \left| h(s, x(h_1(s)), \dots, x(h_l(s))) \right. \\
&\quad \left. - h(s, y(h_1(s)), \dots, y(h_l(s))) \right| ds\, du\, dr \\
&\le (m-1) \frac{H_0}{\tau} \|x-y\| \\
&\quad \times \int_{T}^{+\infty} \int_{r}^{+\infty} \int_{u}^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2} r}{|a(u)|} P(s)\, ds\, du\, dr \\
&\quad + (m-1) \frac{H_i}{\tau} \|x-y\| \\
&\quad \times \int_{T}^{+\infty} \int_{r}^{+\infty} \int_{u}^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2} r}{|a(u)|} R(s)\, ds\, du\, dr \\
&\le \frac{m-1}{\tau} \int_{T}^{+\infty} \int_{r}^{+\infty} \int_{u}^{+\infty} \frac{r u^{m-2}}{|a(u)|} \\
&\quad \times \left[H_0 s^{n-m-1} P(s) + H_i s^{n-m-i-1} R(s) \right] ds\, du\, dr \\
&= \theta \|x-y\|,
\end{aligned}
\tag{95}
$$

which yields that (27) holds. From (17), (94), (98), and Lemma 1, we obtain that for any $x \in \Omega_1(N, M)$ and $t \ge T$

$$
\begin{aligned}
&\left| S_L x(t) - L \right| \\
&\le (m-1) H_0 \sum_{j=1}^{\infty} \int_{t+j\tau}^{+\infty} \int_{r}^{+\infty} \int_{u}^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{|a(u)|} \\
&\quad \times \left[|g(s)| + \left| f(s, x(f_1(s)), \dots, x(f_l(s))) \right| \right] ds\, du\, dr \\
&\quad + (m-1) H_i \sum_{j=1}^{\infty} \int_{t+j\tau}^{+\infty} \int_{r}^{+\infty} \int_{u}^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{|a(u)|} \\
&\quad \times \left| h(s, x(h_1(s)), \dots, x(h_l(s))) \right| ds\, du\, dr \\
&\le (m-1) \frac{H_0}{\tau} \int_{T}^{+\infty} \int_{r}^{+\infty} \int_{u}^{+\infty} \frac{r s^{n-m-1} u^{m-2}}{|a(u)|} \\
&\qquad \times \left[|g(s)| + Q(s) \right] ds\, du\, dr \\
&\quad + (m-1) \frac{H_i}{\tau} \int_{T}^{+\infty} \int_{r}^{+\infty} \int_{u}^{+\infty} \frac{r s^{n-m-i-1} u^{m-2}}{|a(u)|} \\
&\qquad \times W(s)\, ds\, du\, dr \\
&= \frac{m-1}{\tau} \int_{T}^{+\infty} \int_{r}^{+\infty} \int_{u}^{+\infty} \frac{r u^{m-2}}{|a(u)|} \\
&\quad \times \left[H_0 s^{n-m-1} \left(|g(s)| + Q(s) \right) \right. \\
&\quad \left. + H_i s^{n-m-i-1} W(s) \right] ds\, du\, dr \\
&< \min \{ M-L, L-N \},
\end{aligned}
\tag{96}
$$

which means that $S_L(\Omega_1(N, M)) \subseteq \Omega_1(N, M)$. It follows from (27) and (92) that S_L is a contraction mapping and it has a unique fixed point $x \in \Omega_1(N, M)$. It is clear that $x \in \Omega_1(N, M)$ is a positive solution of (5).

On the basis of (27), (90), and (94), we deduce that

$$
\begin{aligned}
&\left| x_{k+1}(t) - x(t) \right| \\
&= \Bigg| (1-\alpha_k) x_k(t) \\
&\quad + \alpha_k \Bigg\{ L + (-1)^{n-1} (m-1) H_0 \\
&\qquad \times \sum_{j=1}^{\infty} \int_{t+j\tau}^{+\infty} \int_{r}^{+\infty} \int_{u}^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\
&\qquad \times \left[g(s) - f(s, x_k(f_1(s)), \dots, \right. \\
&\qquad \qquad \left. x_k(f_l(s))) \right] ds\, du\, dr \\
&\qquad + (-1)^{n-i} (m-1) H_i \\
&\qquad \times \sum_{j=1}^{\infty} \int_{t+j\tau}^{+\infty} \int_{r}^{+\infty} \int_{u}^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\
&\qquad \times h(s, x_k(h_1(s)), \dots, x_k(h_l(s)))\, ds\, du\, dr \Bigg\} \\
&\quad - x(t) \Bigg| \\
&\le (1-\alpha_k) \left| x_k(t) - x(t) \right| + \alpha_k \left| S_L x_k(t) - S_L x(t) \right| \\
&\le (1-\alpha_k) \left| x_k(t) - x(t) \right| + \alpha_k \theta \left| x_k(t) - x(t) \right| \\
&= \left(1 - (1-\theta) \alpha_k \right) \left| x_k(t) - x(t) \right| \\
&\le e^{-(1-\theta)\alpha_k} \left\| x_k - x \right\| \\
&\le e^{-(1-\theta) \sum_{p=0}^{k} \alpha_p} \left\| x_0 - x \right\|, \quad \forall k \in \mathbb{N}_0,\ t \ge T,
\end{aligned}
\tag{97}
$$

which signifies that (20) holds. It follows from (20) and (21) and that $\lim_{k \to \infty} x_k = x$.

