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Research Article

Heteroclinic Solutions for Nonautonomous EFK Equations

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We explore the nonautonomous fourth-order differential equation which has important applications in materials science. By variational approach, we find heteroclinic solutions of the equation. The conditions on the potential function $V(t,u)$ are mild enough to include a broad class of equations. We also consider a separate case where $V(t,u)$ is periodic in t.

1. Introduction

The goal of this paper is to study the nonautonomous extended Fisher-Kolmogorov (EFK) equation

$$q''''(t) - q''(t) + V_q(t,q) = 0, \quad (1)$$

where $V(t,q)$ is a time-dependent potential function and V_q denotes the partial derivative with respect to the second variable q.

Fourth-order differential equations, which often appear in nonlinear elasticity, fluid mechanics, and relating physical problems, have received growing attention from researchers. For example, Peletier and Troy [1–4] studied the EFK equation with odd nonlinearity

$$\mu q'''' - \lambda q'' - q + q^3 = 0, \quad (\mu, \lambda > 0), \quad (2)$$

mainly using shooting argument and variational method. They found the existence of periodic, heteroclinic, and homoclinic solutions and proved the existence of chaotic solutions oscillating between 1 and −1 with all critical points being either minima or maxima. They also explored the boundedness, monotonicity, and other quantitative properties of solutions. Part of Peletier and Troy's work was generalized by Kalies and VanderVorst [5]; they considered the EFK equation with general nonlinear term

$$\mu q'''' - \lambda q'' + V'(q) = 0, \quad (\mu, \lambda > 0), \quad (3)$$

where $V(q)$ is supposed to have symmetric wells with equal depth and grow superquadratically. Then, they investigated the multitransition structure of heteroclinic and homoclinic solutions. Primary examples are

$$V_1(q) = \frac{1}{4}(q^2 - 1)^2, \qquad V_2(q) = \frac{2}{\pi^2}(1 + \cos \pi q). \quad (4)$$

The symmetric property of $V(q)$ is rather restrictive, and many physical problems do not satisfy such conditions. Later, Kalies et al. [6, 7] obtained heteroclinic and homoclinic solutions connecting saddle-focus equilibria, and the potential function $V(q)$ is assumed to grow superquadratically and has at least two nondegenerate global minima, not necessarily symmetric. In case of (3) with $V(q) = V_1(q)$, the presence of saddle-focus equilibria amounts to the requiring of $4\mu > \lambda^2/V_1''(\pm 1)$. Periodic and chaotic solutions of (3) are also explored. These results were generalized in recent works [8, 9], where the author obtained heteroclinic solutions for (3) under very mild conditions of V; in particular, they do not assume that V is symmetric or that saddle-focus equilibria are present.

We have just mentioned a few works that are closely related to our results. A good summary of typical researches in fourth-order differential equations can be found in [10].

In this paper we step forward to study heteroclinic solutions of the nonautonomous EFK equation (1). We are inspired by Yeun [9], Rabinowitz [11, 12], and Izydorek and Janczewska [13]. However, we are emphasizing that the argument of [9, 11] relies on the fact that the equation is autonomous; therefore, the method there cannot be reproduced here to tackle the time-dependent version (1) which is no longer autonomous. The nonautonomous case necessitates careful analysis. Another point should be made is that we are working on the (q, q') phase plane; this is essentially different from [11, 13]. In our argument, we also benefit from analysis of [3, 4] and comments of [5].

In spite of its practical significance [14], there has been few researches on (1) with time-dependent potentials; the paper seems to be the first attempt in the direction and the methods here can be applied to similar equations.

2. Heteroclinic Solutions

By a heteroclinic solution connecting a_- and a_+, we mean a solution $q(t)$ verifying

$$\begin{aligned} \left(q(t), q'(t), q''(t), q'''(t)\right) &\longrightarrow (a_+, 0, 0, 0), \quad t \longrightarrow +\infty, \\ \left(q(t), q'(t), q''(t), q'''(t)\right) &\longrightarrow (a_-, 0, 0, 0), \quad t \longrightarrow -\infty. \end{aligned} \tag{5}$$

Throughout the paper, let $B_\varepsilon(C)$ denote the neighborhood of a set $C \subset \mathbb{R}^n$ defined as below

$$B_\varepsilon(C) = \left\{ w \in \mathbb{R}^n \mid \inf_{v \in C} |w - v| < \varepsilon \right\}. \tag{6}$$

We will make the following assumptions:

(A_1) $V(t, q) \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $V \geqslant 0$;

(A_2) let $\mathscr{V}$ be a set which has at least two points and we assume the points in $\mathscr{V}$ do not collapse together,

$$\gamma \equiv \frac{1}{3} \inf \{ |x - y| \mid x \neq y \in \mathscr{V} \} > 0; \tag{7}$$

(A_3) one of the following is satisfied:

(a) there is $\varepsilon^* \in (0, \gamma]$, such that $\sigma_\varepsilon \triangleq \inf_{z \notin B_\varepsilon(\mathscr{V}), t \in \mathbb{R}} V(t, z) > 0$ for any $\varepsilon \in (0, \varepsilon^*]$;

(b) $\liminf_{|z| \to \infty} |z| V(t, z) > 0$ uniformly for $t \in \mathbb{R}$.

(A_4) $\lim_{|t| \to +\infty} V(t, z) = +\infty$ uniformly for z in any compact subsets of $\mathbb{R} \setminus \mathscr{V}$.

(A_5) For each $z \in \mathscr{V}$,

$$\int_{-\infty}^{\infty} V(t, z)\, dt < \gamma \sigma_\gamma^{1/2}, \tag{8}$$

where σ_γ is defined in (A_3) with $\varepsilon = \gamma$.

Theorem 1. *Let $V(t, q)$ satisfy (A_1)–(A_4). Then for every $z \in \mathscr{V}$, there is a pair of heteroclinic solutions of* (1)*, one emanating from z and the other terminating at z.*

Remark 2. To be noted, we do not assume V has wells of equal depth. By the assumptions of the theorem, we necessarily have $\{z \in \mathbb{R} \mid V(t, z) = 0 \text{ for all } t\} \subset \mathscr{V}$. However, $\mathscr{V}$ may contain other points which are not global minima of $V(t, u)$ but still verify (A_5). In the special case $\mathscr{V} = \{z \in \mathbb{R} \mid V(t, z) = 0 \text{ for all } t\}$, (A_5) is automatically satisfied.

Notation. Define $\widetilde{\mathscr{V}} = \{(z, 0) \in \mathbb{R}^2 \mid z \in \mathscr{V}\}$. For an element $(z, 0)$ in $\widetilde{\mathscr{V}}$, we denote it by $\tilde{z} = (z, 0)$ for brevity. Whence, without further notice, we will use z and $\tilde{z}$ where it is appropriate to denote element in $\mathscr{V}$ and its counterpart in $\widetilde{\mathscr{V}}$.

