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Research Article

A Numerical Approach to Static Deflection Analysis of an Infinite Beam on a Nonlinear Elastic Foundation: One-Way Spring Model

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A numerical procedure proposed by Jang et al. (2011) is applied for the numerical analyzing of static deflection of an infinite beam on a nonlinear elastic foundation. And one-way spring model is used for the modeling of fully nonlinear elastic foundation. The nonlinear procedure involves Green's function technique and an iterative method using the pseudo spring coefficient. The workability of the numerical procedure is demonstrated through showing the validity of the solution and the convergence test with some external loads.

1. Introduction

Accurate modeling of nonlinear deflection of an infinite beam on a nonlinear elastic foundation is crucial for material and structural engineering. The research can be applied to strength analysis and practical engineering design application, say, to curved plate manufacturing. Therefore, many theoretical and experimental studies have been carried out on the nonlinear modeling of an infinite beam on a nonlinear elastic foundation.

The closed-form solutions for the static and dynamic response of a uniform beam resting on a linear elastic foundation can be found in several references [1–3]. Timoshenko [4], Kenney [5], Saito and Murakami [6], and Frýba [7] formulated a closed-form solution using Green's function approach based on a linear assumption. Beaufait and Hoadley [8], Massalas [9], Lakshmanan [10], and Hui [11] proposed the static, dynamic, and elastic stability analysis of a beam resting on a nonlinear elastic foundation. And there are many researches concerning the linear elastic foundation [12–15]. Among the references, Beaufait and Hoadley [8] approximated the relationship of the stress-strain curve to be hyperbolic, but they approximated the bilinear curve to handle the nonlinear problem. The applied nonlinear foundation is active only when the beam is pressing against the foundation, and it is assumed to be inactive in the regions where the beam has been displaced away from the foundation. Soldatos and Selvadurai [16] also applied the hyperbolic-type nonlinear elastic foundation to analyze finite or infinite beams. Lee et al. [17–19] developed the exact and semiexact analysis of a nonuniform beam with general elastic end-restraints. Kuo and Lee [20] derived the static deflection of a general elastically end-restrained, nonuniform beam on a nonlinear elastic foundation under axial and transverse forces.

Recently, Jang et al. [21] proposed a new method for assessing the nonlinear deflection of an infinite beam on a nonlinear elastic foundation. They approach the high nonlinear problems using Green's function technique with an iterative method. Jang and Sung [22] proposed a new functional iterative method for static beam deflection, which has a variable cross-section. Choi and Jang [23] proved the existence and uniqueness of the nonlinear deflections of an infinite beam resting on a nonlinear elastic foundation using the Banach fixed point theorem. Jang [24] also proposed a new iterative method for the large deflection of an infinite beam resting on an elastic foundation based on the v. Karman approximation of geometrical nonlinearity. From the existing

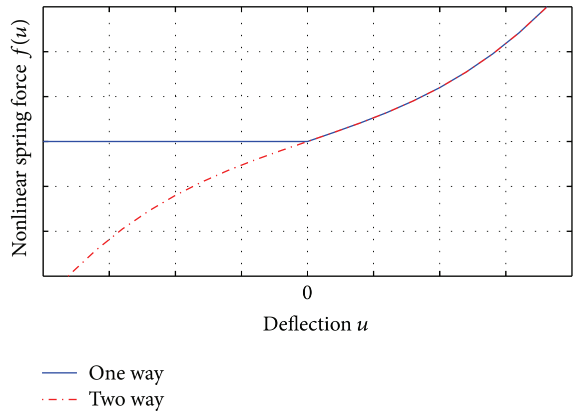

FIGURE 1: Nonlinear elastic foundation model: a nonlinear spring model (one way) and a conventional one (two way).

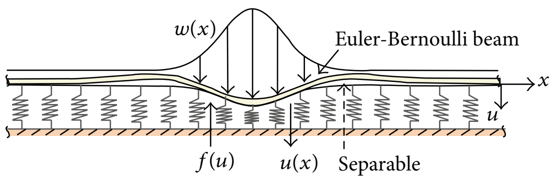

FIGURE 2: An infinite beam on a nonlinear elastic foundation: a nonlinear spring model.

literature, a number of studies have analyzed a beam on an elastic foundation; however, they just use linear plus a nonlinear term of spring force, that is, linear-cubic model. And they are related to the static analysis of nonuniform beams which is resting on a nonlinear elastic foundation, and the recovered solution is not accurate or has many limits. Few studies have fully adopted the nonlinear elastic foundation model, whose spring force is based on one-way spring model, as shown in Figure 1. In the real world, at the steady state, the soil or foundation would not be raised, or they are separable (in Figures 1 and 2). A nonlinear spring force exists when the infinite beam deflects downward but does not exist in case of the other cases.

Although there are many researches, fully nonlinear elastic foundation was not considered. Beaufait and Hoadley [8] and Soldatos and Selvadurai [16] approximated the stress-strain relationship as a bilinear curve. In this paper, one-way spring model is successfully used to examine the real nonlinear elastic foundation, and the nonlinear iterative method proposed by Jang et al. [21] is applied. Some numerical experiments are carried out to report the accuracy of the method, and the convergence of the solution is investigated according to several physical properties of the system.

2. Mathematical Modeling

2.1. Euler-Bernoulli's Beam on a Nonlinear Elastic Foundation: A Nonlinear Spring Model. In this paper, the nonlinear spring force is fully analyzed by the *one-way spring* model instead of the conventional mathematical form of the *two-way* spring model [25].

The well-known classical *Euler-Bernoulli's beam* theory is considered for the solution procedure which is a simplification of elasticity which provides a means of calculating the load-carrying and deflection characteristics of beams. The governing equation for the linear deflection of an infinite beam on an elastic foundation that satisfies the fourth-order differential equation is as follows (the weight of the beam is neglected):

$$EI\frac{d^4u}{dx^4} + f(u) = w(x). \tag{1}$$

And the reaction force $f(u)$,

$$f(u) = k \cdot u + N(u), \tag{2}$$

and E, I, k, $N(u)$ and $w(x)$ are Young's modulus, the mass moment of inertia, a linear spring coefficient, a nonlinear part of spring force, and external load, respectively.

