

Research Article

Strong Convergence Theorems for a Common Fixed Point of a Family of Asymptotically k -Strict Pseudocontractive Mappings

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We provide an iterative process which converges strongly to a common fixed point of finite family of asymptotically k -strict pseudocontractive mappings in Banach spaces. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

1. Introduction

Let E be a real normed linear space with dual E^* . A *gauge* function $\varphi : [0, \infty) \rightarrow \mathbb{R}^+$ is a continuous and strictly increasing function satisfying $\varphi(0) = 0$ and $\varphi(t) \rightarrow \infty$, as $t \rightarrow \infty$. The generalized duality mapping from E to 2^{E^*} associated with the gauge function φ (see, e.g., [1]) is defined by

$$J_\varphi(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \varphi(\|x\|), \quad (1)$$

$$\|x^*\| = \varphi(\|x\|)\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. In the case that $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is called the *normalized duality mapping*.

Following Browder [2], we say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping J_φ is single valued and weak-to-weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in E weakly convergent to a point x , then the sequence $J_\varphi(x_n)$ converges weak* to $J_\varphi(x)$). It is known that J_p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-2}$, for all $1 < p < \infty$.

Let K be a nonempty subset of E . A mapping $T : K \rightarrow K$ is called *asymptotically k -strict pseudocontractive*, with sequence $\{l_n\} \subseteq [1, \infty)$, $\lim_{n \rightarrow \infty} l_n = 1$ (see, e.g., [3–6]) if

for all $x, y \in K$, there exist $j(x - y) \in J(x - y)$ and a constant $k \in [0, 1)$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \quad (2)$$

$$\leq l_n \|x - y\|^2 - k \|(I - T^n)x - (I - T^n)y\|^2,$$

for all $n \geq 1$.

If I denotes the identity operator, then (2) can be equivalently written as

$$\langle (I - T^n)x - (I - T^n)y, j(x - y) \rangle \quad (3)$$

$$\geq k \|(I - T^n)x - (I - T^n)y\|^2 - (l_n - 1) \|x - y\|^2,$$

for all $n \geq 1$.

If $E = H$, a real Hilbert space, it is shown by Osilike et al. [4] that (2) (and hence (3)) is equivalent to the inequality

$$\|T^n x - T^n y\|^2 \leq (1 + 2(l_n - 1)) \|x - y\|^2 \quad (4)$$

$$+ \lambda \|(I - T^n)x - (I - T^n)y\|^2,$$

where $\lambda = (1 - 2k)$. T is called *uniformly Lipschitz* if there exists $L \geq 0$ such that $\|T^n x - T^n y\| \leq L \|x - y\|$ for all $x, y \in D(T)$. It is shown in [4] that an asymptotically k -strict pseudocontractive mapping is uniformly Lipschitz.

The class of asymptotically k -strict pseudocontractive mappings was first introduced in Hilbert spaces by Liu [5]. He proved the following theorem.

Theorem Q (see [5]). Let K be a closed convex and bounded subset of a Hilbert space H . Let $T : K \rightarrow K$ be completely continuous asymptotically k -strict pseudocontractive mapping for some $0 \leq k < 1$ with sequence $\{l_n\} \subset [0, \infty)$ such that $\sum(l_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the modified Mann's iteration method:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \quad n \geq 1, \quad (5)$$

where $\{\alpha_n\}$ is a real sequence satisfying $\epsilon \leq \alpha_n \leq 1 - k - \epsilon$ for all $n \geq 1$ and some $\epsilon > 0$. Then, $\{x_n\}$ converges strongly to a fixed point of T .

The iteration scheme (5) is called *modified Mann's iterative processes* which was introduced by Schu [7, 8] and has been used by several authors (see, e.g., [3-5, 9-17]). We observe that Liu [5] proved *strong convergence* of scheme (5) to a fixed point of asymptotically k -strict pseudocontractive mapping T with additional assumption that T is *completely continuous*, where $T : C \rightarrow C$ is said to be completely continuous if for every bounded sequence $\{x_n\}$, there exists a subsequence, say $\{x_{n_j}\}$ of $\{x_n\}$ such that the sequence $\{Tx_{n_j}\}$ converges strongly to some element of the range of T .

In [12], Kim and Xu studied weak convergence theorem for the class of asymptotically k -strict pseudocontractive mappings in the frame work of Hilbert spaces. In fact, they proved the following.

