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Research Article

Existence Result for Impulsive Differential Equations with Integral Boundary Conditions

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We investigate the following differential equations: $-(y^{[1]}(x))' + q(x)y(x) = \lambda f(x, y(x))$, with impulsive and integral boundary conditions $-\Delta(y^{[1]}(x_i)) = I_i(y(x_i))$, $i = 1, 2, \ldots, m$, $y(0) - ay^{[1]}(0) = \int_0^\omega g_0(s)y(s)ds$, $y(\omega) - by^{[1]}(\omega) = \int_0^\omega g_1(s)y(s)ds$, where $y^{[1]}(x) = p(x)y'(x)$. The expression of Green's function and the existence of positive solution for the system are obtained. Upper and lower bounds for positive solutions are also given. When $p(t)$, $I(\cdot)$, $g_0(s)$, and $g_1(s)$ take different values, the system can be simplified to some forms which has been studied in the works by Guo and LakshmiKantham (1988), Guo et al. (1995), Boucherif (2009), He et al. (2011), and Atici and Guseinov (2001). Our discussion is based on the fixed point index theory in cones.

1. Introduction

The theory of impulsive differential equations in abstract spaces has become a new important branch and has developed rapidly (see [1–4]). As an important aspect, impulsive differential equations with boundary value problems have gained more attention. In recent years, experiments in a variety of different areas (especially in applied mathematics and physics) show that integral boundary conditions can represent the model more accurately. And researchers have obtained many good results in this field.

In this paper, we study the existence of positive solutions for the following system:

$$\begin{gathered} -\left(y^{[1]}(x)\right)' + q(x)\,y(x) = \lambda f(x, y(x)), \quad x \neq x_i,\ x \in J^-, \\ -\Delta\left(y^{[1]}(x_i)\right) = I_i\left(y(x_i)\right), \quad i = 1, 2, \ldots, m, \\ y(0) - ay^{[1]}(0) = \int_0^\omega g_0(s)\,y(s)\,ds, \\ y(\omega) - by^{[1]}(\omega) = \int_0^\omega g_1(s)\,y(s)\,ds, \end{gathered} \tag{1}$$

where $y^{[1]}(x) = p(x)y'(x)$, $J^- = J \setminus \{x_1, x_2, \ldots, x_m\}$, $J = [0, \omega]$, $0 < x_1 < x_2 < \cdots < x_m < \omega$, $f \in C(J \times R^+, R^+)$. $y(x)$, $y^{[1]}(x)$ are left continuous at $x = x_i$, $\Delta(y^{[1]}(x_i)) = y^{[1]}(x_i^+) - y^{[1]}(x_i^-)$. $I_i \in C(R^+, R^+)$. And $a > 0$, $b < 0$, $g_0, g_1 : [0, 1] \to [0, \infty)$ are continuous and positive functions.

When $p(t)$, $I(\cdot)$, $g_0(s)$, and $g_1(s)$ take different values, the system can be simplified to some forms which have been studied. For example, [5–10] discussed the existence of positive solution in case $p(t) = 1$.

Let $p(t) = 1$, $g_0, g_1 = 0$, [11, 12] investigated the system with only one impulse. Reference [13] studied the system when $I(\cdot) = 0$, $g_0, g_1 = 0$. Readers can read the papers in [13] for details.

Throughout the rest of the paper, we assume ω is a fixed positive number, and λ is a parameter. $p(x)$, $q(x)$ are real-valued measurable functions defined on J, and they satisfy the following condition:

(H1) $p(x) > 0$, $q(x) \geq 0$, $q(x) \not\equiv 0$ almost everywhere, and

$$\int_0^\omega \frac{1}{p(x)}dx < \infty, \qquad \int_0^\omega q(x)\,dx < \infty. \tag{2}$$

This paper aims to obtain the positive solution for (1). In Section 2, we introduce some lemmas and notations. In particular, the expression and some properties of Green's functions are investigated. After the preparatory work, we draw the main results in Section 3.