The boundary

$$u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \text{ and } \frac{d^3u}{dx^3} \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty. \tag{3}$$

Therefore, (1) and (3) together form a well-defined boundary value problem. Timoshenko [4], Kenney [5], Saito and Murakami [6], and Frýba [7] derived the general linear solutions neglecting the nonlinear part, $N(u)$, in (2) as follows:

$$u(x) = \int_{-\infty}^{\infty} G(x,\xi;k) \cdot w(\xi)\, d\xi, \tag{4}$$

where Green's function G can be defined as

$$G(x,\xi;k) = \frac{\alpha}{2k}e^{-\alpha|\xi-x|/\sqrt{2}} \times \sin\left(\frac{\alpha|\xi-x|}{\sqrt{2}} + \frac{\pi}{4}\right), \quad \alpha = \sqrt[4]{k/EI}. \tag{5}$$

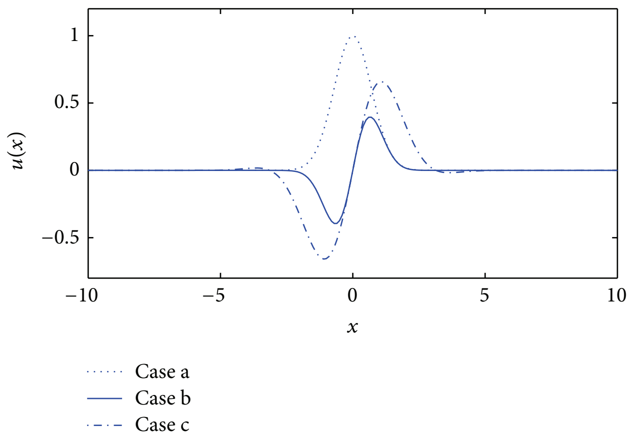

FIGURE 3: Exact solutions in Table 1.

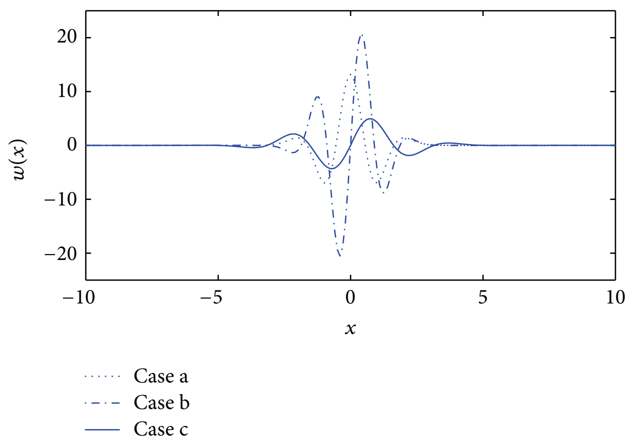

FIGURE 4: Applied loading conditions corresponding to the exact solution in Table 1.

The loading condition is assumed to be localized, so $u(x)$ in (4) satisfies the boundary conditions in (3). Deriving the linear solution in (4), a uniformly upward, nonlinear spring force depends on the beam deflection u.

In this study, the main idea of the present study is proposed by Jang et al. [21]. The nonlinear spring model is used for the formulation of a realistic nonlinear elastic foundation. A pseudo linear spring coefficient k_p and a *(real)* spring force $f(u)$ are used. Therefore, the fourth-order differential equation in (1) is equivalent to the following equation:

$$EI\frac{d^4u}{dx^4} + k_pu + f(u) = w(x) + k_pu, \qquad (6)$$

where the nonlinear spring force f depends on the deflection u:

$$f(u) = \begin{cases} k\cdot u + N(u), & \text{for } u \geq 0 \\ 0, & \text{for } u < 0, \end{cases} \qquad (7)$$

or

$$EI\frac{d^4u}{dx^4} + k_pu = w(x) + k_pu - f(u) \equiv \phi. \qquad (8)$$

TABLE 1: Three cases of the exact solution.

Case	Exact solution $u(x)$
a	e^{-x^2}
b	$\text{Sin}\, x\cdot e^{-x^2}$
c	$\text{Sin}\, x\cdot e^{-x^2/4}$

In (8), k_pu is the pseudolinear spring force term and is finally compensated, so it does not affect the nonlinear solution.

Therefore, (4) shows that (8) must be equivalent to the following equation:

$$u(x) = \int_{-\infty}^{\infty} G(x,\xi;k_p)\cdot\phi(\xi)\,d\xi, \qquad (9)$$

where Green's function with pseudo linear spring coefficient can be expressed as

$$G(x,\xi;k_p) = \frac{\beta}{2k}e^{-\beta|\xi-x|/\sqrt{2}} \times \sin\left(\frac{\beta|\xi-x|}{\sqrt{2}} + \frac{\pi}{4}\right), \quad \beta = \sqrt[4]{k_p/EI}. \qquad (10)$$