We will obtain heteroclinic solutions of (1) by minimizing the energy functional $I(\cdot)$ on an appropriate subset

$$I(q) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2} \left(q''\right)^2 + \frac{1}{2} \left(q'\right)^2 + V(t, q) \right\} dt. \tag{9}$$

Denote the energy on a bounded interval $[r, s]$ by

$$I_{r,s}(q) = \int_r^s \left\{ \frac{1}{2} \left(q''\right)^2 + \frac{1}{2} \left(q'\right)^2 + V(t, q) \right\} dt. \tag{10}$$

We will work on the space

$$E = \left\{ q \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}) \mid \int_{-\infty}^{\infty} \left(\left|q''\right|^2 + \left|q'\right|^2 \right) dt < \infty \right\}, \tag{11}$$

equipped with the norm

$$\|q\|^2 = \int_{-\infty}^{\infty} \left(\left|q''\right|^2 + \left|q'\right|^2 \right) dt + |q(0)|^2. \tag{12}$$

It is readily seen that $(E, \|\cdot\|)$ is a Hilbert space and $q \in E$ implies $q \in C^1(\mathbb{R})$.

Let $\tilde{\vartheta} \in \widetilde{\mathscr{V}} \setminus \{\tilde{0}\}$, $\varepsilon \in (0, 1)$. Denote by $\Gamma_\varepsilon(\vartheta)$ the set of $q \in E$ verifying

(i) $\lim_{t \to -\infty} q(t) = 0$, $\lim_{t \to +\infty} q(t) = \vartheta$;

(ii) $(q(t), q'(t)) \notin B_\varepsilon(\widetilde{\mathscr{V}} \setminus \{\tilde{0}, \tilde{\vartheta}\})$, for all $t \in \mathbb{R}$.

That is, $\Gamma_\varepsilon(\vartheta)$ consists of functions which join 0 and ϑ, and at least keep a distance of ε to the set $\widetilde{\mathscr{V}} \setminus \{\tilde{0}, \tilde{\vartheta}\}$. Clearly, $\Gamma_\varepsilon(\vartheta)$ is not empty, we may consider the minimization problem

$$c_\varepsilon(\vartheta) = \inf_{q \in \Gamma_\varepsilon(\vartheta)} I(q). \tag{13}$$

We will find for each $\varepsilon > 0$ a candidate in $\Gamma_\varepsilon(\vartheta)$ and then send ε to zero to obtain the minimizer, which is actually a heteroclinic solution.

For any $\varepsilon \in (0, \gamma)$, $\nu > 0$ define

$$\sigma_{\varepsilon,\nu} = \inf_{\substack{z \notin B_\varepsilon(\mathscr{V}), |z| \leqslant \nu, \\ t \in \mathbb{R}}} V(t, z), \tag{14}$$

then $\sigma_{\varepsilon,\nu} > 0$ by our assumptions. For $\nu = +\infty$, we take $\sigma_{\varepsilon,+\infty} = \sigma_\varepsilon$ which is defined in (A_3).

From now on, we always assume that

$$0 < \varepsilon < \min\{\varepsilon^*, 1\}. \tag{15}$$

Lemma 3. *Let $q \in E$. Then for any $\nu > 0$, $r < s \in \mathbb{R}$ such that $q(t) \notin B_\varepsilon(\mathcal{V})$ and $|q(t)| \leqslant \nu$ for any $t \in [r,s]$,*

$$I(q) \geqslant \sqrt{2\sigma_{\varepsilon,\nu}}\,|q(r) - q(s)|. \tag{16}$$

In particular, if $\sigma_\varepsilon > 0$, then for any $r < s \in \mathbb{R}$ such that $q(t) \notin B_\varepsilon(\mathcal{V})$ for any $t \in [r,s]$,

$$I(q) \geqslant \sqrt{2\sigma_\varepsilon}\,|q(r) - q(s)|. \tag{17}$$

Proof. Denote $l = |q(r) - q(s)|$ and $\tau = |r - s|$. Then

$$l = \left|\int_r^s q'(t)\,dt\right| \leqslant \int_r^s |q'(t)|\,dt \leqslant \tau^{1/2}\left(\int_r^s |q'(t)|^2 dt\right)^{1/2}. \tag{18}$$

Since $V \geqslant 0$, by assumptions, we have

$$\begin{aligned} I(q) &\geqslant \int_r^s \left\{\frac{1}{2}(q'(t))^2 + V(t, q(t))\right\} dt \\ &\geqslant \frac{l^2}{2\tau} + \sigma_{\varepsilon,\nu}\tau \geqslant \sqrt{2\sigma_{\varepsilon,\nu}}\,l. \end{aligned} \tag{19}$$

Proof for the second part is similar. □

Remark 4. The proof of Lemma 3 indicates that any function $q \in E$ with $I(q) < +\infty$ cannot oscillate too often between different points in $\mathcal{V}$ when the time approaches infinity; precisely, we will show $q(\pm\infty)$ exist and have limits in $\mathcal{V}$.

Lemma 5. *Let $q \in E$. $I(q) < +\infty$ implies $q \in L^\infty(\mathbb{R}, \mathbb{R})$.*

Proof. Suppose to the contrary that $q \notin L^\infty(\mathbb{R})$. Let $\varepsilon > 0$ satisfy (15).

Case 1 ($\sigma_\varepsilon > 0$). If $\mathcal{V}$ is an infinite set, then we can find infinitely many nonoverlapping time intervals $\{(s_i, t_i)\}_{i \geqslant 1}$ such that $|q(s_i) - q(t_i)| \geqslant \gamma$ for $i \geqslant 1$ and $q(t) \notin B_\varepsilon(\mathcal{V})$ for $t \in \cup_{i\geqslant 1}(s_i, t_i)$, by Lemma 3,

$$\begin{aligned} I(q) &> \sum_{i=1}^n I_{s_i,t_i}(q) \\ &\geqslant \sqrt{2\sigma_\varepsilon}\sum_{i=1}^n |q(s_i) - q(t_i)| \geqslant n\gamma\sqrt{2\sigma_\varepsilon}, \end{aligned} \tag{20}$$

for any $n \geqslant 1$, a contradiction.

If $\mathcal{V}$ is a finite set, let $z_\nu = \max_{z\in\mathcal{V}}|z|$, then $z_\nu \in (0,\infty)$. Since $q \notin L^\infty(\mathbb{R})$, there is a sequence of nonoverlapping time intervals, also denoted by $\{(s_i, t_i)\}_{i\geqslant 1}$, such that $|q(s_i) - q(t_i)| \geqslant 1$ for $i \geqslant 1$ and $|q(t)| > \max\{|q(0)|, z_\nu + 1\}$ for $t \in \cup_{i\geqslant 1}(s_i, t_i)$. Again Lemma 3 shows

$$I(q) \geqslant n\sqrt{2\sigma_\varepsilon}, \tag{21}$$

for any $n \geqslant 1$, also a contradiction.