Theorem KX (see [12]). Let K be a closed and convex subset of a Hilbert space H . Let $T : K \rightarrow K$ be an asymptotically k -strict pseudocontractive mapping for some $0 \leq k < 1$ with sequence $\{l_n\} \subset [0, \infty)$ such that $\sum(l_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the modified Mann's iteration method:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \quad n \geq 1, \quad (6)$$

where $\{\alpha_n\}$ is a real sequence satisfying $k + \lambda \leq \alpha_n \leq 1 - \lambda$, for all $n \geq 1$, and $\lambda \in (0, 1)$. Then, $\{x_n\}$ converges weakly to a fixed point of T .

In 2007, Osilike et al. [13] extended Theorem KX by proving *weak convergence* of scheme (6) to a fixed point of T in the frame work of q uniformly smooth Banach spaces which are also uniformly convex under suitable control conditions.

In 2011, Zhang and Xie [17] extended Theorem of Osilike et al. [13] to a more general real uniformly convex Banach space E with Fréchet differentiable norm. In addition, they proved *strong convergence* of scheme (5) to a fixed point of asymptotically k -strict pseudocontractive mapping provided that $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf_{p \in F(T)} \|x_n - p\|$.

However, we observe that the convergence obtained above is either *weak* or requiring *additional assumption* like $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$ or T is completely continuous. But the requirement that $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$ is not easy to verify, as $F(T)$ is in general unknown, and there is also an example of asymptotically k -strict pseudocontractive mapping which is not completely continuous as shown below.

An example of asymptotically k -strict pseudocontractive mapping which is not completely continuous.

Example 1. Let $E = l_2 = \{\bar{x} = \{x_i\}_{i=1}^\infty, x_i \in \mathbb{R}, \sum_{i=1}^\infty |x_i|^2 < \infty\}$ and $\bar{B} = \{\bar{x} \in l_2 : \|\bar{x}\| \leq 1\}$. Define $T : \bar{B} \rightarrow \bar{B}$ by $T\bar{x} = (0, x_1^2, a_2 x_2, a_3 x_3, \dots)$, where $\{a_k\}_{k=1}^\infty$ is a real sequence satisfying $0 < a_k < 1, k \geq 2$, and $\prod_{k=2}^\infty a_k = 1/2$. Then it is shown in [13] that T is asymptotically k -strict pseudocontractive mapping.

Now, we show that T is not completely continuous. Let $\{x_n\}$ be a sequence in \bar{B} defined by $x_1 = (1, 0, 0, \dots)$, $x_2 = (0, 1, 0, 0, \dots)$, $x_3 = (0, 0, 1, 0, 0, \dots)$, \dots . Then $\{x_n\} \subset \bar{B}$ and $\{Tx_n\} = \{y_n\}$ is given by $y_1 = (0, 1, 0, 0, \dots)$, $y_2 = (0, 0, a_2, 0, 0, \dots)$, $y_3 = (0, 0, 0, a_3, 0, 0, \dots)$, \dots . Hence, since $a_k \rightarrow 1$, as $k \rightarrow \infty$, there is no subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{Tx_{n_i}\}$ converges strongly to a point in \bar{B} , as $\|Tx_{n_i} - Tx_{n_j}\| = \|y_{n_i} - y_{n_j}\| = \sqrt{|a_{n_i}|^2 + |a_{n_j}|^2} \not\rightarrow 0$, as $i, j \rightarrow \infty$. Therefore, T is not completely continuous.

Thus, *one question is raised naturally*: can we obtain a scheme that converges strongly to a fixed point of asymptotically k -strict pseudocontractive mappings without those additional assumptions?

It is our purpose in this paper to provide an iterative scheme $\{x_n\}$ which converges strongly to a common fixed point of finite family of asymptotically k -strict pseudocontractive mappings in Banach spaces. The assumption that $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$ or T is completely continuous is not required.

2. Preliminaries

We need the following definitions from [18]. The Banach space E is said to be *uniformly convex* if, given $\epsilon > 0$, there exists $\delta > 0$, such that, for all $x, y \in E$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon, \|(1/2)(x + y)\| \leq 1 - \delta$. It is well known that L_p, ℓ_p , and Sobolev spaces $W_m^p, (1 < p < \infty)$, are uniformly convex.