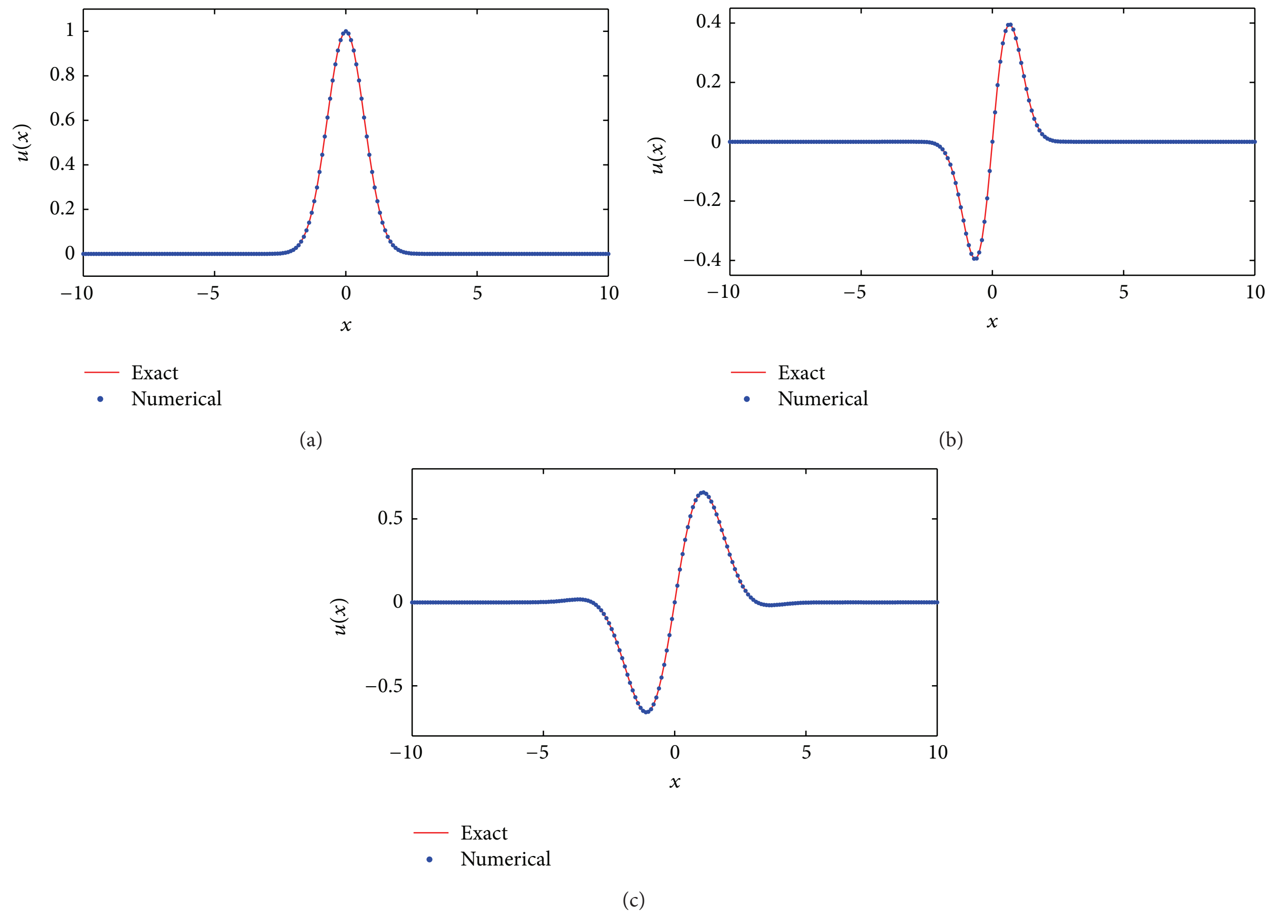

Figure 5: Numerical solutions compared to the exact solutions.

Regarding (8), the nonlinear relation for u can be derived as follows:

$$u(x) = \int_{-\infty}^{\infty} G(x, \xi; k_p) \cdot w(\xi)\, d\xi + \int_{-\infty}^{\infty} G(x, \xi; k_p) \cdot K[u(\xi)]\, d\xi, \tag{11}$$

where the function K can be written as

$$K(u) = k_p \cdot u - f(u). \tag{12}$$

Equation (11) is a nonlinear Fredholm integral equation of the second kind for u.

2.2. Iterative Procedure. From (11), the nonlinear iterative procedure is applied [21–24]

$$u_{n+1}(x) = \int_{-\infty}^{\infty} G(x, \xi; k_p) \cdot w(\xi)\, d\xi + \int_{-\infty}^{\infty} G(x, \xi; k_p) \cdot K[u_n(\xi)]\, d\xi, \tag{13}$$

where $K(u)$ satisfies (7) and (12).

To examine the nonlinear iterative procedure in (13) by simulation, it should be discretized as follows:

$$u_{n+1}(x) = \sum_{j=1}^{\mathrm{N}} W_j \left\{ G(x, \xi_j; k_p) \cdot w(\xi_j) + G(x, \xi; k_p) \cdot K[u_n(\xi_j)] \right\}, \quad j = 0, 1, 2, \ldots, N, \tag{14}$$

where W_j denotes the weights for the integration rule. The number N in the summation of (14) denotes the total segments of the interval $(-R, R)$, and R is a sufficiently large value satisfying (3).

3. Numerical Experiments

In this section, numerical experiments are performed to determine the validity of the iterative method. This study assumes a nonlinear spring force $f(u)$ in (7) and examines how accurate the solution converges to an exact solution. The convergence of the method is investigated with some external loading conditions.