Case 2 ($\liminf_{|z|\to\infty}|z|V(t,z) > 0$ uniformly in $t \in \mathbb{R}$). There are $z_0 > 0$ and $c_1 > 0$ such that $V(t,z) \geqslant c_1|z|^{-1}$ for $|z| > z_0$ and $t \in \mathbb{R}$. Let $n_q = \max\{[|q(0)|] + 1, [z_0] + 1\}$ ($[x]$ denotes the integral part of x). Since $q \notin L^\infty(\mathbb{R})$, there is a sequence of nonoverlapping time intervals $\{(s_j, t_j)\}_{j\geqslant n_q}$ with $|q(s_j)| = j$, $|q(t_j)| = j + 1$ and $|q(t)| \in [j, j+1]$ on (s_j, t_j) this is possible via the continuity of $q(t)$. We claim that $\inf_{j\geqslant n_q}|t_j - s_j| > 0$. Indeed, for $j \geqslant n_q$,

$$\begin{aligned} 1 &= \left||q(t_j)| - |q(s_j)|\right| \\ &\leqslant |q(t_j) - q(s_j)| \\ &\leqslant \int_{s_j}^{t_j} |q'(t)|\,dt \\ &\leqslant |t_j - s_j|^{1/2}\left(\int_{s_j}^{t_j} |q'(t)|^2 dt\right)^{1/2}, \end{aligned} \tag{22}$$

so

$$\begin{aligned} 1 &\leqslant |t_j - s_j| \int_r^s |q'(t)|^2 dt \\ &\leqslant 2|t_j - s_j|\, I(q), \quad \forall j \geqslant n_q, \end{aligned} \tag{23}$$

which implies

$$\inf_{j\geqslant n_q} |t_j - s_j| > 0. \tag{24}$$

Hence,

$$\begin{aligned} I(q) &\geqslant \sum_{j=n_q}^n \int_{s_j}^{t_j} V(t, q(t))\,dt \\ &\geqslant c_1 \sum_{j=n_q}^n \int_{s_j}^{t_j} \frac{1}{|q(t)|} dt \geqslant c_1\left(\inf_{j\geqslant n_q}|t_j - s_j|\right)\sum_{j=n_q}^n \frac{1}{j+1}, \end{aligned} \tag{25}$$

for any $n \geqslant n_q$, a contradiction.

Therefore, in either case, we must have $q \in L^\infty(\mathbb{R})$. □

The preceding lemma has an important corollary which will be used later; we prove it here.

Lemma 6. *For $M > 0$ and $u \in \mathcal{V}$, define*

$$Q_{u,M} = \{q \in E \mid q(-\infty) = u, I(q) \leqslant M\}. \tag{26}$$

Then $Q_{u,M}$ is a bounded set in $C(\mathbb{R})$ with the usual uniform norm.

Proof. Without loss of generality, assume $u = 0$. Let $\varepsilon > 0$ satisfy (15). Suppose to the contrary that, for any $n \in \mathbb{N}$, there is an orbit q_n such that $q_n(-\infty) = 0, I(q_n) \leqslant M$ and $\sup_{t\in\mathbb{R}}|q_n(t)| > n$. Since $q_n(t)$ is continuous, there is $t_n^* \in \mathbb{R}$ for which $|q_n(t_n^*)| = n$ and $|q_n(t)| \leqslant n$ for $t \leqslant t_n^*$; this can be accomplished by taking as t_n^* the first hitting time of $|q_n(t)|$ on the level n.

Case 1 ($\sigma_\varepsilon > 0$). If $\mathcal{V}$ is an infinite set, then for each $n \geqslant [3\gamma] + 1$, $n \in \mathbb{N}$, there are l_n nonoverlapping time intervals

$\{(s_{n,i}, t_{n,i})\}_{i\leq l_n}$ such that $|q_n(s_{n,i}) - q_n(t_{n,i})| \geqslant \gamma$ for $1 \leqslant i \leqslant l_n$, $q_n(t) \notin B_\varepsilon(\mathscr{V})$ for $t \in \cup_{i=1}^{l_n}(s_{n,i}, t_{n,i})$; moreover, the number of intervals $l_n \to \infty$ as $n \to \infty$. Then by Lemma 3,

$$\begin{aligned} I(q_n) &> \sum_{i=1}^{l_n} I_{s_{n,i},t_{n,i}}(q_n) \geqslant \sqrt{2\sigma_\varepsilon} \sum_{i=1}^{l_n} |q_n(s_{n,i}) - q_n(t_{n,i})| \\ &\geqslant l_n \gamma \sqrt{2\sigma_\varepsilon}, \end{aligned} \tag{27}$$

for any $n \geqslant [3\gamma] + 1$, a contradiction.

If $\mathscr{V}$ is a finite set, let $z_v = \max_{z\in\mathscr{V}} |z|$, then $z_v \in (0, \infty)$. For each $n \geqslant [z_v] + 1$, $n \in \mathbb{N}$, there is an $s_n < t_n^*$ with $|q_n(s_n)| = z_v + 1$ and $|q_n(t)| z_v + 1$ for $t \in (s_n, t_n^*)$. Then Lemma 3 indicates

$$I(q_n) \geqslant (n - (z_v + 1)) \sqrt{2\sigma_\varepsilon}, \tag{28}$$

for any $n \geqslant [z_v] + 1$, also a contradiction.

Case 2 ($\liminf_{|z|\to\infty} |z| V(t, z) > 0$ *uniformly in* $t \in \mathbb{R}$). There are $z_0 > 0$ and $c_1 > 0$ such that $V(t, z) \geqslant c_1 |z|^{-1}$ for $|z| > z_0$ and $t \in \mathbb{R}$. For each $n \geqslant 2$, there are $n-1$ nonoverlapping time intervals $\{(s_{n,i}, t_{n,i})\}_{2\leq i\leq n}$ such that $|q_n(s_{n,i})| = i-1$, $|q_n(t_{n,i})| = i$ and $|q_n(t)| \in [i - 1, i]$ on $(s_{n,i}, t_{n,i})$. The same argument that we used to obtain (24) shows

$$\inf_{n\geqslant 2} \inf_{n\geqslant j\geqslant 2} |t_{n,j} - s_{n,j}| > 0. \tag{29}$$

Hence

$$\begin{aligned} I(q_n) &\geqslant \sum_{j=[z_0]+2}^{n} \int_{s_{n,j}}^{t_{n,j}} V(t, q_j(t))\, dt \\ &\geqslant c_1 \sum_{j=[z_0]+2}^{n} \int_{s_{n,j}}^{t_{n,j}} \frac{1}{|q_j(t)|} dt \\ &\geqslant c_1 \left(\inf_{n\geqslant 2} \inf_{n\geqslant j\geqslant 2} |t_{n,j} - s_{n,j}| \right) \sum_{j=[z_0]+2}^{n} \frac{1}{j}, \end{aligned} \tag{30}$$

for any $n \geqslant [z_0] + 2$, $n \in \mathbb{N}$, a contradiction.

Therefore, a constant $C_0 > 0$ depending only on M and u exists, for which $\sup_{t\in\mathbb{R}} |q(t)| \leqslant C_0$ for all $q \in Q_{u,M}$. □

Lemma 7. *Let* $q \in E$. *If* $I(q) < +\infty$, *then there are* $\alpha, \beta \in \mathscr{V}$ *such that* $\lim_{t\to-\infty} q(t) = \alpha$ *and* $\lim_{t\to+\infty} q(t) = \beta$.