2. Preliminaries

Theorem 1 (Krasnoselskii's fixed point theorem). *Let E be a Banach space and $C \in E$. Assume Ω_1, Ω_2 are open sets in E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and $S : C\bigcap(\overline{\Omega}_2 \setminus \Omega_1) \to C$ be a completely continuous operator such that either*

(i) $\|s(y)\| \leq \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|s(y)\| \geq \|y\|$, $y \in C \cap \partial\Omega_2$; *or*

(ii) $\|s(y)\| \geq \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|s(y)\| \leq \|y\|$, $y \in C \cap \partial\Omega_2$.

Then S has a fixed point in $C\bigcap(\overline{\Omega}_2 \setminus \Omega_1)$.

Definition 2. For two differential functions y and z, we defined their Wronskian by

$$W_x(y,z) = y(x)z^{[1]}(x) - y^{[1]}(x)z(x) = p(x)\left[y(x)z'(x) - y'(x)z(x)\right]. \quad (3)$$

Consider the linear nonhomogeneous problem of the form

$$-\left(y^{[1]}(x)\right)' + q(x)y(x) = h(x), \quad x \in J. \quad (4)$$

Its corresponding homogeneous equation is

$$-\left(y^{[1]}(x)\right)' + q(x)y(x) = 0, \quad x \in J. \quad (5)$$

Lemma 3. *Suppose that y_1 and y_2 form a fundamental set of solutions for the homogeneous problem (5). Then the general solution of the nonhomogeneous problem (4) is given by*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int_0^x \frac{y_1(x)y_2(s) - y_1(s)y_2(x)}{w_s(y_1,y_2)} h(s)\,ds, \quad (6)$$

where c_1 and c_2 are arbitrary constants.

Proof. We just need to show that the function

$$z(x) = \int_0^x \frac{y_1(x)y_2(s) - y_1(s)y_2(x)}{w_s(y_1,y_2)} h(s)\,ds \quad (7)$$

is a particular solution of (4). From (7), we have for $x \in [0,\omega]$,

$$z'(x) = \int_0^x \frac{y_1'(x)y_2(s) - y_1(s)y_2'(x)}{w_s(y_1,y_2)} h(s)\,ds, \quad (8)$$

$$\left[p(x)z'(x)\right]' = -h(x) + q(x)z(x). \quad (9)$$

Besides, from (7) and (8), we have

$$z(0) = 0, \qquad z^{[1]}(0) = 0. \quad (10)$$

Thus, $z(x)$ satisfies (4). □

Consider the following boundary value problem with integral boundary conditions:

$$-\left(y^{[1]}(x)\right)' + q(x)y(x) = h(x), \quad x \in J,$$
$$y(0) - ay^{[1]}(0) = \int_0^\omega g_0(s)\sigma_0(s)\,ds, \quad (11)$$
$$y(\omega) - by^{[1]}(\omega) = \int_0^\omega g_1(s)\sigma_1(s)\,ds.$$

Denote by $u(x)$ and $v(x)$ the solutions of the homogenous equation (5) satisfying the initial conditions

$$u(0) = a, \qquad u^{[1]}(0) = 1,$$
$$v(\omega) = -b, \qquad v^{[1]}(\omega) = -1. \quad (12)$$

(H2) Let $x, s \in J$, denote a function

$$\phi(x,s) = \frac{u(x)}{u(\omega) - bu^{[1]}(\omega)} g_1(s) + \frac{v(x)}{v(0) - av^{[1]}(0)} g_0(s) \quad (13)$$

satisfies $0 \leq \phi(x,s) < 1/\omega$.

For convenience, we denote $m := \min\{\phi(x,s); x, s \in J\}$, $M := \max\{\phi(x,s); x, s \in J\}$.