Now we show that (b) holds. Let $L_1, L_2 \in (N, M)$ and $L_1 \ne L_2$. As in the proof of (a), we conclude that for each $p \in \{1, 2\}$, there exist $\theta_p \in (0, 1)$, $T_p > 1 + |t_0| + \tau + |\gamma|$ and $S_{L_p} : \Omega_1(N, M) \to \Omega_1(N, M)$ satisfying (91)–(94), where L, θ, T, and S_L are replaced by L_p, θ_p, T_p, and S_{L_p}, respectively, and S_{L_p} has a unique fixed point $z_p \in \Omega_1(N,M)$, which is a positive solution of (5) in $\Omega_1(N,M)$, that is,

$$\begin{aligned} z_p(t) &= L_p + (-1)^{n-1}(m-1)H_0 \\ &\quad \times \sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\ &\quad \times \left[g(s) - f\left(s, z_p(f_1(s)),\ldots,z_p(f_l(s))\right)\right] ds\,du\,dr \\ &\quad + (-1)^{n-i}(m-1)H_i \\ &\quad \times \sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\ &\quad \times h\left(s, z_p(h_1(s)),\ldots,z_p(h_l(s))\right) ds\,du\,dr, \\ &\qquad \forall t \ge T_p,\ p \in \{1,2\}. \end{aligned} \tag{98}$$

In order to prove (b), it is sufficient to show that $z_1 \neq z_2$. Note that (16), (92), (98), and Lemma 1 lead to

$$\begin{aligned} &\left|z_1(t) - z_2(t)\right| \\ &\ge \left|L_1 - L_2\right| - (m-1)\frac{H_0}{\tau}\left\|z_1 - z_2\right\| \\ &\quad \times \int_{t+\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{rs^{n-m-1}u^{m-2}}{|a(u)|}P(s)\,ds\,du\,dr \\ &\quad - (m-1)\frac{H_i}{\tau}\left\|z_1 - z_2\right\| \\ &\quad \times \int_{t+\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{rs^{n-m-i-1}u^{m-2}}{|a(u)|}R(s)\,ds\,du\,dr \\ &\ge \left|L_1 - L_2\right| - \frac{(m-1)\left\|z_1 - z_2\right\|}{\tau} \\ &\quad \times \int_{\max\{T_1,T_2\}}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{ru^{m-2}}{|a(u)|} \\ &\quad \times \left[H_0 s^{n-m-1}P(s) + H_i s^{n-m-i-1}R(s)\right] ds\,du\,dr \\ &> \left|L_1 - L_2\right| - \max\{\theta_1,\theta_2\}\left\|z_1 - z_2\right\|, \quad \forall t \ge \max\{T_1,T_2\}, \end{aligned} \tag{99}$$

which means that

$$\left\|z_1 - z_2\right\| \ge \frac{\left|L_1 - L_2\right|}{1 + \max\{\theta_1,\theta_2\}} > 0, \tag{100}$$

that is, $z_1 \neq z_2$. This completes the proof. □

Theorem 10. *Let $m = 1$. Assume that there exist two constants M, N with $M > N > 0$ and four functions $P, Q, R, W \in C([t_0,+\infty),\mathbb{R}^+)$ satisfying (16), (17), (89), and*

$$\begin{aligned} \int_{t_0}^{+\infty}\int_u^{+\infty}\frac{|u|}{|a(u)|}\Big[|s|^{n-2}\max\{P(s),Q(s),|g(s)|\} \\ + |s|^{n-i-2}\max\{R(s),W(s)\}\Big] ds\,du \\ < +\infty. \end{aligned} \tag{101}$$

Then

(*a*) *for any $L \in (N,M)$, there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that for each $x_0 \in \Omega_1(N,M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by the following scheme*

$$x_{k+1}(t) = \begin{cases} (1-\alpha_k)x_k(t) \\ \quad +\alpha_k\Bigg\{L + \dfrac{(-1)^{n-1}}{(n-2)!}\displaystyle\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-2}}{a(u)} \\ \qquad \times\left[g(s) - f(s, x_k(f_1(s)),\ldots,x_k(f_l(s)))\right] ds\,du \\ \qquad + \dfrac{(-1)^{n-i}}{(n-i-2)!}\displaystyle\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)} \\ \qquad \times h(s, x_k(h_1(s)),\ldots,x_k(h_l(s)))\,ds\,du\Bigg\}, \\ \qquad\qquad t \ge T,\ k \in \mathbb{N}_0, \\ (1-\alpha_k)x_k(T) \\ \quad +\alpha_k\Bigg\{L + \dfrac{(-1)^{n-1}}{(n-2)!}\displaystyle\sum_{j=1}^{\infty}\int_{T+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-2}}{a(u)} \\ \qquad \times\left[g(s) - f(s, x_k(f_1(s)),\ldots,x_k(f_l(s)))\right] ds\,du \\ \qquad + \dfrac{(-1)^{n-i}}{(n-i-2)!}\displaystyle\sum_{j=1}^{\infty}\int_{T+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)} \\ \qquad \times h(s, x_k(h_1(s)),\ldots,x_k(h_l(s)))\,ds\,du\Bigg\}, \\ \qquad\qquad \gamma \le t < T,\ k \in \mathbb{N}_0, \end{cases} \tag{102}$$

converges to a positive solution $x \in \Omega_1(N,M)$ of (5) *and has the error estimate* (20), *where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in* $[0,1]$ *with* (21);

(*b*) *Equation* (5) *has uncountably many positive solutions in $\Omega_1(N,M)$.*

Proof. Set $L \in (N, M)$. Due to (101), there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying (91),

$$\theta = \frac{1}{\tau}\int_T^{+\infty}\int_u^{+\infty}\frac{u}{|a(u)|}\times\left[\frac{s^{n-2}}{(n-2)!}P(s)+\frac{s^{n-i-2}}{(n-i-2)!}R(s)\right]ds\,du, \tag{103}$$

$$\frac{1}{\tau}\int_T^{+\infty}\int_u^{+\infty}\frac{u}{|a(u)|}\times\left[\frac{s^{n-2}}{(n-2)!}\left(|g(s)|+Q(s)\right)+\frac{s^{n-i-2}}{(n-i-2)!}W(s)\right]ds\,du < \min\{M-L, L-N\}. \tag{104}$$