Proof. By Lemma 5, $q \in L^\infty(\mathbb{R}, \mathbb{R})$. Since $q \in C(\mathbb{R})$, then $\sup_{t\in\mathbb{R}} |q(t)| \leqslant Q$ for some $Q > 0$. Therefore, the set

$$\Omega(q, -\infty) = \{x : q(s_k) \longrightarrow x \text{ for some } s_k \longrightarrow -\infty\} \tag{31}$$

of accumulation points of q at $-\infty$ is not empty. Moreover, $\Omega(q, -\infty) \cap \mathscr{V} \neq \emptyset$. Indeed, suppose there are $\delta \in (0, \gamma)$ and $r_0 > 0$ such that $q(t) \notin B_\delta(\mathscr{V})$ for all $t \leqslant -r_0$. Then, by definition of $\sigma_{\delta,Q}$ in (14) with $\varepsilon = \delta$ and $v = Q$,

$$I(q) > \int_{-\infty}^{-r_0} \{V(t, q)\}\, dt > \int_{-\infty}^{-r_0} \sigma_{\delta,Q} dt, \tag{32}$$

which shows $I(q) = +\infty$, a contradiction. Hence, $\Omega(q, -\infty) \cap \mathscr{V}$ is not empty and there exists an $\alpha \in \Omega(q, -\infty) \cap \mathscr{V}$. Finally we claim $q(-\infty) = \alpha$. Suppose otherwise, there must be some $\delta_1 \in (0, \varepsilon^*)$ such that $q(t)$ intersects $\partial B_{\delta_1}(\alpha)$ and $\partial B_{2\delta_1}(\alpha)$ infinitely many times, then Lemma 3 implies that the energy along the orbit $q(t)$ would tend to infinity, contradicting the hypothesis. Thus, we have shown $q(t)$ tends to α as $t \to -\infty$. The other limit can be proved similarly. □

Theorem 8. *For any* $\varepsilon \in (0, \varepsilon^*]$, $\vartheta \in \mathscr{V} \setminus \{0\}$. *There exists* $q_{\varepsilon,\vartheta} \in \Gamma_\varepsilon(\vartheta)$ *such that* $q_{\varepsilon,\vartheta}$ *minimizes the functional* $I(q)$ *on the set* $\Gamma_\varepsilon(\vartheta)$, *that is,* $I(q_{\varepsilon,\vartheta}) = c_\varepsilon(\vartheta)$.

Proof. Let $\{q_{\varepsilon,\vartheta,n}\} \in \Gamma_\varepsilon(\vartheta)$ be a minimizing sequence for (13),

$$I(q_{\varepsilon,\vartheta,n}) \longrightarrow c_\varepsilon(\vartheta) = \inf_{u\in\Gamma_\varepsilon(\vartheta)} I(u). \tag{33}$$

For notational convenience, we will suppress subscripts ε, ϑ in what follows and simply write $q_{\varepsilon,\vartheta,n} = q_n$. Now since $\{q_n\}$ is a minimizing sequence, we have $I(q_n) \leqslant M$ for some $M > 0$. Moreover, Lemma 6 shows that $\{q_n(0)\}$ is a bounded sequence; therefore, $\{q_n\}$ is bounded in E. Going if necessary to a subsequence, we may suppose that $\{q_n\}$ converges weakly in E to an element $q \in E$. Sobolev imbedding theorem shows $q_n \to q$ in $C^1_{\text{loc}}(\mathbb{R})$. To show q is the minimizer we are looking for, we need firstly to prove

$$I(q) < +\infty. \tag{34}$$

Indeed, for any $r < s \in \mathbb{R}$,

$$I_{r,s}(q_n) \leqslant I(q_n) \leqslant M, \tag{35}$$

by the weakly lower semicontinuity of $I_{r,s}(\cdot)$,

$$I_{r,s}(q) = \liminf_{n\to+\infty} I_{r,s}(q_n) \leqslant \liminf_{n\to+\infty} I(q_n) = \inf_{u\in\Gamma_\varepsilon(\vartheta)} I(u) \leqslant M. \tag{36}$$

Clearly, the constant M is independent of r and s. Since $q \in E$, the above inequality implies $V(\cdot, q(\cdot)) \in L^1(\mathbb{R}, \mathbb{R})$, and

$$I(q) \leqslant \inf_{u\in\Gamma_\varepsilon(\vartheta)} I(u) \leqslant M. \tag{37}$$

Hence, (34) holds.

Next we show

$$\lim_{t\to-\infty} q(t) = 0, \qquad \lim_{t\to+\infty} q(t) = \vartheta. \tag{38}$$

Inequality (34) and Lemma 7 imply that there are $\alpha, \beta \in \mathscr{V}$ verifying

$$\lim_{t\to-\infty} q(t) = \alpha, \qquad \lim_{t\to+\infty} q(t) = \beta. \tag{39}$$

But for each n, $(q_n(t), q_n'(t)) \notin B_\varepsilon(\widetilde{\mathscr{V}} \setminus \{\tilde{0}, \tilde{\vartheta}\})$ for all $t \in \mathbb{R}$ and $q_n \to q$ in $C^1_{\text{loc}}(\mathbb{R})$, so $(q(t), q'(t)) \notin B_\varepsilon(\widetilde{\mathscr{V}} \setminus \{\tilde{0}, \tilde{\vartheta}\})$ for all $t \in \mathbb{R}$. Therefore, $\alpha, \beta \in \{0, \vartheta\}$. For each $t \in \mathbb{R}$, there is $n(t) \in \mathbb{R}$ such that

$$|q_n(t) - q(t)| < \varepsilon, \quad \forall n \geqslant n(t). \tag{40}$$

We claim $\beta = \vartheta$. Suppose to the contrary that

$$\lim_{t\to+\infty} q(t) = 0. \tag{41}$$

Then, there is $t_0 > 0$ such that

$$|q(t)| < \varepsilon, \quad \forall t \geqslant t_0. \tag{42}$$

Hence,

$$|q_n(t)| \leqslant |q_n(t) - q(t)| + |q(t)| < 2\varepsilon, \tag{43}$$

whenever $t \geqslant t_0$ and $n \geqslant n(t)$. In particular

$$|q_{n(t)}(t)| \leqslant 2\varepsilon, \quad \forall t \geqslant t_0. \tag{44}$$

Let

$$\begin{aligned} s_n &= s_n(\varepsilon) = \sup\{t \in \mathbb{R} \mid q_n(t) \in \partial B_{2\varepsilon}(0)\}, \\ \bar{s}_n &= \bar{s}_n(\varepsilon) = \inf\{t > s_n \mid q_n(t) \in \partial B_{4\varepsilon}(0)\}. \end{aligned} \tag{45}$$

Since $q_n \in \Gamma_\varepsilon(\vartheta)$, $s_n < \bar{s}_n$ are well defined for each n. The same argument that we used to obtain (24) shows

$$\inf_n |\bar{s}_n - s_n| \geqslant c_0, \tag{46}$$

for some constant $c_0 > 0$. Whence, for $t \geqslant t_0$, by the definition of s_n and c_0, $|q_{n(t)}(\tau_{n(t)})| \in (2\varepsilon, 4\varepsilon)$, and the mean value theorem,

$$I(q_{n(t)}) \geqslant \int_{s_{n(t)}}^{s_{n(t)}+c_0} V(s, q_{n(t)})\, ds = c_0 V(\tau_{n(t)}, q_{n(t)}(\tau_{n(t)})), \tag{47}$$

where $\tau_{n(t)} \in (s_{n(t)}, s_{n(t)} + c_0)$. Inequality (44) also implies $t \leqslant s_{n(t)} < \tau_{n(t)}$. Hence, by (A_4), $V(\tau_{n(t)}, q_{n(t)}(\tau_{n(t)})) \to +\infty$ as $t \to +\infty$, which leads to a contradiction that $I(q_{n(t)}) \to +\infty$ as $t \to +\infty$. Similar argument shows that $\lim_{t\to-\infty} q(t) = 0$ and the proof is completed. □