A Banach space E is said to have a *Fréchet differentiable norm* if for all $x \in B = \{x \in E : \|x\| = 1\}$

$$\lim_{t \rightarrow 0} \frac{\|x + t\gamma\| - \|x\|}{t} \quad (7)$$

exists and is attained uniformly in $\gamma \in B$. It is well known that uniformly smooth Banach spaces has a Fréchet differentiable norm.

In order to prove our results, we need the following lemmas.

Lemma 2 (see [19]). Let C be a nonempty close convex subset of a real Banach space E which has the Fréchet differentiable norm. For $x \in E$, let ρ be defined for $0 < t < \infty$ by

$$\rho(t) = \sup_{\gamma \in B} \left| \frac{\|x + t\gamma\|^2 - \|x\|^2}{t} - 2 \langle \gamma, j(x) \rangle \right|. \quad (8)$$

Then, $\lim_{t \rightarrow 0} \rho(t) = 0$ and

$$\|x + h\|^2 \leq \|x\|^2 + 2 \langle h, j(x) \rangle + \|h\| \rho(\|h\|), \quad \forall h \in E \setminus \{0\}. \quad (9)$$

It is shown in [19] that if $E = H$, a real Hilbert space, then $\rho(t) = t$, for $t > 0$. In our general setting, throughout this paper we assume that $\rho(t) \leq 2t$.

Lemma 3. *Let E be a real Banach space. Then the following inequality holds:*

$$\begin{aligned} \|x + y\|^2 &\leq \|x\|^2 + \langle y, j(x + y) \rangle, \\ \forall x, y \in E, \quad j(x + y) &\in J(x + y). \end{aligned} \tag{10}$$

Lemma 4 (see [20]). *Let E be a uniformly convex Banach space and $B_R(0)$ a closed ball of E . Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\begin{aligned} &\|\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_k x_k\|^2 \\ &\leq \sum_{i=0}^k \alpha_i \|x_i\|^2 - \alpha_i \alpha_j g(\|x_i - x_j\|), \end{aligned} \tag{11}$$

for each $\alpha_i \in (0, 1)$ and for $x_i \in B_R(0) := \{x \in E : \|x\| \leq R\}$, $i = 0, 1, 2, \dots, k$ with $\sum_{i=0}^k \alpha_i = 1$.

Lemma 5 (see [21]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \delta_n, \quad n \geq n_0, \tag{12}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$. Then, $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 6 (see [17]). *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E which has the Fréchet differentiable norm. Let $T : C \rightarrow C$ be an asymptotically k -strict pseudocontractive mapping with fixed point of T , $F(T) := \{x \in C : Tx = x\} \neq \emptyset$. Then $(I - T)$ is demiclosed at zero, that is, if $x_n \rightarrow x$ and $Tx_n - x_n \rightarrow 0$, as $n \rightarrow \infty$, then $x = T(x)$.*

Lemma 7 (see [22]). *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \rightarrow \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1}, \quad a_k \leq a_{m_k+1}. \tag{13}$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3. Main Results

We now prove our main theorem.

Theorem 8. *Let C be a nonempty, closed, and convex subset of a real uniformly convex Banach space E which has Fréchet differentiable norm possessing a weakly sequentially continuous duality mapping from E into E^* . Let $T_i : C \rightarrow C$ be asymptotically k_i -strict pseudocontractive mappings for $0 \leq k_i < 1$*

with sequences $\{l_{n,i}\} \subset [1, \infty)$, for $i = 1, 2, \dots, N$. Assume that $F := \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) ((1 - \beta_n) x_n + \beta_n S_n x_n), \quad n \geq 1, \tag{14}$$

where $S_n := \theta_{n,1} T_1^n + \theta_{n,2} T_2^n + \dots + \theta_{n,N} T_N^n$, such that $\theta_{n,1} + \theta_{n,2} + \dots + \theta_{n,N} = 1$, for each $n \geq 1$, $\{\alpha_n\}, \{\theta_{n,i}\} \subset (0, c) \subset (0, 1)$, satisfying $\liminf_n \theta_{n,i} > 0$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum \alpha_n = \infty$, $\lim_{n \rightarrow \infty} ((l_{n,i} - 1)/\alpha_n) = 0$, for $i = 1, 2, \dots, N$ and $\{\beta_n\} \subset [a, b] \subset (0, k)$ (a, b , and c constants), for $k = \min_{1 \leq i \leq N} \{k_i\}$. Then the sequence $\{x_n\}$ generated by (14) converges strongly to a common fixed point of $\{T_i : i = 1, 2, \dots, N\}$.