Define a mapping $S_L : \Omega_1(N,M) \to \mathrm{CB}([\gamma,+\infty),\mathbb{R})$ by

$$S_L x(t) = \begin{cases} L + \dfrac{(-1)^{n-1}}{(n-2)!}\displaystyle\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-2}}{a(u)} \\ \times\left[g(s)-f\left(s,x\left(f_1(s)\right),\ldots,x\left(f_l(s)\right)\right)\right]ds\,du \\ +\dfrac{(-1)^{n-i}}{(n-i-2)!}\displaystyle\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)} \\ \times h\left(s,x\left(h_1(s)\right),\ldots,x\left(h_l(s)\right)\right)ds\,du, \\ \qquad t \ge T,\ x \in \Omega_1(N,M), \\ S_L x(T), \quad \gamma \le t < T,\ x \in \Omega_1(N,M). \end{cases} \tag{105}$$

In view of (16), (103), (105), and Lemma 1, we achieve that for any $x, y \in \Omega_1(N,M)$ and $t \ge T$

$$\begin{aligned}
&\left|S_L x(t) - S_L y(t)\right| \\
&\quad\le \frac{1}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-2}}{|a(u)|} \\
&\qquad\times\left|f\left(s,x\left(f_1(s)\right),\ldots,x\left(f_l(s)\right)\right)\right. \\
&\qquad\quad\left.-f\left(s,y\left(f_1(s)\right),\ldots,y\left(f_l(s)\right)\right)\right|ds\,du \\
&\qquad+\frac{1}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{|a(u)|} \\
&\qquad\times\left|h\left(s,x\left(h_1(s)\right),\ldots,x\left(h_l(s)\right)\right)\right. \\
&\qquad\quad\left.-h\left(s,y\left(h_1(s)\right),\ldots,y\left(h_l(s)\right)\right)\right|ds\,du \\
&\quad\le \frac{\|x-y\|}{\tau(n-2)!}\int_{t+\tau}^{+\infty}\int_u^{+\infty}\frac{us^{n-2}}{|a(u)|}P(s)\,ds\,du \\
&\qquad+\frac{\|x-y\|}{\tau(n-i-2)!}\int_{t+\tau}^{+\infty}\int_u^{+\infty}\frac{us^{n-i-2}}{|a(u)|}R(s)\,ds\,du \\
&\quad\le \frac{\|x-y\|}{\tau}\int_T^{+\infty}\int_u^{+\infty}\frac{u}{|a(u)|} \\
&\qquad\times\left[\frac{s^{n-2}}{(n-2)!}P(s)+\frac{s^{n-i-2}}{(n-i-2)!}R(s)\right]ds\,du \\
&\quad=\theta\|x-y\|,
\end{aligned} \tag{106}$$

which means that (27) holds. It follows from (17), (104), (105), and Lemma 1 that for any $x \in \Omega_1(N,M)$ and $t \ge T$

$$\begin{aligned}
&\left|S_L x(t) - L\right| \\
&\quad\le \frac{1}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-2}}{|a(u)|} \\
&\qquad\times\left[|g(s)|+\left|f\left(s,x\left(f_1(s)\right),\ldots,x\left(f_l(s)\right)\right)\right|\right]ds\,du \\
&\qquad+\frac{1}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{|a(u)|} \\
&\qquad\times\left|h\left(s,x\left(h_1(s)\right),\ldots,x\left(h_l(s)\right)\right)\right|ds\,du \\
&\quad\le \frac{1}{\tau(n-2)!}\int_{t+\tau}^{+\infty}\int_u^{+\infty}\frac{us^{n-2}}{|a(u)|}\left(|g(s)|+Q(s)\right)ds\,du \\
&\qquad+\frac{1}{\tau(n-i-2)!}\int_{t+\tau}^{+\infty}\int_u^{+\infty}\frac{us^{n-i-2}}{|a(u)|}W(s)\,ds\,du \\
&\quad\le \frac{1}{\tau}\int_T^{+\infty}\int_u^{+\infty}\frac{u}{|a(u)|} \\
&\qquad\times\left[\frac{s^{n-2}}{(n-2)!}\left(|g(s)|+Q(s)\right)+\frac{s^{n-i-2}}{(n-i-2)!}W(s)\right]ds\,du \\
&\quad<\min\{M-L, L-N\},
\end{aligned} \tag{107}$$

which means that $S_L(\Omega_1(N,M)) \subseteq \Omega_1(N,M)$. Coupled with (27), we know that S_L is a contraction mapping and it has a unique fixed point $x \in \Omega_1(N,M)$. It follows that $x \in \Omega_1(N,M)$ is a positive solution of (5).

In view of (27), (102), and (105), we deduce that

$$\begin{aligned}
&\left|x_{k+1}(t) - x(t)\right| \\
&\quad=\left|\left(1-\alpha_k\right)x_k(t)\right. \\
&\qquad+\alpha_k\left\{L+\frac{(-1)^{n-1}}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-2}}{a(u)}\right. \\
&\qquad\quad\times\left[g(s)-f\left(s,x_k\left(f_1(s)\right),\ldots,x_k\left(f_l(s)\right)\right)\right]ds\,du
\end{aligned}$$

$$+\frac{(-1)^{n-i}}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)}$$
$$\left.\times h\left(s,x_k\left(h_1(s)\right),\ldots,x_k\left(h_l(s)\right)\right)ds\,du\right\}-x(t)\Bigg|$$
$$\leq\left(1-\alpha_k\right)\left|x_k(t)-x(t)\right|+\alpha_k\left|S_Lx_k(t)-S_Lx(t)\right|$$
$$\leq\left(1-\alpha_k\right)\left|x_k(t)-x(t)\right|+\alpha_k\theta\left|x_k(t)-x(t)\right|$$
$$=\left(1-(1-\theta)\alpha_k\right)\left|x_k(t)-x(t)\right|$$
$$\leq e^{-(1-\theta)\alpha_k}\left\|x_k-x\right\|$$
$$\leq e^{-(1-\theta)\sum_{p=0}^{k}\alpha_p}\left\|x_0-x\right\|,\quad\forall k\in\mathbb{N}_0,\ t\geq T,\tag{108}$$

which signifies that (20) holds. It follows from (20) and (21) that $\lim_{k\to\infty}x_k=x$.