Denote

$$\begin{aligned} \mathcal{G} &= \mathcal{G}(\varepsilon, \vartheta) \\ &= \{t \in \mathbb{R} \mid (q_{\varepsilon,\vartheta}(t), q'_{\varepsilon,\vartheta}(t)) \in \partial B_\varepsilon(\widetilde{\mathcal{V}} \setminus \{\tilde{0}, \tilde{\vartheta}\})\}. \end{aligned} \tag{48}$$

Theorem 9. *The minimizer $q = q_{\varepsilon,\vartheta}$ obtained in Theorem 8 is a classical solution on $\mathbb{R} \setminus \mathcal{G}$.*

Proof. For any $t \in \mathbb{R} \setminus \mathcal{G}$, since $\mathbb{R} \setminus \mathcal{G}$ is an open set, there is a maximal open interval $\mathfrak{I} \subset \mathbb{R} \setminus \mathcal{G}$ containing t. Let $\varphi \in C_0^\infty(\mathfrak{I})$, then for $\delta > 0$ sufficiently small $q + \delta\varphi \in \Gamma_\varepsilon(\vartheta)$. Since q minimize $I(\cdot)$ on $\Gamma_\varepsilon(\vartheta)$, then, for all $\varphi \in C_0^\infty(\mathfrak{I})$,

$$(I'(q), \varphi) = \int_{-\infty}^{\infty} \{q''\varphi'' + q'\varphi' + V_q(t, q)\varphi\}\, dt = 0. \tag{49}$$

Fix any $r < s \in \mathfrak{I}$, this equality also holds for any $\varphi \in W_0^{2,2}([r,s])$. Hence, q is a weak solution of the following system:

$$\begin{gathered} u''''(t) - u''(t) + V_q(t, q) = 0, \quad r < t < s; \\ u(r) = q(r), \quad u(s) = q(s); \\ u'(r) = q'(r), \quad u'(s) = q'(s); \end{gathered} \tag{50}$$

the system has a unique C^4 solution, denoted by v. Then

$$\int_r^s \{v''\varphi'' + v'\varphi' + V_q(t, q)\varphi\}\, dt = 0. \tag{51}$$

Combining (49) and (51) yields

$$\int_r^s \{(q'' - v'')\varphi'' + (q' - v')\varphi'\}\, dt = 0, \tag{52}$$

for all $\varphi \in W_0^{2,2}([r,s])$. Since $q - v$ belongs to E, it follows that $q - v$ is constant. Noting the boundary conditions that q coincides with v at r and s, we have $q = v \in C^4([r,s])$. Since $r, s \in \mathfrak{I}$ are arbitrary, it follows that $q \in C^4(\mathfrak{I})$, which completes the proof. □

Define

$$c_\varepsilon = \inf_{\vartheta \in \mathcal{V}\setminus\{0\}} c_\varepsilon(\vartheta). \tag{53}$$

Since c_ε is nonincreasing as a function of $\varepsilon > 0$, $\inf_{\varepsilon>0} c_\varepsilon$ exists and we may choose a decreasing sequence $\varepsilon_i \to 0$ such that

$$c_{\varepsilon_i} \longrightarrow \inf_{\varepsilon>0} c_\varepsilon \quad \text{as } i \longrightarrow \infty. \tag{54}$$

By Lemma 6, for each i, the set of candidates for the infimum c_{ε_i} is finite; hence, it is achieved by an element $\kappa_i \in \mathcal{V} \setminus \{0\}$, namely,

$$c_{\varepsilon_i} = c_{\varepsilon_i}(\kappa_i) = \inf_{q \in \Gamma_{\varepsilon_i}(\kappa_i)} I(q). \tag{55}$$

Theorem 8 shows, for each i, there is a $q_i \in \Gamma_{\varepsilon_i}(\kappa_i)$ such that

$$I(q_i) = \inf_{q \in \Gamma_{\varepsilon_i}(\kappa_i)} I(q). \tag{56}$$

Again by Lemma 6, $\{\kappa_i, i \geqslant 1\} \subset \mathcal{V} \setminus \{0\}$ must be a finite set and thus contains a constant subsequence $\{\kappa_{i_k}, k \geqslant 1\}$; in other words, there is a $\kappa \in \{\kappa_i, i \geqslant 1\}$ for which $\kappa_{i_k} = \kappa$, for all $k \geqslant 1$. Without loss of generality, we also denote this constant sequence by $\{\kappa_i, i \geqslant 1\}$ (correspondingly $\{\varepsilon_i, i \geqslant 1\}$), then

$$c_{\varepsilon_i} = \inf_{q \in \Gamma_{\varepsilon_i}(\kappa)} I(q) = I(q_i), \tag{57}$$

where $q_i \in \Gamma_{\varepsilon_i}(\kappa)$.

Theorem 10. *For i sufficiently large, q_i is a heteroclinic solution connecting 0 and κ.*

Proof. To finish the proof, we show that q_i never touches $\partial B_{\varepsilon_i}(\widetilde{\mathcal{V}} \setminus \{\tilde{0}, \tilde{\kappa}\})$ for i large enough. Assume not, then there are $\{\chi_j\} \subset \mathcal{V} \setminus \{0, \kappa\}$ and $\{\tau_j\} \in \mathbb{R}$ such that $(q_j(\tau_j), q_j'(\tau_j)) \in \partial B_{\varepsilon_j}(\tilde{\chi}_j)$ and $(q_j(t), q_j'(t)) \notin \partial B_{\varepsilon_j}(\tilde{\chi}_j)$ for $t < \tau_j$; precisely, τ_j is the first hitting time of q_j on $\partial B_{\varepsilon_j}(\tilde{\chi}_j)$. An application of Lemma 6 shows that $\{\chi_j\}$ is bounded, so it must contain a constant subsequence which we still denote by $\{\chi_j\}$, that is, $\chi_j \equiv \chi \in \mathcal{V} \setminus \{0, \kappa\}$ for $j \geqslant 1$. Now, two situations may occur, for infinitely many j, (i) (q_j, q_j') reaches $\partial B_{\varepsilon_j}(\widetilde{\mathcal{V}} \setminus \{\tilde{\chi}\})$ before $\partial B_{\varepsilon_j}(\widetilde{\mathcal{V}} \setminus \{\tilde{\kappa}\})$ or (ii) touches $\partial B_{\varepsilon_j}(\widetilde{\mathcal{V}} \setminus \{\tilde{\kappa}\})$ before $\partial B_{\varepsilon_j}(\widetilde{\mathcal{V}} \setminus \{\tilde{\chi}\})$.