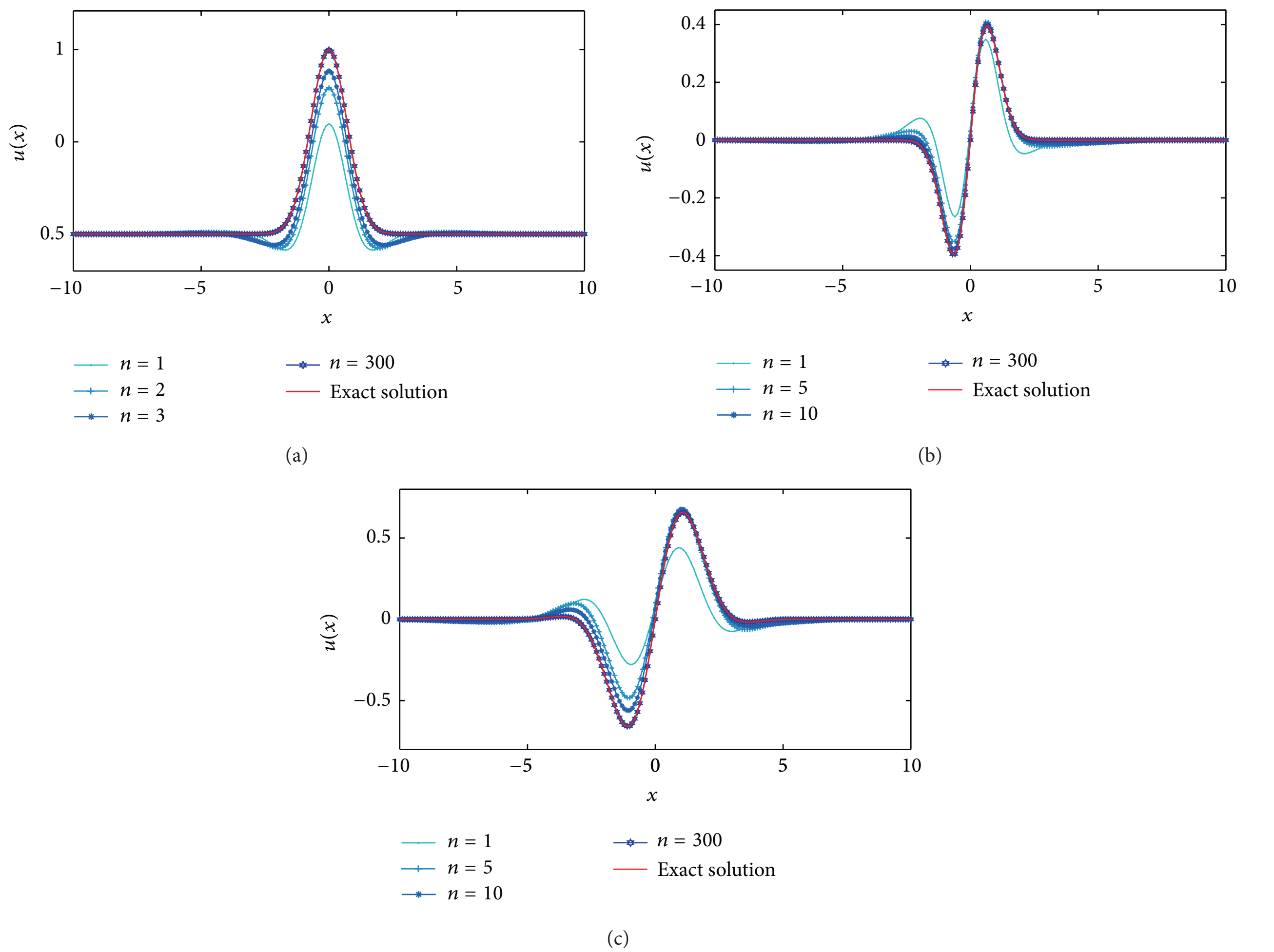

Figure 6: Convergence behaviors of the iterative solutions.

3.1. Nonlinear Spring Model. For simplicity, the present study considers an infinite beam on a nonlinear elastic foundation, whose spring force is derived as follows:

$$f(u) = \begin{cases} k \cdot u + \gamma \cdot u^3, & \text{for } u \geq 0 \\ 0, & \text{for } u < 0, \end{cases} \quad (15)$$

where $N(u)$ in (7) is chosen immediately as a cubic form

$$N(u) = \gamma \cdot u^3. \quad (16)$$

3.2. Comparison with Exact Solution. To determine if the iterative method converges to an exact solution, the 3 cases of exact solutions listed in Table 1 are first assumed and are illustrated in Figure 3. These cases are chosen to be infinitely differentiable and satisfy the conditions in (3). The following are assumed: $E = I = k = 1$, $k_p = 3$, and $\gamma = 0.2$, and the initial guess of the deflection is $u_0 = 0$. Simpson's integration rule is applied to the numerical integration. Three cases of external loads, shown in Figure 4 are obtained by substituting the exact solutions in Table 1 to (6). Under the loading condition, the nonlinear iterative method in Section 2 is applied to obtain the solution. Figure 5 compares the exact and numerical solutions, and Figure 6 shows the convergence behavior of the solutions in Figure 5. The errors of the solutions at the nth iteration are defined as follows:

$$\text{Error}(n) = \frac{\|u_{\text{exact}} - u_n\|_2}{\|u_{\text{exact}}\|_2}, \quad \text{where } \|z\|_2 \equiv \left(\sum_{i=1}^{N} |z_i|^2\right)^{1/2}. \quad (17)$$

Three cases of Error(n) are plotted as a function of the iteration number in Figure 7. The solutions converge at 200–300 iterations.

3.3. Convergence of the Procedure. The accuracy of the applied iterative method for the nonlinear spring model is proven in Section 3.2. In this section, 2 cases of loading conditions are investigated to show the convergence of the solutions.

The locally distributed rectangular-type loadings in Figure 8 are taken

$$w_{\text{normal (single)}}(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