Now we show that (b) holds. Let $L_1,L_2\in(N,M)$ and $L_1\neq L_2$. As in the proof of (a), we conclude that for each $p\in\{1,2\}$, there exist $\theta_p\in(0,1)$, $T_p>1+|t_0|+\tau+|\gamma|$ and $S_{L_p}:\Omega_1(N,M)\to\Omega_1(N,M)$ satisfying (91) and (103)–(105), where L,θ,T, and S_L are replaced by L_p,θ_p,T_p and S_{L_p}, respectively, and S_{L_p} has a unique fixed point $z_p\in\Omega_1(N,M)$, which is a positive solution of (5) in $\Omega_1(N,M)$. It follows that for any $t\geq T_p$ and $p\in\{1,2\}$

$$z_p(t)=L_p+\frac{(-1)^{n-1}}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-2}}{a(u)}$$
$$\times\left[g(s)-f\left(s,z_p\left(f_1(s)\right),\ldots,z_p\left(f_l(s)\right)\right)\right]ds\,du$$
$$+\frac{(-1)^{n-i}}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)}$$
$$\times h\left(s,z_p\left(h_1(s)\right),\ldots,z_p\left(h_l(s)\right)\right)ds\,du.\tag{109}$$

In order to prove (b), we just need to show that $z_1\neq z_2$. Notice that (16), (103), (109), and Lemma 1 ensure that

$$\left|z_1(t)-z_2(t)\right|$$
$$\geq\left|L_1-L_2\right|-\frac{1}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-2}}{|a(u)|}$$
$$\times\left|f\left(s,z_2\left(f_1(s)\right),\ldots,z_2\left(f_l(s)\right)\right)\right.$$
$$\left.-f\left(s,z_1\left(f_1(s)\right),\ldots,z_1\left(f_l(s)\right)\right)\right|ds\,du$$
$$-\frac{1}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-i-2}}{|a(u)|}$$
$$\times\left|h\left(s,z_1\left(h_1(s)\right),\ldots,z_1\left(h_l(s)\right)\right)\right.$$
$$\left.-h\left(s,z_2\left(h_1(s)\right),\ldots,z_2\left(h_l(s)\right)\right)\right|ds\,du$$
$$\geq\left|L_1-L_2\right|-\frac{\left\|z_1-z_2\right\|}{\tau(n-2)!}\int_{t+\tau}^{+\infty}\int_{u}^{+\infty}\frac{us^{n-2}}{|a(u)|}P(s)\,ds\,du$$
$$-\frac{\left\|z_1-z_2\right\|}{\tau(n-i-2)!}\int_{t+\tau}^{+\infty}\int_{u}^{+\infty}\frac{us^{n-i-2}}{|a(u)|}R(s)\,ds\,du$$
$$\geq\left|L_1-L_2\right|-\frac{\left\|z_1-z_2\right\|}{\tau}\int_{\max\{T_1,T_2\}}^{+\infty}\int_{u}^{+\infty}\frac{u}{|a(u)|}$$
$$\times\left[\frac{s^{n-2}}{(n-2)!}P(s)+\frac{s^{n-i-2}}{(n-i-2)!}R(s)\right]ds\,du$$
$$>\left|L_1-L_2\right|-\max\left\{\theta_1,\theta_2\right\}\left\|z_1-z_2\right\|,\tag{110}$$

which yields that

$$\left\|z_1-z_2\right\|\geq\frac{\left|L_1-L_2\right|}{1+\max\left\{\theta_1,\theta_2\right\}}>0,\tag{111}$$

that is, $z_1\neq z_2$. This completes the proof. □

3. Remark and Examples

Remark 11. Theorems 2–10 extend, improve, and unifies Theorems 1–4 in [1], the theorem in [2], Theorems 2.1–2.4 in [4], Theorems 2.1–2.5 in [5, 8], Theorems 1–3 in [9], and Theorems 1–4 in [11], respectively. The examples below prove that Theorems 2–10 extend substantially the corresponding results in [1, 2, 4, 5, 8, 9, 11]. Note that none of the known results can be applied to these examples.

Example 12. Consider the higher order nonlinear neutral delay differential equation

$$\left[\left(t^{m+1}+1\right)\left(x(t)+\frac{\sin\left(2t^2\right)-\cos\left(t^5-1\right)}{7+2\sin\left(8t^3+2t-1\right)}x(t-\tau)\right)^{(m)}\right]^{(n-m)}$$
$$+\left(\frac{t^2x(t-3)\,x^2(t-4)}{t^{n-m-i+3}+t^2+1}\right)^{(i)}$$
$$+\frac{t^3x^3\left(t^2-t\right)-x^4(t-1)}{t^{n-m+4}+t+2}$$
$$=\frac{t\ln\left(1+t^2\right)-\cos^2\left(t^2-t+1\right)}{t^{2n-m+3}+1},\quad\forall t\geq 2,\tag{112}$$

where $\tau > 0$ and $i \le n-m-1$. Let $l = 2$, $t_0 = 2$, $\gamma = \min\{2-\tau, -2\}$, $M = 10$, $N = 1$, $b_0 = 2/5$ and

$$\begin{gathered} h_1(t) = t-3, \quad h_2(t) = t-4, \\ f_1(t) = t^2 - t, \quad f_2(t) = t-1, \\ a(t) = t^{m+1}+1, \quad b(t) = \frac{\sin(2t^2) - \cos(t^5-1)}{7+2\sin(8t^3+2t-1)}, \\ h(t,u,v) = \frac{t^2uv^2}{t^{n-m-i+3}+t^2+1}, \\ f(t,u,v) = \frac{t^3u^3 - v^4}{t^{n-m+4}+t+2}, \\ g(t) = \frac{t\ln(1+t^2) - \cos^2(t^2-t+1)}{t^{2n-m+3}+1}, \\ P(t) = \frac{M^2(3t^3+4M)}{t^{n-m+4}+t+2}, \quad Q(t) = \frac{M^3(t^3+M)}{t^{n-m+4}+t+2}, \\ R(t) = \frac{3M^2t^2}{t^{n-m-i+3}+t^2+1}, \\ W(t) = \frac{M^3t^2}{t^{n-m-i+3}+t^2+1}, \\ \forall (t,u,v) \in [t_0, +\infty) \times [N,M]^2. \end{gathered} \tag{113}$$

It is easy to verify that the conditions of Theorem 2 are satisfied. Thus Theorem 2 ensures that (112) has uncountably many positive solutions in $\Omega_1(1,10)$, and for any $L \in (5,6)$, there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (19) and (21) converges to a positive solution $x \in \Omega_1(1,10)$ of (112) and has the error estimate (20).