Proof. Fix $x^* \in F$. Let $y_n = (1 - \beta_n)x_n + \beta_n S_n x_n$ and $l_n := \max\{l_{n,i} : i = 1, 2, \dots, N\}$. Then, using Lemma 2 and (3) we have that

$$\begin{aligned} &\|y_n - x^*\|^2 \\ &= \|(x_n - x^*) - \beta_n (x_n - S_n x_n)\|^2 \\ &= \|(x_n - x^*) \\ &\quad - \beta_n (x_n - (\theta_{n,1} T_1^n + \theta_{n,2} T_2^n + \dots + \theta_{n,N} T_N^n) x_n)\|^2 \\ &\leq \|x_n - x^*\|^2 \\ &\quad - 2\beta_n \langle \theta_{n,1} (x_n - T_1^n x_n) + \theta_{n,2} (x_n - T_2^n x_n) \\ &\quad \quad + \dots + \theta_{n,N} (x_n - T_N^n x_n), j(x_n - x^*) \rangle \\ &\quad + \beta_n \|x_n - S_n x_n\| \rho(\beta_n \|x_n - S_n x_n\|) \\ &\leq \|x_n - x^*\|^2 - 2\beta_n \theta_{n,1} \langle x_n - T_1^n x_n, j(x_n - x^*) \rangle \\ &\quad - 2\beta_n \theta_{n,2} \langle x_n - T_2^n x_n, j(x_n - x^*) \rangle \\ &\quad - \dots - 2\beta_n \theta_{n,N} \langle x_n - T_N^n x_n, j(x_n - x^*) \rangle \\ &\quad + 2\beta_n^2 \|x_n - S_n x_n\|^2 \\ &\leq \|x_n - x^*\|^2 \\ &\quad - 2\beta_n \theta_{n,1} [k \|x_n - T_1^n x_n\|^2 - (l_n - 1) \|x_n - x^*\|^2] \\ &\quad - 2\beta_n \theta_{n,2} [k \|x_n - T_2^n x_n\|^2 - (l_n - 1) \|x_n - x^*\|^2] \\ &\quad - \dots - 2\beta_n \theta_{n,N} [k \|x_n - T_N^n x_n\|^2 - (l_n - 1) \|x_n - x^*\|^2] \\ &\quad + 2\beta_n^2 \|x_n - S_n x_n\|^2 \\ &\leq [1 + 2\beta_n (l_n - 1)] \|x_n - x^*\|^2 - 2\beta_n \theta_{n,1} k \|x_n - T_1^n x_n\|^2 \\ &\quad - 2\beta_n \theta_{n,2} k \|x_n - T_2^n x_n\|^2 \\ &\quad - \dots - 2\beta_n \theta_{n,N} k \|x_n - T_N^n x_n\|^2 \\ &\quad + 2\beta_n^2 \|x_n - S_n x_n\|^2. \end{aligned} \tag{15}$$

On the other hand using Lemma 4 we get that

$$\begin{aligned}
\|x_n - S_n x_n\|^2 &= \|x_n - (\theta_{n,1} T_1^n + \theta_{n,2} T_2^n + \cdots + \theta_{n,N} T_N^n) x_n\|^2 \\
&= \|\theta_{n,1} (x_n - T_1^n x_n) + \theta_{n,2} (x_n - T_2^n x_n) \\
&\quad + \cdots + \theta_{n,N} (x_n - T_N^n x_n)\|^2 \\
&\leq \theta_{n,1} \|x_n - T_1^n x_n\|^2 + \theta_{n,2} \|x_n - T_2^n x_n\|^2 \\
&\quad + \cdots + \theta_{n,N} \|x_n - T_N^n x_n\|^2.
\end{aligned} \tag{16}$$

Now substituting (16) into (15) we obtain that

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq [1 + 2\beta_n(l_n - 1)] \|x_n - x^*\|^2 \\
&\quad - 2\beta_n \theta_{n,1} (k - \beta_n) \|x_n - T_1^n x_n\|^2 \\
&\quad - 2\beta_n \theta_{n,2} (k - \beta_n) \|x_n - T_2^n x_n\|^2 \\
&\quad - \cdots - 2\beta_n \theta_{n,N} (k - \beta_n) \|x_n - T_N^n x_n\|^2 \\
&\leq [1 + 2\beta_n(l_n - 1)] \|x_n - x^*\|^2,
\end{aligned} \tag{17}$$