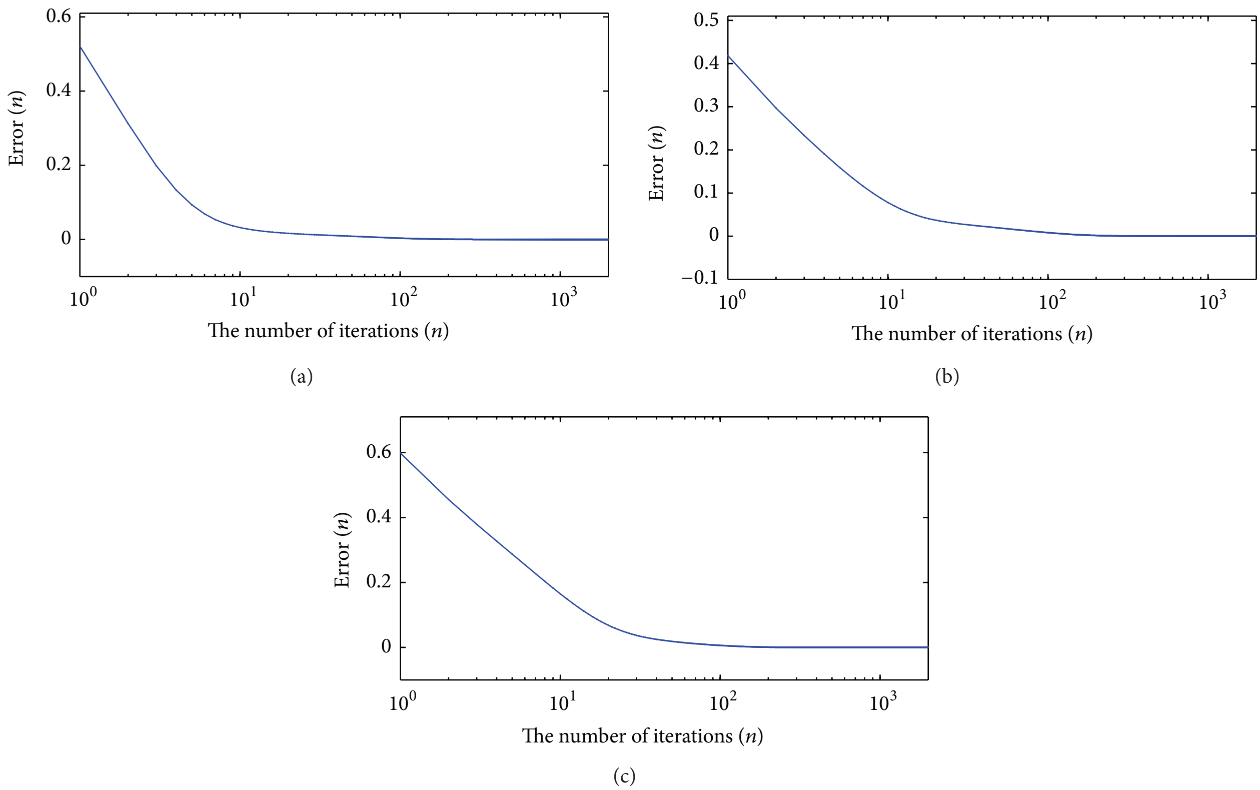

FIGURE 7: Errors between the exact solutions and iterative solutions.

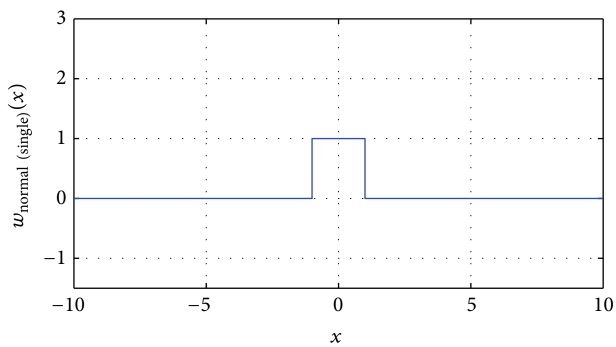

FIGURE 8: Applied loading.

Figure 9 presents the convergence behavior of the solutions according to the 3 cases of k_p: $E = I = k = 1$ and $\gamma = 0.2$. As specified in Section 2.1, k_p does not affect the converged iterative solutions, but the convergence characteristics changed. The solution converged to a steady state faster when k_p is small.

In Figure 10, two cases of converged solutions are compared: the numerical results using the one-way and two-way spring models. The loading condition is $w_{\text{normal (single)}}(x)$ in (18). They are different only when the beam deflected upward; that is, the deflections near $x = \pm 5$ are larger when the beam is separable (or free) from the foundation, whereas the one-way spring model is not. Of course, at $x = 0$, the solutions are equivalent because the physical system is the same in Figure 1 for $u > 0$. Therefore, Figure 10 shows the validity of the applied iterative method for the nonlinear spring model.

Another loading condition in Figure 11 is taken as follows:

$$w_{\text{normal}}(x) = \begin{cases} 1 & 4 \le |x| \le 5,\ 1 \le |x| \le 2 \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Other numerical experiments are also conducted according to the 3 cases of a linear spring coefficient k, whereas the properties ($E = I = k = 1$ and $\gamma = 0.2$) are fixed, and the solution converged (Figure 12). Figure 13 compares the solutions from the one-way and two-way spring models: $E = I = k = 1$, $\gamma = 0.2$, and $k_p = 3$.