Since $q_j \in \Gamma_{\varepsilon_j}(\kappa)$, ε_j is a decreasing sequence of j and c_{ε_j} is nonincreasing in j; hence,

$$I(q_j) = c_{\varepsilon_j} \leqslant c_{\varepsilon_1} < +\infty, \tag{58}$$

which by Lemma 6 implies that $\{q_j\}$ must be uniformly bounded on $\mathbb{R}$, say $\sup_{s\in\mathbb{R}} |q_j(s)| < M$ for all j.

Case 1 (($(q_j(t), q_j'(t)) \notin \overline{B}_{\varepsilon_j}(\widetilde{\kappa})$ *for all* $t < \tau_j$). Before stepping forward, we need to show $\{\tau_j\}$ is bounded both from above and below. Suppose $\{\tau_j\}$ is unbounded from above. Let

$$\begin{aligned} s_j &= \sup\left\{t \in \mathbb{R} \mid (q_j(t), q_j'(t)) \in \partial B_\gamma(\widetilde{\chi})\right\}, \\ \bar{s}_j &= \inf\left\{t > s_j \mid (q_j(t), q_j'(t)) \in \partial B_\gamma(\widetilde{\kappa})\right\}. \end{aligned} \tag{59}$$

For j large enough, $\gamma > \varepsilon_j$, so $s_j > \tau_j$ by definition of τ_j. Hence, $\{s_j\}$ is unbounded from above. Without loss of generality; we may suppose $\lim_{j\to\infty} s_j = +\infty$. Similar to (46) it can be shown that

$$\inf_j |s_j - \bar{s}_j| > c_1, \tag{60}$$

for some positive constant $c_1 > 0$. Hence,

$$\begin{aligned} +\infty > c_{\varepsilon_j} &= I(q_j) \\ &\geqslant \int_{s_j}^{\bar{s}_j} V(s, q_j(s))\, ds \\ &= (\bar{s}_j - s_j) V(\hat{s}_j, q_j(\hat{s}_j)) \\ &> c_1 V(\hat{s}_j, q_j(\hat{s}_j)), \end{aligned} \tag{61}$$

where $s_j < \hat{s}_j < \bar{s}_j$. Now assumption (A_4) and the uniform boundedness of $\{q_j\}$ indicate that $\{\hat{s}_j\}$ is bounded, which is a contradiction. Thus, $\{\tau_j\}$ must have an upper bound. Similarly it can be proved that $\{\tau_j\}$ is also bounded from below by considering $r_j = \sup\{t \in \mathbb{R} \mid (q_j(t), q_j'(t)) \in \partial B_\gamma(\widetilde{0})\}$ and $\bar{r}_j = \inf\{t > r_j \mid (q_j(t), q_j'(t)) \in \partial B_\gamma(\widetilde{\kappa})\}$. Hence, $\{\tau_j\}$ is bounded.

Define

$$\varsigma_j(t) = \begin{cases} \chi, & \text{for } t \geqslant \tau_j + \varepsilon_j^{1/3}; \\ P_j(t - \tau_j), & \text{for } t \in [\tau_j, \tau_j + \varepsilon_j^{1/3}); \\ q_j(t), & \text{for } t < \tau_j, \end{cases} \tag{62}$$

where P_j is a polynomial defined as

$$\begin{aligned} P_j(t) &= a_j + b_j t + c_j t^2 + d_j t^3, \\ a_j &= q_j(\tau_j), \qquad b_j = q_j'(\tau_j), \\ c_j &= -\varepsilon_j^{-2/3}\left[3(q_j(\tau_j) - \chi) + 2\varepsilon_j^{1/3} q_j'(\tau_j)\right], \\ d_j &= \varepsilon_j^{-1}\left[2(q_j(\tau_j) - \chi) + \varepsilon_j^{1/3} q_j'(\tau_j)\right]. \end{aligned} \tag{63}$$

It is easy to verify the boundary conditions

$$(P_j(0), P_j'(0)) = (q_j(\tau_j), q_j'(\tau_j)), \tag{64}$$

$$(P_j(\varepsilon_j^{1/3}), P_j'(\varepsilon_j^{1/3})) = (\chi, 0). \tag{65}$$

In addition, $\lim_{j\to\infty} a_j = 0$, $\lim_{j\to\infty} b_j = 0$, $\lim_{j\to\infty} c_j = 0$, and $\{d_j\}$ is bounded.

Comparing the energy of q_j and ς_j, we have

$$\begin{aligned} I(q_j) - I(\varsigma_j) &= I_{\tau_j,+\infty}(q_j) - I_{\tau_j,+\infty}(\varsigma_j) \\ &= I_{\tau_j,+\infty}(q_j) - I_{\tau_j,\tau_j+\varepsilon_j^{1/3}}(\varsigma_j) - I_{\tau_j+\varepsilon_j^{1/3},+\infty}(\varsigma_j) \\ &\geqslant \gamma\sqrt{2\sigma_\gamma} - I_{\tau_j,\tau_j+\varepsilon_j^{1/3}}(\varsigma_j) - I_{\tau_j+\varepsilon_j^{1/3},+\infty}(\varsigma_j). \end{aligned} \tag{66}$$

By assumption (A_5),

$$I_{\tau_j+\varepsilon_j^{1/3},+\infty}(\varsigma_j) < \int_{-\infty}^{\infty} V(t, \chi)\, dt < \gamma\sqrt{\sigma_\gamma}. \tag{67}$$

Moreover, since $\{\varsigma_j\}$ is uniformly bounded on $\mathbb{R}$ and $\{\tau_j\}$ is bounded, it readily follows that

$$\int_{\tau_j}^{\tau_j+\varepsilon_j^{1/3}} V(t, \varsigma_j)\, dt \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{68}$$

Meanwhile, by construction,

$$\begin{aligned} \int_{\tau_j}^{\tau_j+\varepsilon_j^{1/3}} (\varsigma_j'')^2 &= \int_{\tau_j}^{\tau_j+\varepsilon_j^{1/3}} (2c_j + 6d_j t)^2 dt \longrightarrow 0, \\ \int_{\tau_j}^{\tau_j+\varepsilon_j^{1/3}} (\varsigma_j')^2 &= \int_{\tau_j}^{\tau_j+\varepsilon_j^{1/3}} (b_j + 2c_j t + 3d_j t^2)^2 dt \longrightarrow 0. \end{aligned} \tag{69}$$

So

$$\lim_{j\to+\infty} I_{\tau_j,\tau_j+\varepsilon_j^{1/3}}(\varsigma_j) = 0. \tag{70}$$

Therefore, for j large enough,

$$\begin{aligned} I(q_j) - I(\varsigma_j) &\geqslant \gamma\sqrt{2\sigma_\gamma} - \frac{\sqrt{2}-1}{2}\gamma\sqrt{\sigma_\gamma} - \gamma\sqrt{\sigma_\gamma} \\ &> \frac{\sqrt{2}-1}{2}\gamma\sqrt{\sigma_\gamma}, \end{aligned} \tag{71}$$

which is impossible.

Case 2 (($(q_j(t), q_j'(t)) \in \partial B_{\varepsilon_j}(\{\widetilde{\kappa}\})$ *for some* $s_j < \tau_j$). A contradiction can also be reached by similar arguments.