since $(k - \beta_n) \geq 0$ for each $n \geq 1$. Then now, from (14) and (18) we get that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(u - x^*) + (1 - \alpha_n)(y_n - x^*)\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|y_n - x^*\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 \\
&\quad + (1 - \alpha_n) [1 + 2\beta_n(l_n - 1)] \|x_n - x^*\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 \\
&\quad + (1 - \alpha_n + \epsilon \alpha_n) \|x_n - x^*\|^2, \quad \forall n \geq N_0, \\
&\leq \alpha_n \|u - x^*\|^2 \\
&\quad + (1 - \alpha_n(1 - \epsilon)) \|x_n - x^*\|^2, \quad \forall n \geq N_0,
\end{aligned} \tag{19}$$

where N_0 is a positive integer such that $2(1 - \alpha_n)\beta_n(l_n - 1)/\alpha_n < \epsilon$, for all $n \geq N_0$, for some $\epsilon > 0$. Therefore, by induction,

$$\|x_{n+1} - x^*\|^2 \leq \max \left\{ \|x_{N_0} - x^*\|^2, (1 - \epsilon)^{-1} \|u - x^*\|^2 \right\}, \quad \forall n \geq N_0, \tag{20}$$

which implies that $\{x_n\}$ and hence $\{y_n\}$ are bounded.

Furthermore, from (14), Lemma 3, and (17) we get that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n(u - x^*) + (1 - \alpha_n)(y_n - x^*)\|^2 \\
&\leq (1 - \alpha_n) \|y_n - x^*\|^2 \\
&\quad + 2\alpha_n \langle u - x^*, J(x_{n+1} - x^*) \rangle \\
&\leq (1 - \alpha_n) [1 + 2\beta_n(l_n - 1)] \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \langle u - x^*, J(x_{n+1} - x^*) \rangle \\
&\quad - 2\beta_n \theta_{n,1} (k - \beta_n) (1 - \alpha_n) \|x_n - T_1^n x_n\|^2 \\
&\quad - 2\beta_n \theta_{n,2} (k - \beta_n) (1 - \alpha_n) \|x_n - T_2^n x_n\|^2 \\
&\quad - \cdots - 2\beta_n \theta_{n,N} (k - \beta_n) (1 - \alpha_n) \|x_n - T_N^n x_n\|^2 \\
&\leq (1 - \alpha_n) \|x_n - x^*\|^2 + 2\alpha_n \langle u - x^*, J(x_{n+1} - x^*) \rangle \\
&\quad + 2\beta_n M(l_n - 1) \\
&\quad - 2\beta_n \theta_{n,1} (1 - \alpha_n) (k - \beta_n) \|x_n - T_1^n x_n\|^2 \\
&\quad - 2\beta_n \theta_{n,2} (k - \beta_n) (1 - \alpha_n) \|x_n - T_2^n x_n\|^2 \\
&\quad - \cdots - 2\beta_n \theta_{n,N} (k - \beta_n) (1 - \alpha_n) \|x_n - T_N^n x_n\|^2 \\
&\leq (1 - \alpha_n) \|x_n - x^*\|^2 + 2\alpha_n \langle u - x^*, J(x_{n+1} - x^*) \rangle \\
&\quad + 2\beta_n M(l_n - 1),
\end{aligned} \tag{21}$$

for some $M > 0$.

Now, the rest of the proof is divided into two parts.

Case 1. Suppose that there exists $N_1 \geq 0$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n \geq N_1$. Then we have that $\{\|x_n - x^*\|\}$ is convergent. Then from (21) and the assumptions on $\{\beta_n\}$, $\{\alpha_n\}$, and $\{l_n\}$ we have that $\beta_n \theta_{n,i} (1 - c)(k - \beta_n) \|x_n - T_i^n x_n\|^2 \rightarrow 0$, as $n \rightarrow \infty$, which implies that

$$x_n - T_i^n x_n \rightarrow 0, \quad \text{as } n \rightarrow \infty, \tag{23}$$

for $i = 1, 2, \dots, N$. Then from (14) we obtain that

$$x_{n+1} - y_n = \alpha_n (u - y_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{24}$$