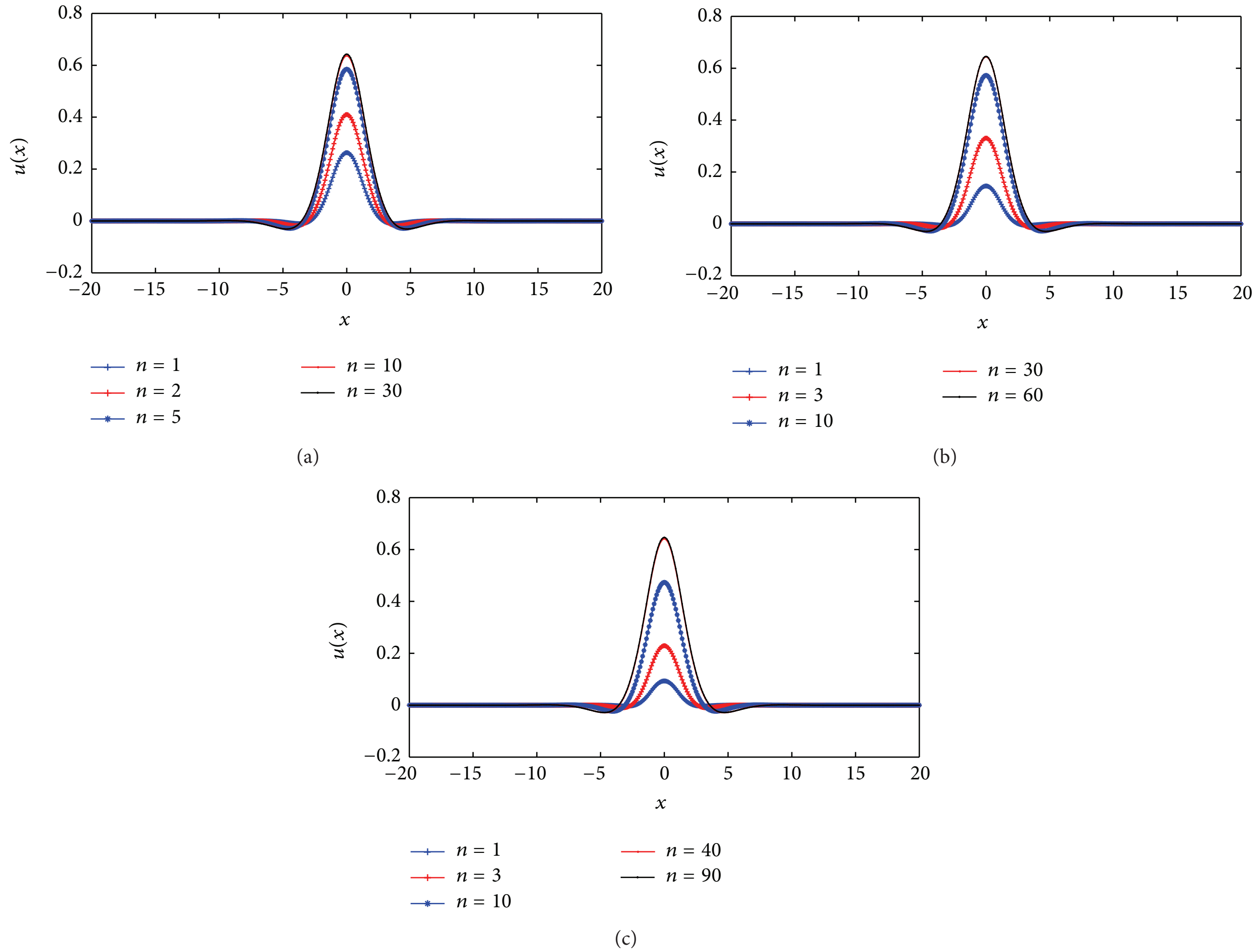

FIGURE 9: Convergence behavior of the applied iterative method $E = I = k = 1$, $\gamma = 0.2$: (a) $k_p = 3$, (b) $k_p = 6$, and (c) $k_p = 10$.

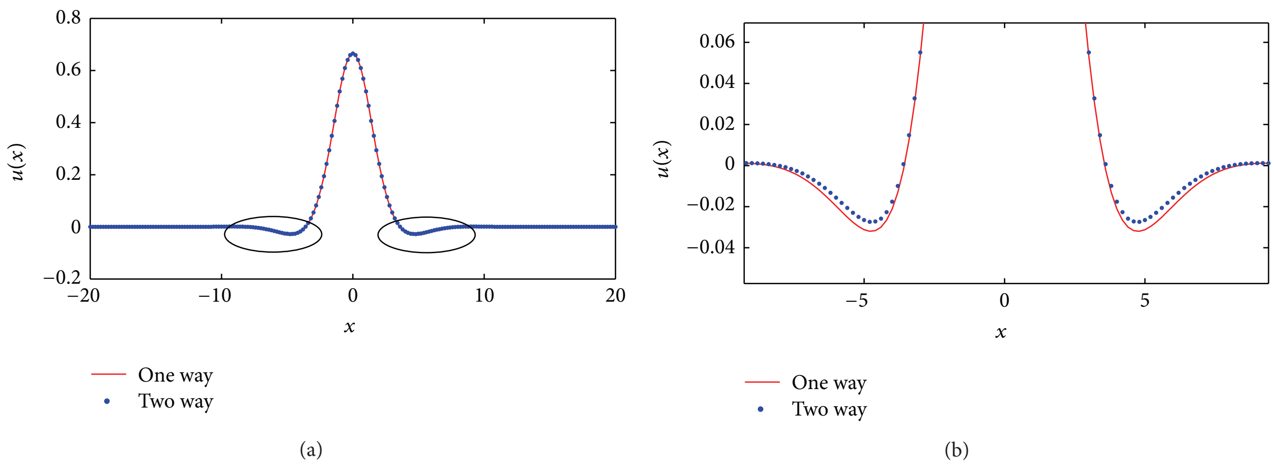

FIGURE 10: Validity of the solution.

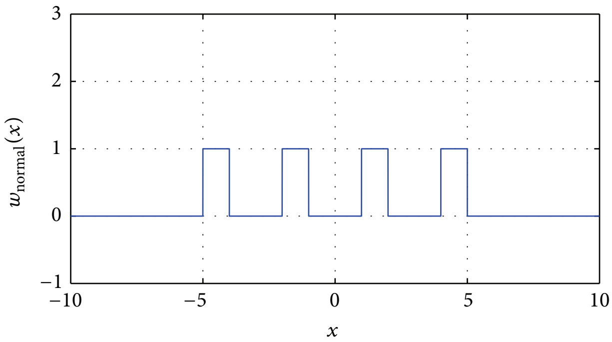

FIGURE 11: Applied loading conditions.