Example 13. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned} &\left[(t^n+1)\left(x(t) + \frac{3t^2}{4t^2+3}x(t-\tau)\right)^{(m)}\right]^{(n-m)} \\ &\quad + \left(\frac{t^2x(t\ln t) - (t+1)x(t\ln t)x(\sqrt{2t})}{t^{n+m-i+3}}\right)^{(i)} \\ &\quad + \frac{(2-t^2)\arctan t + tx^3(t^3+t^2)x^2(t^2)}{t^{n+m+3}+x^2(t^2)} \\ &= \frac{\sqrt{1-8t^3+13t^5+5t^6}\cos(t^3-1)}{t^{2n+m+4}}, \quad \forall t \ge 1, \end{aligned} \tag{114}$$

where $\tau > 0$ and $i \le n-m-1$. Let $l = 2$, $t_0 = 1$, $\gamma = \min\{1-\tau, 0\}$, $M = 6$, $N = 1$, $b_0 = 3/4$ and

$$\begin{gathered} h_1(t) = t\ln t, \quad h_2(t) = \sqrt{2t}, \\ f_1(t) = t^3 + t^2, \quad f_2(t) = t^2, \\ a(t) = t^n + 1, \quad b(t) = \frac{3t^2}{4t^2+3}, \\ h(t,u,v) = \frac{t^2u - (t+1)uv}{t^{n+m-i+3}}, \\ f(t,u,v) = \frac{(2-t^2)\arctan t + tu^3v^2}{t^{n+m+3}+v^2}, \\ g(t) = \frac{\sqrt{1-8t^3+13t^5+5t^6}\cos(t^3-1)}{t^{2n+m+4}}, \\ P(t) = \frac{5M^4t^{n+m+4} + 2M(2+t^2)\arctan t + 5M^6t}{(t^{n+m+3}+N^2)^2}, \\ Q(t) = \frac{(2+t^2)\arctan t + M^5t}{t^{n+m+3}+N^2}, \\ R(t) = \frac{t^2+2M(t+1)}{t^{n+m-i+3}}, \quad W(t) = \frac{Mt^2+M^2(t+1)}{t^{n+m-i+3}}, \\ \forall (t,u,v) \in [t_0, +\infty) \times [N,M]^2. \end{gathered} \tag{115}$$

It is easy to check that the conditions of Theorem 3 are satisfied. Therefore (114) has uncountably many positive solutions in $\Omega_1(1,6)$, and for any $L \in (11/2, 6)$, there exist $\theta \in (0,1)$ and $T > 1+|t_0|+\tau+|\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (19) and (21) converges to a positive solution $x \in \Omega_1(1,6)$ of (114) and has the error estimate (20).

Example 14. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned} &\left[(t+1)\left(x(t) - \frac{\arctan t}{2}x(t-\tau)\right)^{(m)}\right]^{(n-m)} \\ &\quad + \left(\frac{t^2x(t^3+t) - x^2(t^3-1)}{t^{2n+m-i+3}+x^2(t^3+t)}\right)^{(i)} \\ &\quad + \frac{tx(t-1)\sin(tx(t-\sin t))}{t^{2n+m+3}+t^3+2} \\ &= \frac{t\sqrt{t+1}\sin^2(t^2+2t+1)}{t^{2n+m+3}+1}, \quad \forall t \ge 0, \end{aligned} \tag{116}$$

where $\tau > 0$ and $i \le n - m - 1$. Let $l = 2$, $t_0 = 0$, $\gamma = \min\{-\tau, -1\}$, $M = 8$, $N = 1/2$, $b_0 = 7/8$ and

$$\begin{gathered} h_1(t) = t^3 + t, \quad h_2(t) = t^3 - 1, \\ f_1(t) = t - 1, \quad f_2(t) = t - \sin t, \\ a(t) = t + 1, \quad b(t) = -\frac{1}{2}\arctan t, \\ h(t, u, v) = \frac{t^2 u - v^2}{t^{2n+m-i+3} + u^2}, \\ f(t, u, v) = \frac{tu\sin(tv)}{t^{2n+m+3} + t^3 + 2}, \\ g(t) = \frac{t\sqrt{t+1}\sin^2\left(t^2 + 2t + 1\right)}{t^{2n+m+3} + 1}, \\ P(t) = \frac{Mt^2 + t}{t^{2n+m+3} + t^3 + 2}, \quad Q(t) = \frac{Mt}{t^{2n+m+3} + t^3 + 2}, \\ R(t) = \frac{t^{2n+m-i+5} + M^2 t^2 + 2Mt^{2n+m-i+3} + 4M^3}{\left(t^{2n+m-i+3} + N^2\right)^2}, \\ W(t) = \frac{Mt^2 + M^2}{t^{2n+m-i+3} + N^2}, \quad \forall (t, u, v) \in \left[t_0, +\infty\right) \times [N, M]^2. \end{gathered} \tag{117}$$

It is easy to prove that the conditions of Theorem 4 are satisfied. Hence (116) has uncountably many positive solutions in $\Omega_1(1/2, 8)$, and for any $L \in (1/2, 1)$, there exist $\theta \in (0, 1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (19) and (21) converges to a positive solution $x \in \Omega_1(1/2, 8)$ of (116) and has the error estimate (20).