Again, from (23) we get that

$$\begin{aligned}
\|y_n - x_n\| &= \|\beta_n (S_n x_n - x_n)\| \leq \|S_n x_n - x_n\| \\
&\leq \theta_{n,1} \|T_1^n x_n - x_n\| + \theta_{n,2} \|T_2^n x_n - x_n\| \\
&\quad + \cdots + \theta_{n,N} \|T_N^n x_n - x_n\| \rightarrow 0,
\end{aligned} \tag{25}$$

as $n \rightarrow \infty$. Thus, (24) and (25) imply that

$$x_{n+1} - x_n \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{26}$$

Therefore, since each T_i , for $i = 1, 2, \dots, N$, is uniformly L -Lipschitzian and

$$\begin{aligned} & \|x_n - T_i x_n\| \\ & \leq \|x_n - x_{n+1}\| \\ & \quad + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\| \\ & \quad + \|T_i^{n+1} x_n - T_i x_n\| \\ & \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + L \|x_{n+1} - x_n\| \\ & \quad + \|T_i (T_i^n x_n) - T_i x_n\| \\ & = (1 + L) \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| \\ & \quad + \|T_i (T_i^n x_n) - T_i x_n\|, \end{aligned} \tag{27}$$

we have from (23), (26), and uniform continuity of T_i that

$$\|x_n - T_i x_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty, \tag{28}$$

for each $i = 1, 2, \dots, N$. Furthermore, the fact that $\{x_n\}$ is bounded and E is reflexive implies that we can choose a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that $x_{n_i+1} \rightarrow z$ and

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \langle u - x^*, J(x_{n+1} - x^*) \rangle \\ & = \lim_{i \rightarrow \infty} \langle u - x^*, J(x_{n_i+1} - x^*) \rangle. \end{aligned} \tag{29}$$

Now, from (26) we get that $x_{n_i} \rightarrow z$ and from Lemma 6 we have that $z \in F(T_i)$, for each $i = 1, 2, \dots, N$. Hence, $z \in \bigcap_{i=1}^N F(T_i)$. Therefore, putting $x^* = z$ in (29) and using the fact that J is weakly sequentially continuous we immediately obtain that $\limsup_{n \rightarrow \infty} \langle u - z, J(x_{n+1} - z) \rangle = \lim_{i \rightarrow \infty} \langle u - z, J(x_{n_i+1} - z) \rangle = \langle u - z, J(z - z) \rangle = 0$. Again, putting $x^* = z$ in inequality (22), we get that

$$\begin{aligned} \|x_{n+1} - z\|^2 & \leq (1 - \alpha_n) \|x_n - z\|^2 \\ & \quad + 2\alpha_n \langle u - z, J(x_{n+1} - z) \rangle \\ & \quad + 2\beta_n M(l_n - 1), \end{aligned} \tag{30}$$

and, hence, it follows from (30) and Lemma 5 that $\|x_n - z\| \rightarrow 0$, as $n \rightarrow \infty$. Consequently, $x_n \rightarrow z$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| < \|x_{n_i+1} - x^*\|, \tag{31}$$

for all $i \in N$. Then, by Lemma 7, there exists a nondecreasing sequence $\{m_j\} \subset N$ such that $m_j \rightarrow \infty$, $\|x_{m_j} - x^*\| \leq \|x_{m_j+1} - x^*\|$ and $\|x_j - x^*\| \leq \|x_{m_j+1} - x^*\|$ for all $j \in N$. Then from (21) and following the method of Case 1, we get that

$$\|x_{m_j} - T_i^{m_j} x_{m_j}\| \rightarrow 0, \quad \text{as } j \rightarrow \infty, \tag{32}$$

for each $i = 1, 2, \dots, N$. Thus, again following the method of Case 1, we obtain that $x_{m_j+1} - x_{m_j} \rightarrow 0$ and $x_{m_j} - T_i x_{m_j} \rightarrow 0$, as $j \rightarrow \infty$, for each $i = 1, 2, \dots, N$ and there exists $z^* \in \bigcap_{i=1}^N F(T_i)$ such that

$$\limsup_{j \rightarrow \infty} \langle u - z^*, J(x_{m_j+1} - z^*) \rangle = 0. \tag{33}$$

Then now, putting $x^* = z^*$ in (22) we have that

$$\begin{aligned} \|x_{m_j+1} - z^*\|^2 & \leq (1 - \alpha_{m_j}) \|x_{m_j} - z^*\|^2 \\ & \quad + 2\alpha_{m_j} \langle u - z^*, J(x_{m_j+1} - z^*) \rangle \\ & \quad + 2\beta_{m_j} M(l_{m_j} - 1). \end{aligned} \tag{34}$$