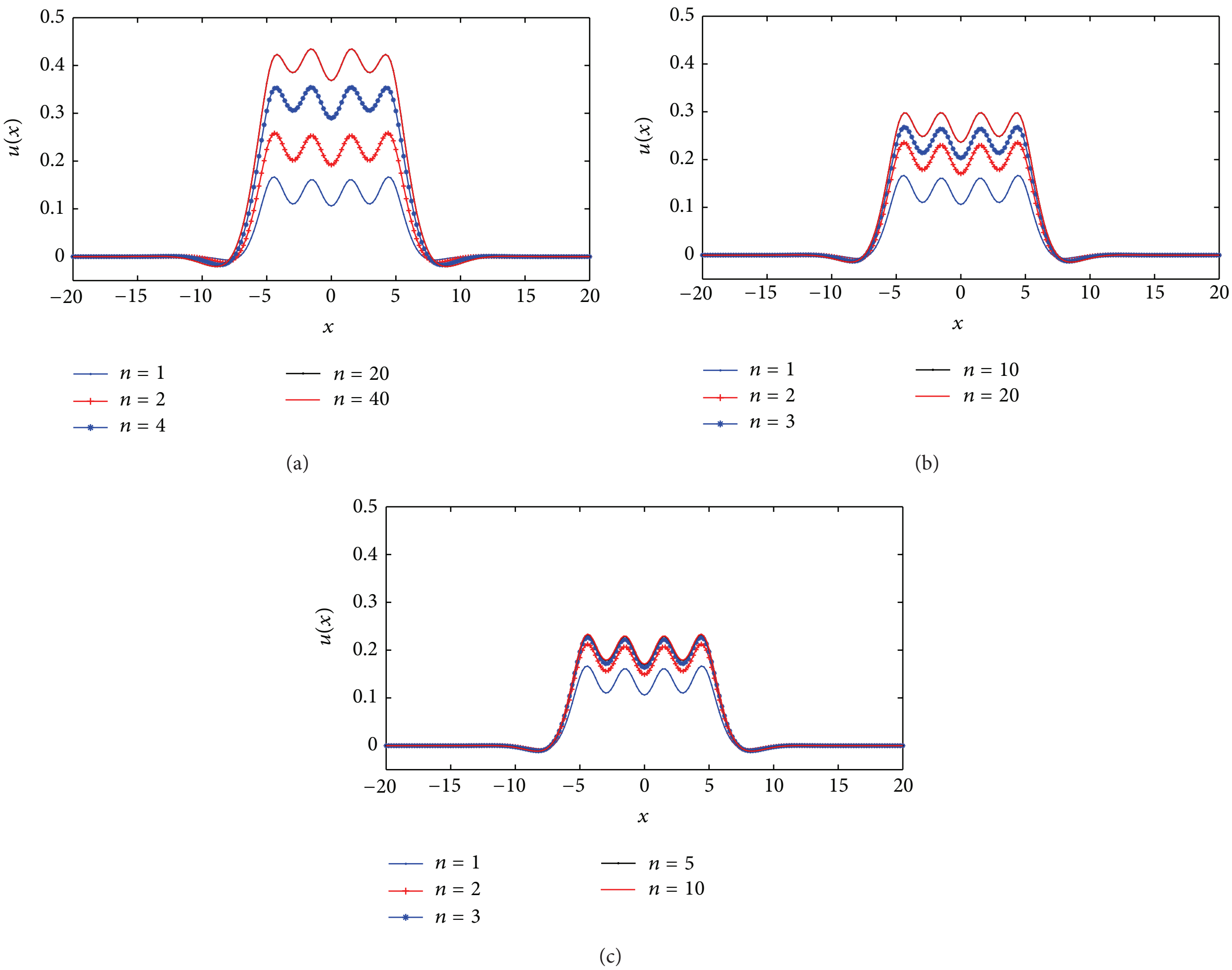

FIGURE 12: Convergence behavior of the solutions $E = I = 1$, $\gamma = 0.2$, and $k_p = 3$: (a) $k = 1$, (b) $k = 1.5$, and (c) $k = 2$.

Finally, the validity and accuracy of the applied iterative procedure are investigated. The nonlinear spring force is considered, and the iterative procedure is applied successfully for the solution. Convergence of the deflections according to the external loading conditions is observed for the effects of the pseudo k_p and real k.

4. Conclusion

In this work, we succeeded in applying the numerical method proposed by Jang et al. [21] to find the static deflection of an infinite beam on a full nonlinear elastic foundation. For that, one-way spring model is considered for the formulation

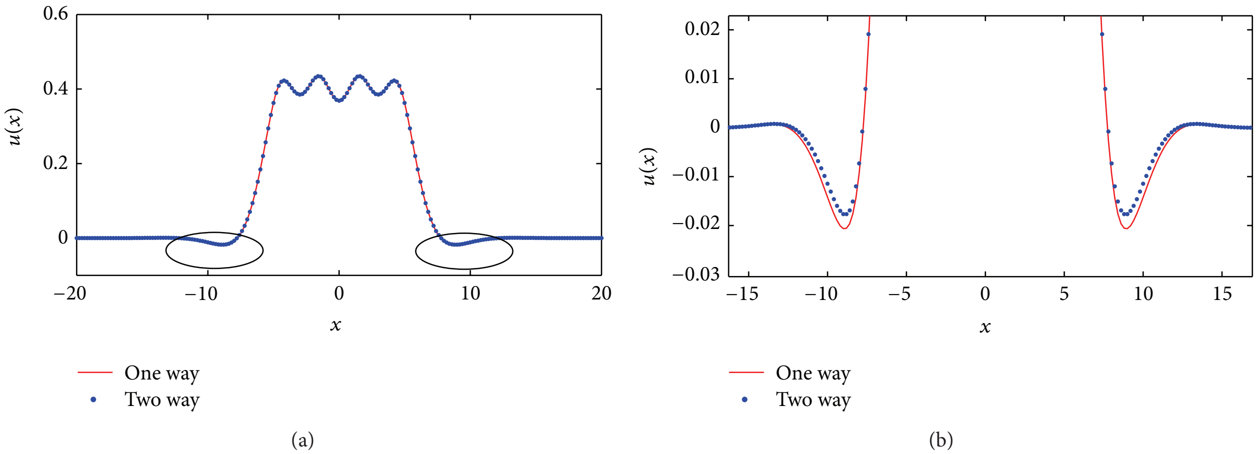

Figure 13: Validity of the solution in Figure 12(a).

of the nonlinear elastic one. Since the problem concerns one-way spring force, the governing equation for the static beam deflection is transformed as in (11). For the solution, an iterative procedure is applied for the calculation of the high nonlinearity. Some numerical experiments are carried out for showing the validity and the fast convergence of the applied numerical method. And we can also find that the results converge to the solutions fast for certain external loads.