Example 15. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned} &\left[\left(t^{2i} + 1\right)\left(x(t) + 2^{t^2+1} x(t - \tau)\right)^{(m)}\right]^{(n-m)} \\ &\quad + \left(\frac{\sqrt{t+1}\,x^2(t-1)\,x\left(t^2\right)}{t^{n+i+1} + t\ln(1 + |x(t-2)|) + 4}\right)^{(i)} \\ &\quad + \frac{t^2 x(t - 12)\, x\left(t^2 - 9\right)}{t^{2n+3} + t\,|x(t-3)| + 3} \\ &= \frac{t^2\cos(2t) + \arctan t^3}{t^{n+4} + \sin^2\left(1 - t^3 + t^4\right) + 1}, \quad \forall t \ge 3, \end{aligned} \tag{118}$$

where $\tau > 0$ and $i \le n - m - 1$. Let $l = 3$, $t_0 = 0$, $\gamma = \min\{3 - \tau, -9\}$, $M = 12$, $N = 5$, $b_0 = 2$ and

$$\begin{gathered} h_1(t) = t - 1, \quad h_2(t) = t^2, \quad h_3(t) = t - 2, \\ f_1(t) = t - 12, \quad f_2(t) = t^2 - 9, \quad f_3(t) = t - 3, \\ a(t) = t^{2i} + 1, \quad b(t) = 2^{t^2+1}, \\ h(t, u, v, w) = \frac{\sqrt{t+1}\,u^2 v}{t^{n+i+1} + t\ln(1 + |w|) + 4}, \\ g(t) = \frac{t^2\cos(2t) + \arctan t^3}{t^{n+4} + \sin^2\left(1 - t^3 + t^4\right) + 1}, \\ f(t, u, v, w) = \frac{t^2 uv}{t^{2n+3} + t\,|w| + 3}, \\ P(t) = \frac{Mt^2\left(6b_0 + 3M^2 t + 2b_0 t^{2n+3}\right)}{b_0^2\left(t^{2n+3} + 3\right)^2}, \\ Q(t) = \frac{M^2 t^2}{b_0^2\left(t^{2n+3} + 3\right)}, \\ R(t) = \frac{M^2\sqrt{t+1}}{b_0^2\left(t^{n+i+1} + 4\right)^2} \times \left[3t^{n+i+1} + 12 + \frac{M}{b_0}t + 3t\ln\left(1 + \frac{M}{b_0}\right)\right], \\ W(t) = \frac{M^3\sqrt{t+1}}{b_0^3\left(t^{n+i+1} + 4\right)}, \\ \forall (t, u, v, w) \in \left[t_0, +\infty\right) \times \left[0, \frac{M}{b_0}\right]^3. \end{gathered} \tag{119}$$

It is easy to verify that the conditions of Theorem 5 are satisfied. Hence Theorem 5 ensures that (118) has uncountably many positive solutions in $\Omega_2(5, 12)$, and, for any $L \in (11, 12)$, there exist $\theta \in (0, 1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (48) and (21) converges to a positive solution $x \in \Omega_2(5, 12)$ of (118) and has the error estimate (20).

Example 16. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned} &\left[\left(t^{3n} + 2t^{n+1} + 1\right)\left(x(t) - \left(t^2 + 2t + 4\right)x(t - \tau)\right)^{(m)}\right]^{(n-m)} \\ &\quad + \left(\frac{tx(t-3)\,x^2(t-4)}{t^{n+10} + x^2(t-3)}\right)^{(i)} \end{aligned}$$

$$+\frac{\sqrt{t+1}x^3(t\ln t)-tx^4(t^2-t)}{(t+1)^{n+3}+tx^2(t\ln t)}$$
$$=\frac{t\ln(1+t^2)-\sqrt{t+3}\cos^3(t^3+1)}{t^{n+5}+4t^3-1-t\sin^5(t^2-3)},\quad \forall t\geq 1, \tag{120}$$

where $\tau>0$, and $i\leq n-m-1$. Let $l=2$, $t_0=1$, $\gamma=\min\{1-\tau,-3\}$, $M=6$, $N=2$, $b_0=3$ and

$$\begin{gathered}
h_1(t)=t-3,\quad h_2(t)=t-4,\\
f_1(t)=t\ln t,\quad f_2(t)=t^2-t,\\
a(t)=t^{3n}+2t^{n+1}+1,\quad b(t)=-t^2-2t-4,\\
h(t,u,v)=\frac{tuv^2}{t^{n+10}+u^2},\\
f(t,u,v)=\frac{\sqrt{t+1}u^3-tv^4}{(t+1)^{n+3}+tu^2},\\
g(t)=\frac{t\ln(1+t^2)-\sqrt{t+3}\cos^3(t^3+1)}{t^{n+5}+4t^3-1-t\sin^5(t^2-3)},
\end{gathered}$$
$$P(t)=\frac{M^2}{b_0^2(t+1)^{2n+6}}\times\left[3(t+1)^{n+7/2}+\frac{M^2}{b_0^2}t\sqrt{t+1}+4\frac{M}{b_0}t(t+1)^{n+3}+6\frac{M^3}{b_0^3}t^2\right],$$
$$\begin{gathered}
Q(t)=\frac{M^3}{b_0^3(t+1)^{n+8}}\left(\sqrt{t+1}+\frac{M}{b_0}t\right),\\
R(t)=\frac{3M^2}{b_0^2t^{2n+19}}\left(t^{n+10}+\frac{M^2}{b_0^2}\right),\\
W(t)=\frac{M^3}{b_0^3t^{n+9}},\\
\forall(t,u,v)\in[t_0,+\infty)\times\left[0,\frac{M}{b_0}\right]^2.
\end{gathered}\tag{121}$$

It is easy to check that the conditions of Theorem 6 are satisfied. Thus Theorem 6 ensures that (120) has uncountably many positive solutions in $\Omega_3(2,6)$, and, for any $L\in(2,4)$, there exist $\theta\in(0,1)$ and $T>1+|t_0|+\tau+|\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (61) and (21) converges to a positive solution $x\in\Omega_3(2,6)$ of (120) and has the error estimate (20).