Since $\|x_{m_j} - z^*\|^2 \leq \|x_{m_j+1} - z^*\|^2$, (34) implies that

$$\begin{aligned} \alpha_{m_j} \|x_{m_j} - z^*\|^2 & \leq \|x_{m_j} - z^*\|^2 - \|x_{m_j+1} - z^*\|^2 \\ & \quad + 2\alpha_{m_j} \langle u - z^*, J(x_{m_j+1} - z^*) \rangle \\ & \quad + 2\beta_{m_j} M(l_{m_j} - 1). \end{aligned} \tag{35}$$

Moreover, since $\alpha_{m_j} > 0$, inequality (35) gives that

$$\begin{aligned} \|x_{m_j} - z^*\|^2 & \leq 2 \langle u - z^*, J(x_{m_j+1} - z^*) \rangle \\ & \quad + \frac{2\beta_{m_j} M(l_{m_j} - 1)}{\alpha_{m_j}}. \end{aligned} \tag{36}$$

Then, from (33) and the fact that $2\beta_{m_j} M(l_{m_j} - 1)/\alpha_{m_j} \rightarrow 0$, we obtain that $\|x_{m_j} - z^*\| \rightarrow 0$, as $j \rightarrow \infty$. This together with (34) gives that $\|x_{m_j+1} - z^*\| \rightarrow 0$, as $j \rightarrow \infty$. But $\|x_j - z^*\| \leq \|x_{m_j+1} - z^*\|$, for all $j \in N$; thus we obtain that $x_j \rightarrow z^*$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to an element of F and the proof is complete. \square

If, in Theorem 8, we assume a single asymptotically k -strict pseudocontractive mapping we get the following corollary.

Corollary 9. *Let C be a nonempty, closed, and convex subset of a real uniformly convex Banach space E which has Fréchet differentiable norm possessing a weakly sequentially continuous duality mapping from E into E^* . Let $T : C \rightarrow C$ be an asymptotically k -strict pseudocontractive mapping for $0 \leq k < 1$ with sequences $\{l_n\} \subset [1, \infty)$. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = u \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) ((1 - \beta_n) x_n + \beta_n T^n x_n), \quad n \geq 1, \tag{37}$$

where $\{\alpha_n\} \subset (0, c) \subset (0, 1)$, satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum \alpha_n = \infty$, $\lim_{n \rightarrow \infty} ((l_n - 1)/\alpha_n) = 0$, and $\{\beta_n\} \subset [a, b] \subset (0, k)$ (a, b , and c constants). Then the sequence $\{x_n\}$ generated by (37) converges strongly to a fixed point of T .

Proof. Putting $T = T_1 = T_2 = \dots = T_N$ in (14), we get that $S_n = T^n$ and the scheme reduces to scheme (37) and following the method of proof of Theorem 8 we get that (see (21) and (22))

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n) \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \langle u - x^*, J(x_{n+1} - x^*) \rangle \\
&\quad - 2\beta_n (k - \beta_n) (1 - \alpha_n) \|x_n - T^n x_n\|^2 \\
&\quad + 2\beta_n M' (l_n - 1) \\
&\leq (1 - \alpha_n) \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \langle u - x^*, J(x_{n+1} - x^*) \rangle \quad (38) \\
&\quad - 2\beta_n (k - \beta_n) (1 - c) \|x_n - T^n x_n\|^2 \\
&\quad + 2\beta_n M' (l_n - 1) \\
&\leq (1 - \alpha_n) \|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \langle u - x^*, J(x_{n+1} - x^*) \rangle \\
&\quad + 2\beta_n M' (l_n - 1),
\end{aligned}$$

for some $M' > 0$. Now, considering cases, as in the proof of Theorem 8, we obtain the required result. \square

Corollary 10. Let K be a nonempty, closed, and convex subset of l_p , $1 < p < \infty$. Let $T_i : C \rightarrow C$ be asymptotically k_i -strict pseudocontractive mappings for $0 \leq k_i < 1$ with sequences $\{l_{n,i}\} \subset [1, \infty)$, for $i = 1, 2, \dots, N$. Assume that $F := \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) ((1 - \beta_n) x_n + \beta_n S_n x_n), \quad n \geq 1, \quad (39)$$