Acknowledgment

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References

[1] M. Hetenyi, *Beams on Elastic Foundation*, The University of Michigan Press, Ann Arbor, Mich, USA, 1946.

[2] C. Miranda and K. Nair, "Finite beams on elastic foundation," *Journal of the Structural Division*, vol. 92, pp. 131–142, 1966.

[3] B. Y. Ting, "Finite beams on elastic foundation with restraints," *Journal of the Structural Division*, vol. 108, no. 3, pp. 611–621, 1982.

[4] S. Timoshenko, "Method of analysis of statistical and dynamical stress in rail," in *Proceeding of the International Congress of Applied Mechanics*, pp. 407–418, Zurich, Switzerland, 1926.

[5] J. T. Kenney, "Steady-state vibrations of beam on elastic foundation for moving load," *Journal of Applied Mechanics*, vol. 21, pp. 359–364, 1954.

[6] H. Saito and T. Murakami, "Vibrations of an infinite beam on an elastic foundation with consideration of mass of a foundation," *The Japan Society of Mechanical Engineers*, vol. 12, pp. 200–205, 1969.

[7] L. Frýba, "Infinitely long beam on elastic foundation under moving load," *Aplikace Matematiky*, vol. 2, pp. 105–132, 1957.

[8] F. W. Beaufait and P. W. Hoadley, "Analysis of elastic beams on nonlinear foundations," *Computers & Structures*, vol. 12, no. 5, pp. 669–676, 1980.

[9] C. Massalas, "Fundamental frequency of vibration of a beam on a non-linear elastic foundation," *Journal of Sound and Vibration*, vol. 54, no. 4, pp. 613–615, 1977.

[10] M. Lakshmanan, "Comment on the fundamental frequency of vibration of a beam on a non-linear elastic foundation," *Journal of Sound and Vibration*, vol. 58, no. 3, pp. 455–457, 1978.

[11] D. Hui, "Postbuckling behavior of infinite beams on elastic foundations using Koiter's improved theory," *International Journal of Non-Linear Mechanics*, vol. 23, no. 2, pp. 113–123, 1988.

[12] D. Z. Yankelevsky and M. Eisenberger, "Analysis of a beam column on elastic foundation," *Computers & Structures*, vol. 23, no. 3, pp. 351–356, 1986.

[13] M. Eisenberger and J. Clastornik, "Beams on variable two-parameter elastic foundation," *Journal of Engineering Mechanics*, vol. 113, no. 10, pp. 1454–1466, 1987.

[14] D. Karamanlidis and V. Prakash, "Exact transfer and stiffness matrices for a beam/column resting on a two-parameter foundation," *Computer Methods in Applied Mechanics and Engineering*, vol. 72, no. 1, pp. 77–89, 1989.

[15] S. N. Sirosh, A. Ghali, and A. G. Razaqpur, "A general finite element for beams or beam-columns with or without an elastic foundation," *International Journal for Numerical Methods in Engineering*, vol. 28, no. 5, pp. 1061–1076, 1989.

[16] K. P. Soldatos and A. P. S. Selvadurai, "Flexure of beams resting on hyperbolic elastic foundations," *International Journal of Solids & Structures*, vol. 21, no. 4, pp. 373–388, 1985.

[17] S. Y. Lee, H. Y. Ke, and Y. H. Kuo, "Exact static deflection of a non-uniform bernoulli-euler beam with general elastic end restraints," *Computers & Structures*, vol. 36, no. 1, pp. 91–97, 1990.

[18] S. Y. Lee and H. Y. Ke, "Free vibrations of a non-uniform beam with general elastically restrained boundary conditions," *Journal of Sound and Vibration*, vol. 136, no. 3, pp. 425–437, 1990.

[19] S. Y. Lee, H. Y. Ke, and Y. H. Kuo, "Analysis of non-uniform beam vibration," *Journal of Sound and Vibration*, vol. 142, no. 1, pp. 15–29, 1990.

[20] Y. H. Kuo and S. Y. Lee, "Deflection of nonuniform beams resting on a nonlinear elastic foundation," *Computers & Structures*, vol. 51, no. 5, pp. 513–519, 1994.

[21] T. S. Jang, H. S. Baek, and J. K. Paik, "A new method for the nonlinear deflection analysis of an infinite beam resting on a nonlinear elastic foundation," *International Journal of Non-Linear Mechanics*, vol. 46, no. 1, pp. 339–346, 2011.

[22] T. S. Jang and H. G. Sung, "A new semi-analytical method for the non-linear static analysis of an infinite beam on a non-linear

elastic foundation: a general approach to a variable beam cross-section," *International Journal of Non-Linear Mechanics*, vol. 47, pp. 132–139, 2012.

[23] S. W. Choi and T. S. Jang, "Existence and uniqueness of nonlinear deflections of an infinite beam resting on a non-uniform nonlinear elastic foundation," *Boundary Value Problems*, vol. 2012, article 5, 2012.

[24] T. S. Jang, "A new semi-analytical approach to large deflections of Bernoulli–Euler-v. Karman beams on a linear elastic foundation: nonlinear analysis of infinite beams," *International Journal of Mechanical Sciences*, vol. 66, pp. 22–32, 2013.

[25] M. D. Greenberg, *Foundations of Applied Mathematics*, Prentice-Hall, New Jersey, NJ, USA, 1978.