Example 17. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned}
&\left[(t^2\ln t)(x(t)+x(t-\tau))^{(m)}\right]^{(n-m)}\\
&\quad+\left(\frac{x(\sqrt{t}-2)+x(2t-1)}{t^{3n-m-i+2}+x^2(t-\cos t)}\right)^{(i)}\\
&\quad+\frac{x^2(t-4)+x(\sqrt{t-1})x^2(t-\sin(t^9+1))}{t^{2n-m+3}+1}\\
&=\frac{\sin^{13}(t^5-\sqrt{t}+1)}{t^{n+7/2}+1},\quad\forall t\geq 4,
\end{aligned}\tag{122}$$

where $\tau>0$, $m\geq 2$ and $i\leq n-m-1$. Let $l=3$, $t_0=4$, $\gamma=\min\{4-\tau,0\}$, $M=100$, $N=1$ and

$$\begin{gathered}
h_1(t)=\sqrt{t}-2,\quad h_2(t)=2t-1,\quad h_3(t)=t-\cos t,\\
f_1(t)=t-4,\quad f_2(t)=\sqrt{t-1},\quad f_3(t)=t-\sin(t^9+1),\\
a(t)=t^2\ln t,\quad b(t)=1,\\
h(t,u,v,w)=\frac{u+v}{t^{3n-m-i+2}+w^2},\\
f(t,u,v,w)=\frac{u^2+vw^2}{t^{2n-m+3}+1},\\
g(t)=\frac{\sin^{13}(t^5-\sqrt{t}+1)}{t^{n+7/2}+1},\quad P(t)=\frac{2M+3M^2}{t^{2n-m+3}+1},\\
Q(t)=\frac{M^2+M^3}{t^{2n-m+3}+1},\quad R(t)=\frac{2t^{3n-m-i+2}+6M^2}{(t^{3n-m-i+2}+N^2)^2},\\
W(t)=\frac{2M}{t^{3n-m-i+2}+N^2},\\
\forall(t,u,v,w)\in[t_0,+\infty)\times[N,M]^3.
\end{gathered}\tag{123}$$

It is easy to check that the conditions of Theorem 7 are satisfied. Thus (122) has uncountably many positive solutions in $\Omega_1(1,100)$, and, for any $L\in(1,100)$, there exist $\theta\in(0,1)$ and $T>1+|t_0|+\tau+|\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (66) and (21) converges to a positive solution $x\in\Omega_1(1,100)$ of (122) and has the error estimate (20).

Example 18. Consider the higher order nonlinear neutral delay differential equation

$$\left[\frac{2+\sin\left(t+\sqrt{t}\right)}{t^2}(x(t)+x(t-\tau))'\right]^{(n-1)} + \left(\frac{x^3(t-2)-t^2x^4\left(\sqrt{t+1}-1\right)}{t^{5n+3}+t+1}\right)^{(i)} + \frac{x^3(t-3)-t}{t^{3n+4}+x^2\left(t-(-1)^n\right)} = \frac{\sin\left(t^4-\sqrt{t^2+1}\right)}{t^{n+3}+\ln t}, \quad \forall t \ge 3, \tag{124}$$

where $\tau > 0$, $m = 1$ and $i \le n-2$. Let $l = 2$, $t_0 = 3$, $\gamma = \min\{3-\tau, 0\}$, $M = 10$, $N = 9$ and

$$\begin{gathered} h_1(t) = t-2, \quad h_2(t) = \sqrt{t+1}-1, \\ f_1(t) = t-3, \quad f_2(t) = t-(-1)^n, \\ a(t) = \frac{2+\sin\left(t+\sqrt{t}\right)}{t^2}, \quad b(t) = 1, \\ h(t,u,v) = \frac{u^3-t^2v^4}{t^{5n+3}+t+1}, \\ f(t,u,v) = \frac{u^3-t}{t^{3n+4}+v^2}, \\ g(t) = \frac{\sin\left(t^4-\sqrt{t^2+1}\right)}{t^{n+3}+\ln t}, \\ P(t) = \frac{M\left(5M^3+2t+2Mt^{3n+4}\right)}{\left(t^{3n+4}+N^2\right)^2}, \\ Q(t) = \frac{M^3+t}{t^{3n+4}+N^2}, \\ R(t) = \frac{M^2\left(3+4Mt^2\right)}{t^{5n+3}+t+1}, \quad W(t) = \frac{M^3\left(1+Mt^2\right)}{t^{5n+3}+t+1}, \\ \forall (t,u,v) \in [t_0,+\infty) \times [N,M]^2. \end{gathered} \tag{125}$$

It is easy to check that the conditions of Theorem 8 are satisfied. Thus Theorem 8 ensures that (124) has uncountably many positive solutions in $\Omega_1(9,10)$, and, for any $L \in (9,10)$, there exist $\theta \in (0,1)$ and $T > 1+|t_0|+\tau+|\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (78) and (21) converges to a positive solution $x \in \Omega_1(9,10)$ of (124) and has the error estimate (20).

Example 19. Consider the higher order nonlinear neutral delay differential equation

$$\left[t^{m+1}\ln\left(4+\sin\left(t^2-\sqrt{t}\right)\right)(x(t)-x(t-\tau))^{(m)}\right]^{(n-m)} + \left(\frac{t}{t^{n-m+4}+x^4\left(t-\sqrt{t}\right)}\right)^{(i)} + \frac{\sin\left(t^3-2t+\sqrt{t^3+1}\right)}{t^{n+m}+|t-x(t-3)|} = \frac{t\cos^5\left(t^7-t^4+1\right)}{t^n+2t-\cos^3\left(t^2-3\right)}, \quad \forall t \ge 4, \tag{126}$$

where $\tau > 0$, $m \ge 2$ and $i \le n-m-1$. Let $l = 1$, $t_0 = 4$, $\gamma = \min\{4-\tau, 1\}$, $M = 7$, $N = 5$ and

$$\begin{gathered} h_1(t) = t-\sqrt{t}, \quad f_1(t) = t-3, \\ a(t) = t^{m+1}\ln\left(4+\sin\left(t^2-\sqrt{t}\right)\right), \\ b(t) = -1, \quad h(t,u) = \frac{t}{t^{n-m+4}+u^4}, \\ f(t,u) = \frac{\sin t^3\left(-2t+\sqrt{t^3+1}\right)}{t^{n+m}+|t-u|}, \\ g(t) = \frac{t\cos^5\left(t^7-t^4+1\right)}{t^n+2t-\cos^3\left(t^2-3\right)}, \\ P(t) = \frac{1}{t^{2n+2m}}, \quad Q(t) = \frac{1}{t^{n+m}}, \\ R(t) = \frac{4M^3}{t^{2n-2m+7}}, \quad W(t) = \frac{1}{t^{n-m+3}}, \\ \forall (t,u) \in [t_0,+\infty) \times [N,M]^2. \end{gathered} \tag{127}$$