Either case leads to a contradiction; therefore, for i sufficiently large, the orbit of $q_i = q_{\varepsilon_i,\kappa}$ must keep some distance away from $\widetilde{\mathscr{V}} \setminus \{\widetilde{0}, \widetilde{\kappa}\}$, and hence it is a classical solution, and the proof is complete. □

Proof of Theorem 1. Theorem 10 shows for every $z \in \mathscr{V}$, there is heteroclinic solution of (1) emanating from z. If we consider the set $\overline{\Gamma}_\varepsilon(\vartheta) \subset E$ of function $q(t)$ for which $q(+\infty) = 0$, $q(-\infty) = \vartheta \in \mathscr{V} \setminus \{0\}$ and $(q(t), q'(t)) \notin B_\varepsilon(\widetilde{\mathscr{V}} \setminus \{\widetilde{0}, \widetilde{\vartheta}\})$, for all $t \in \mathbb{R}$; then the proof for $\Gamma_\varepsilon(\vartheta)$ works equally well for $\overline{\Gamma}_\varepsilon(\vartheta)$, and so, for every $z \in \mathscr{V}$, there is heteroclinic solution of (1) terminating at z. □

3. Heteroclinic Solution in Periodic Case

In the last section, we consider the case where $V(t,z)$ is periodic in t with period $T > 0$. In this case, the assumptions and proof can be simplified. For completeness, we devote this section to the periodic case. Since most results in the last section can be carried out verbatim, we just present those that are different. We make the following assumptions:

(H_1) $V(t,q) \in C^2(\mathbb{R} \times \mathbb{R})$, $V \geqslant 0$;

(H_2) the set $\mathscr{V} = \{z \in \mathbb{R} \mid \text{for all } t, V(t,z) = 0, V_q(t,z) = 0 \text{ and } V_{qq}(t,z) \geqslant 0\}$ is discrete and has at least two points, and

$$\gamma \equiv \frac{1}{3} \inf \{|x - y| \mid x \neq y \in \mathscr{V}\} > 0. \tag{72}$$

We also assume without no loss of generality $0 \in \mathscr{V}$.

(H_3) There is a constant $V_0 > 0$ such that $\liminf_{|q| \to \infty} V(t,q) \geqslant V_0$ uniformly for $t \in \mathbb{R}$.

(H_4) For each $z \in \mathscr{V}$,

$$\int_{-\infty}^{\infty} V(t,z)\, dt < \gamma \sigma_\gamma^{1/2}. \tag{73}$$

Note that (A_4) is no longer needed in the periodic case. (H_3) is indeed a special case of (A_3), we are using (H_3) instead of (A_3) only for illustrative purpose. As before, we work on the space

$$E = \left\{ q \in W^{2,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}) \mid \int_{-\infty}^{\infty} \left(|q''|^2 + |q'|^2 \right) dt < \infty \right\}, \tag{74}$$

with the norm

$$\|q\|^2 = \int_{-\infty}^{\infty} \left(|q''|^2 + |q'|^2 \right) dt + |q(0)|^2. \tag{75}$$

We will use the shift operator σ_j defined for each j as

$$\sigma_j q = q(t - jT). \tag{76}$$

Clearly, if q is a heteroclinic solution, so is $\tau_j q$ for all j. Recall $\Gamma_\varepsilon(\zeta), \zeta \in \mathscr{V} \setminus \{0\}$ is the set of $q \in E$ verifying

(i) $\lim_{t \to -\infty} q(t) = 0$, $\lim_{t \to +\infty} q(t) = \zeta$;

(ii) $(q(t), q'(t)) \notin B_\varepsilon(\widetilde{\mathscr{V}} / \{\widetilde{0}, \widetilde{\vartheta}\})$, for all $t \in \mathbb{R}$; where $\widetilde{\mathscr{V}} = \{(z,0) \in \mathbb{R}^2 \mid z \in \mathscr{V}\}$.

$\Gamma_\varepsilon(\zeta)$ is not empty, we may consider the minimization problem

$$c_\varepsilon(\zeta) = \inf_{q \in \Gamma_\varepsilon(\zeta)} I(q). \tag{77}$$

We shall follow the lines of argument in previous section, and similar proofs will be omitted. For example, Lemmas 3, 5, 6, and 7 remain valid in the setting of this section.

Theorem 11. *For any $\varepsilon \in (0, \gamma]$, $\widetilde{\zeta} \in \widetilde{\mathscr{V}} \setminus \{\widetilde{0}\}$. There exists $q_{\varepsilon,\zeta} \in \Gamma_\varepsilon(\zeta)$ such that $q_{\varepsilon,\zeta}$ minimizes the functional $I(q)$ on the set $\Gamma_\varepsilon(\zeta)$, that is, $I(q_{\varepsilon,\zeta}) = c_\varepsilon(\zeta)$.*

Proof. Let $\{q_n\}$ be a minimizing sequence for (77). As in the preceding section, it is bounded in E and we may choose a subsequence, also denoted by $\{q_n\}$, converging weakly in E to some $q \in E$ and $q_n \to q$ in $C^1_{\text{loc}}(\mathbb{R})$. Moreover,

$$I(q) < +\infty. \tag{78}$$

It remains to show

$$\lim_{t \to -\infty} q(t) = 0, \qquad \lim_{t \to +\infty} q(t) = \zeta. \tag{79}$$

Inequality (78) and Lemma 7 imply that there are $\widetilde{\alpha}, \widetilde{\beta} \in \widetilde{\mathscr{V}}$ verifying

$$\lim_{t \to -\infty} q(t) = \alpha, \qquad \lim_{t \to +\infty} q(t) = \beta. \tag{80}$$

Since $V(t,u)$ is periodic in t, if $\{q_n\}$ is the minimizing sequence, so is $\sigma_{j_n} q_n$ for any sequence $\{j_n\} \subset \mathbb{N}$ and $I(q_n) = I(\sigma_{j_n} q_n)$. Whence, with $\{j_n\}$ being appropriately chosen and $\{q_n\}$ replaced with $\{\sigma_{j_n} q_n\}$, we are safe to suppose that $q_n(t) \in B_\varepsilon(0)$ for $t \leqslant 0$ and $q_n(s) \in \partial B_\varepsilon(0)$ for some $s \in [0,T]$. Noting for each n, $(q_n(t), q_n'(t)) \notin B_\varepsilon(\widetilde{\mathscr{V}} \setminus \{\widetilde{0}, \widetilde{\zeta}\})$ for all $t \in \mathbb{R}$ and $q_n \to q$ in $C^1_{\text{loc}}(\mathbb{R})$, we have $(q(t), q'(t)) \notin B_\varepsilon(\widetilde{\mathscr{V}} \setminus \{\widetilde{0}, \widetilde{\zeta}\})$ for all $t \in \mathbb{R}$ and so $\widetilde{\alpha}, \widetilde{\beta} \in \{\widetilde{0}, \widetilde{\zeta}\}$. Since $q_n(t) \in B_\varepsilon(0)$ for $t \leqslant 0$, $\alpha \in \{0, \zeta\} \cap \overline{B}_\varepsilon(0) = \{0\}$, so $\alpha = 0$.