where $S_n := \theta_{n,1} T_1^n + \theta_{n,2} T_2^n + \dots + \theta_{n,N} T_N^n$, such that $\theta_{n,1} + \theta_{n,2} + \dots + \theta_{n,N} = 1$, for each $n \geq 1$, $\{\alpha_n\}, \{\theta_{n,i}\} \subset (0, c) \subset (0, 1)$, satisfying $\liminf_{n \rightarrow \infty} \theta_{n,i} > 0$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum \alpha_n = \infty$, $\lim_{n \rightarrow \infty} ((l_{n,i} - 1)/\alpha_n) = 0$ and $\{\beta_n\} \subset [a, b] \subset (0, k)$ (for a, b , and c constants), for $k = \min_{1 \leq i \leq N} \{k_i\}$. Then the sequence $\{x_n\}$ generated by (39) converges strongly to a common fixed point of $\{T_i : i = 1, 2, \dots, N\}$.

Proof. We note that l_p , $1 < p < \infty$, spaces are uniformly convex which have Fréchet differentiable norm possessing a weakly sequentially continuous duality mapping from E into E^* (see, e.g., [18]). Thus, the result follows from Theorem 8. \square

Corollary 11. Let K be a nonempty, closed, and convex subset of l_p , $1 < p < \infty$. Let $T : C \rightarrow C$ be an asymptotically k -strict pseudocontractive mapping for some $0 \leq k < 1$ with sequences $\{l_n\} \subset [1, \infty)$. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a

sequence defined by $x_1 = u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) ((1 - \beta_n) x_n + \beta_n T^n x_n), \quad n \geq 1, \quad (40)$$

where $\{\alpha_n\} \subset (0, c) \subset (0, 1)$, and $\{\beta_n\} \subset [a, b] \subset (0, k)$ (for a, b , and c constants) satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} ((l_n - 1)/\alpha_n) = 0$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T .

If in Theorem 8 we have that $E = H$, a real Hilbert space, then E is uniformly convex with Fréchet differentiable norm possessing a weakly sequentially continuous duality mapping. Thus, we have the following corollary.

Corollary 12. Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $T_i : C \rightarrow C$ be asymptotically k_i -strict pseudocontractive mappings for $0 \leq k_i < 1$ with sequences $\{l_{n,i}\} \subset [1, \infty)$, for $i = 1, 2, \dots, N$. Assume that $F := \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) ((1 - \beta_n) x_n + \beta_n S_n x_n), \quad n \geq 1, \quad (41)$$

where $S_n := \theta_{n,1} T_1^n + \theta_{n,2} T_2^n + \dots + \theta_{n,N} T_N^n$, such that $\theta_{n,1} + \theta_{n,2} + \dots + \theta_{n,N} = 1$, for each $n \geq 1$, $\{\alpha_n\}, \{\theta_{n,i}\} \subset (0, c) \subset (0, 1)$, satisfying $\liminf_{n \rightarrow \infty} \theta_{n,i} > 0$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum \alpha_n = \infty$, $\lim_{n \rightarrow \infty} ((l_{n,i} - 1)/\alpha_n) = 0$ and $\{\beta_n\} \subset [a, b] \subset (0, k)$ (for a, b , and c constants), for $k = \min_{1 \leq i \leq N} \{k_i\}$. Then the sequence $\{x_n\}$ generated by (41) converges strongly to a common fixed point of $\{T_i : i = 1, 2, \dots, N\}$.

Corollary 13. Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ be an asymptotically k -strict pseudocontractive mapping for some $0 \leq k < 1$ with sequences $\{l_n\} \subset [1, \infty)$. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) ((1 - \beta_n) x_n + \beta_n T^n x_n), \quad n \geq 1, \quad (42)$$

where $\{\alpha_n\} \subset (0, c) \subset (0, 1)$, and $\{\beta_n\} \subset [a, b] \subset (0, k)$ (for a, b , and c constants) satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} ((l_n - 1)/\alpha_n) = 0$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T .

Remark 14. We note that Corollary 9 generalizes several recent results of this nature. Particularly, it extends Theorem KX of [12], Theorem 2 of Liu [5], and corresponding theorem of Schu [7] in the sense that our convergence is *strong* in more general Banach spaces possessing weakly sequentially continuous duality mappings without the requirement that T be completely continuous.

Remark 15. Corollary 9 is an improvement of Theorem 3.2 of Osilike et al. [13] and Theorems 3.1 and 3.2 of Zhang and Xie [17] in the sense that our convergence is *strong* without the requirement that $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$, provided that E possesses weakly sequentially continuous duality mappings.

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