It is easy to check that the conditions of Theorem 9 are satisfied. Thus Theorem 9 ensures that (126) has uncountably many positive solutions in $\Omega_1(5,7)$, and, for any $L \in (5,7)$, there exist $\theta \in (0,1)$ and $T > 1+|t_0|+\tau+|\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (90) and (21) converges to a positive solution $x \in \Omega_1(5,7)$ of (126) and has the error estimate (20).

Example 20. Consider the higher order nonlinear neutral delay differential equation

$$\left[\frac{1}{t^3}(x(t)-x(t-\tau))'\right]^{(n-1)} + \left(\frac{t-\sin\left(t^8-4t^5-1\right)}{t^{n+7}+\left|x(t-1)-x^3(t-2)\right|}\right)^{(i)}$$

$$+\frac{\ln\left(1+x^2(t-\arctan t)\right)}{t^{2n+6}+x^2(t-4)}$$
$$=\frac{t\ln\left(t+\cos\left(t^3-1\right)\right)}{t^{n+8}+1}, \quad \forall t \geq 5, \tag{128}$$

where $\tau > 0$, $m = 1$ and $i \leq n-2$. Let $l = 2$, $t_0 = 5$, $\gamma = \min\{5-\tau, 1\}$, $M = 4$, $N = 2$ and

$$h_1(t) = t-1, \quad h_2(t) = t-2, \quad f_1(t) = t - \arctan t,$$
$$f_2(t) = t-4, \quad a(t) = \frac{1}{t^3}, \quad b(t) = -1,$$
$$h(t,u,v) = \frac{t - \sin\left(t^8 - 4t^5 - 1\right)}{t^{n+7} + \left|u - v^3\right|},$$
$$f(t,u,v) = \frac{\ln\left(1+u^2\right)}{t^{2n+6}+v^2}, \quad g(t) = \frac{t\ln\left(t+\cos\left(t^3-1\right)\right)}{t^{n+8}+1},$$
$$P(t) = \frac{2M\left(2M^2+t^{2n+6}\right)}{t^{4n+12}}, \quad Q(t) = \frac{M^2}{t^{2n+6}},$$
$$R(t) = \frac{2+6M^2}{t^{2n+13}}, \quad W(t) = \frac{2}{t^{n+6}},$$
$$\forall (t,u,v) \in \left[t_0, +\infty\right) \times [N,M]^2. \tag{129}$$

It is easy to check that the conditions of Theorem 10 are satisfied. Thus Theorem 10 ensures that (128) has uncountably many positive solutions in $\Omega_1(2,4)$, and, for any $L \in (2,4)$, there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (102) and (21) converges to a positive solution $x \in \Omega_1(2,4)$ of (128) and has the error estimate (20).

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References

[1] T. Candan, "The existence of nonoscillatory solutions of higher order nonlinear neutral equations," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 412–416, 2012.

[2] M. R. S. Kulenović and S. Hadžiomerspahić, "Existence of nonoscillatory solution of second order linear neutral delay equation," *Journal of Mathematical Analysis and Applications*, vol. 228, no. 2, pp. 436–448, 1998.

[3] Z. Liu, L. Chen, S. M. Kang, and S. Y. Cho, "Existence of nonoscillatory solutions for a third-order nonlinear neutral delay differential equation," *Abstract and Applied Analysis*, vol. 2011, Article ID 693890, 23 pages, 2011.

[4] Z. Liu, H. Gao, S. M. Kang, and S. H. Shim, "Existence and Mann iterative approximations of nonoscillatory solutions of *n*th-order neutral delay differential equations," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 515–529, 2007.

[5] Z. Liu and S. M. Kang, "Infinitely many nonoscillatory solutions for second order nonlinear neutral delay differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 12, pp. 4274–4293, 2009.

[6] Z. Liu, S. M. Kang, and J. S. Ume, "Existence and iterative approximations of nonoscillatory solutions of higher order nonlinear neutral delay differential equations," *Applied Mathematics and Computation*, vol. 193, no. 1, pp. 73–88, 2007.

[7] Z. Liu, S. M. Kang, and J. S. Ume, "Existence of bounded nonoscillatory solutions of first-order nonlinear neutral delay differential equations," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3535–3547, 2010.

[8] Z. Liu, L. Wang, S. M. Kang, and J. S. Ume, "Solvability and iterative algorithms for a higher order nonlinear neutral delay differential equation," *Applied Mathematics and Computation*, vol. 215, no. 7, pp. 2534–2543, 2009.

[9] W. Zhang, W. Feng, J. Yan, and J. Song, "Existence of nonoscillatory solutions of first-order linear neutral delay differential equations," *Computers & Mathematics with Applications*, vol. 49, no. 7-8, pp. 1021–1027, 2005.

[10] Y. Zhou, "Existence for nonoscillatory solutions of second-order nonlinear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 91–96, 2007.

[11] Y. Zhou and B. G. Zhang, "Existence of nonoscillatory solutions of higher-order neutral differential equations with positive and negative coefficients," *Applied Mathematics Letters*, vol. 15, no. 7, pp. 867–874, 2002.

[12] Y. Zhou, B. G. Zhang, and Y. Q. Huang, "Existence for nonoscillatory solutions of higher order nonlinear neutral differential equations," *Czechoslovak Mathematical Journal*, vol. 55, no. 1, pp. 237–253, 2005.