It remains to show $\beta = \zeta$. Note $\beta = q(+\infty) \in \{0, \zeta\}$. Suppose that $q(+\infty) = 0$. Choose $\delta > 0$, for the time being we only assume that $4\delta < \varepsilon$. Since $q'(+\infty) = 0$, there is a $t_\delta > T$ such that $q(t), q'(t) \in B_\delta(0)$ for all $t > t_\delta$. Since $q_n \to q$ in $C^1_{\text{loc}}(\mathbb{R})$, so there is n_δ, which depends on δ, such that $q_n(t_\delta), q_n'(t_\delta) \in B_{2\delta}(0)$ for n larger than n_δ. But $q_n(s) \in \partial B_\varepsilon(0)$ for some $s \in [0,T]$, so q_n intersects $\partial B_{2\delta}(0)$ and $\partial B_\varepsilon(0)$. By Lemma 3, we have

$$I(q_n) \geqslant \frac{\varepsilon}{2} \sqrt{2\sigma_{\varepsilon/2}} + I_{t_\delta, +\infty}(q_n). \tag{81}$$

Now construct a sequence as follows:

$$\varsigma_n(t) = \begin{cases} 0, & \text{for } t \leqslant t_\delta - t_n; \\ P_n(t - t_\delta + t_n), & \text{for } t \in (t_\delta - t_n, t_\delta]; \\ q_n(t), & \text{for } t > t_\delta, \end{cases} \tag{82}$$

where

$$t_n = \begin{cases} \sqrt[3]{|q_n(t_\delta)| + |q_n'(t_\delta)|}, & \text{if } \sqrt[3]{|q_n(t_\delta)| + |q_n'(t_\delta)|} \neq 0, \\ 1, & \text{if } \sqrt[3]{|q_n(t_\delta)| + |q_n'(t_\delta)|} = 0, \end{cases} \tag{83}$$

and $P_n(t) = g_n t^2 + h_n t^3$ is a polynomial with coefficients

$$g_n = t_n^{-2}\left[3q_n(t_\delta) - t_n q_n'(t_\delta)\right],$$
$$h_n = t_n^{-3}\left[t_n q_n'(t_\delta) - 2q_n(t_\delta)\right]. \tag{84}$$

It is easy to check that $P_n(t)$ satisfies

$$\left(P_n(0), P_n'(0)\right) = (0,0),$$
$$\left(P_n(t_n), P_n'(t_n)\right) = \left(q_n(t_\delta), q_n'(t_\delta)\right). \tag{85}$$

Moreover, $q_n(t_\delta), q_n'(t_\delta)$ tends to zero if $\delta \to 0$ and $n > n_\delta$. By construction $\varsigma_n \in \Gamma_\varepsilon(\zeta)$. So far, we only used the fact $0 < 4\delta < \varepsilon$, we can fix δ smaller still such that, for all $n > n_\delta$,

$$I_{t_\delta - t_n, t_\delta}(\varsigma_n) = \int_{t_\delta - t_n}^{t_\delta}\left\{\frac{1}{2}(2g_n + 6h_n t)^2 + \frac{1}{2}\left(2g_n t + 3h_n t^2\right)^2 + V(t,\varsigma_n)\right\} dt,$$
$$I_{t_\delta - t_n, t_\delta}(\varsigma_n) < \frac{\varepsilon}{4}\sqrt{2\sigma_{\varepsilon/2}}. \tag{86}$$

Then

$$\begin{aligned} I(q_n) - I(\varsigma_n) &= I_{-\infty, t_\delta}(q_n) - I_{t_\delta - t_n, t_\delta}(\varsigma_n) \\ &\geqslant \frac{\varepsilon}{2}\sqrt{2\sigma_{\varepsilon/2}} - I_{t_\delta - t_n, t_\delta}(\varsigma_n) \\ &> \frac{\varepsilon}{4}\sqrt{2\sigma_{\varepsilon/2}}, \end{aligned} \tag{87}$$

for $n > n_\delta$. This implies

$$\begin{aligned} \liminf_{n\to+\infty} I(q_n) - \frac{\varepsilon}{4}\sqrt{2\sigma_{\varepsilon/2}} &\geqslant \liminf_{n\to+\infty} I(\varsigma_n) \\ &\geqslant \inf_{q\in\Gamma_\varepsilon(\zeta)} I(q) \\ &= \liminf_{n\to+\infty} I(q_n), \end{aligned} \tag{88}$$

which is impossible. The proof is complete. □

Theorem 12. *There is a pair of orbits $q_{0,\zeta}, q_{\zeta,0}$ which connect* 0 *and ζ, one emanating from* 0 *and the other terminating at* 0.

Proof. The proof is similar to Theorems 8 and 10 in [9]. □

Lastly, we observe the following.

Theorem 13. *For the solution $q = q_{0,\zeta}$ or $q_{\zeta,0}$ obtained above,* $\lim_{n\to\pm\infty} q''(t) = 0$, $\lim_{n\to\pm\infty} q'''(t) = 0$.

Proof. Consider $q = q_{0,\zeta}$. By (H_2), there are positive constants r^*, c_1, c_2 such that

$$V(t,u)\,c_1 \geqslant |u-\zeta|^2,$$
$$\left|V_z(t,u)\right| \leqslant c_2\,|u-\zeta|, \tag{89}$$

whenever $|u-\zeta| \leqslant r^*$. Since $q(+\infty) = \zeta$, $|q(t)-\zeta| \leqslant r^*$ for t larger than some $t^* > 0$.

From (89) and the Euler equation,

$$q'''' - q'' + V_z(t,q) = 0, \tag{90}$$

we have

$$\begin{aligned} \int_{t^*}^{+\infty} \left|q''''(t)\right|^2 dt &\leqslant 2\int_{t^*}^{+\infty}\left|q''(t)\right|^2 dt + 2\int_{t^*}^{+\infty}\left|V_z(t,q)\right|^2 dt \\ &\leqslant 2\int_{t^*}^{+\infty}\left|q''(t)\right|^2 dt + 2c_2^2\int_{t^*}^{+\infty}\left|q(t)-\zeta\right|^2 dt \\ &= 2\int_{t^*}^{+\infty}\left|q''(t)\right|^2 dt + 2\frac{c_2^2}{c_1}\int_{t^*}^{+\infty} c_1\left|q(t)-\zeta\right|^2 dt \\ &\leqslant 2\int_{t^*}^{+\infty}\left|q''(t)\right|^2 dt + 2\frac{c_2^2}{c_1}\int_{t^*}^{+\infty} V(t,q(t))\,dt \\ &< \infty. \end{aligned} \tag{91}$$

An application of the interpolation inequality yields

$$\int_{t^*}^{+\infty}\left|q'''(t)\right|^2 dt \leqslant K\int_{t^*}^{+\infty}\left|q''(t)\right|^2 dt + K\int_{t^*}^{+\infty}\left|q''''(t)\right|^2 dt < \infty, \tag{92}$$

where K is a positive constant.

Therefore, we have proved that

$$\lim_{n\to+\infty} q''(t) = \lim_{n\to+\infty} q'''(t) = 0. \tag{93}$$

The limit for negative infinity can be derived similarly. The proof is complete. □

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