Lemma 4. *Let $K(x,s)$ be a nonnegative continuous function defined for $-\infty < x_1 \leq x, s \leq x_2 < \infty$ and $\psi(x)$ a nonnegative integrable function on $[x_1,x_2]$. Then for arbitrary nonnegative continuous function $\varphi(x)$ defined on $[x_1,x_2]$, the Volterra integral equation*

$$y(x) = \varphi(x) + \int_{x_1}^x K(x,s)\psi(s)y(s)\,ds, \quad x_1 \leq x \leq x_2 \quad (14)$$

has a unique solution $y(x)$. Moreover, this solution is continuous and satisfied the inequality

$$y(x) \geq \varphi(x), \quad x_1 \leq x \leq x_2. \quad (15)$$

Proof. We solve (14) by the method of successive approximations setting

$$y_0(x) = \varphi(x),$$
$$y_n = \int_{x_1}^x K(x,s)\psi(s)y_{n-1}(s)\,ds, \quad n = 1,2,\ldots. \quad (16)$$

If the series $\sum_{n=0}^{\infty} y_n(x)$ converges uniformly with respect to $x \in [x_1,x_2]$, then its sum will be, obviously, a continuous solution of (14). To prove the uniform convergence of this series, we put

$$\max_{x_1 \leqslant x \leqslant x_2} \varphi(x) = c, \qquad \max_{x_1 \leqslant x,s \leqslant x_2} K(x,s) = c_1. \quad (17)$$

Then it is easy to get from (16) that

$$0 \leqslant y_n(x) \leqslant c\frac{c_1^n}{n!}\left[\int_{x_1}^x \psi(s)\,ds\right]^n, \quad n = 0,1,2,\ldots. \quad (18)$$

Hence it follows that (14) has a continuous solution

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{19}$$

and because $y_0 = \varphi(x)$, $y_n \geq 0$, $n = 1, 2, \ldots$, for this solution the inequality (15) holds. Uniqueness of the solution of (14) can be proved in a usual way. The proof is complete. □

Remark 5. Evidently, the statement of Lemma 4 is also valid for the Volterra equation of the form

$$y(x) = \varphi(x) + \int_x^{x_2} K(x,s)\psi(s) y(s)\, ds, \quad x_1 \leq x \leq x_2. \tag{20}$$

Lemma 6. *For the solution $y(x)$ of the BVP* (11), *the formula*

$$y(x) = w(x) + \int_0^{\omega} G(x,s) h(s)\, ds, \quad x \in J \tag{21}$$

holds, where

$$\begin{aligned} w(x) = {}& \frac{u(x)}{u(\omega) - bu^{[1]}(\omega)} \int_0^{\omega} g_1(s)\sigma_1(s)\, ds \\ &+ \frac{v(x)}{v(0) - av^{[1]}(0)} \int_0^{\omega} g_0(s)\sigma_0(s)\, ds, \end{aligned} \tag{22}$$

$$G(x,s) = -\frac{1}{w_s(u,v)} \begin{cases} u(s)v(x), & 0 \leq s \leq x \leq \omega, \\ u(x)v(s), & 0 \leq x \leq s \leq \omega. \end{cases}$$

Proof. By Lemma 3, the general solutions of the nonhomogeneous problem (4) has the form

$$\begin{aligned} y(x) = {}& c_1 u(x) + c_2 v(x) \\ &+ \int_0^x \frac{u(x)v(s) - u(s)v(x)}{W_s(u,v)} h(s)\, ds, \end{aligned} \tag{23}$$

where c_1 and c_2 are arbitrary constants. Now we try to choose the constants c_1 and c_2 so that the function $y(x)$ satisfies the boundary conditions of (11).

From (23), we have

$$\begin{aligned} y^{[1]}(x) = {}& c_1 u^{[1]}(x) + c_2 v^{[1]}(x) \\ &+ \int_0^x \frac{u^{[1]}(x)v(s) - u(s)v^{[1]}(x)}{W_s(u,v)} h(s)\, ds. \end{aligned} \tag{24}$$

Consequently,

$$\begin{aligned} y(0) &= c_1 a + c_2 v(0), \\ y^{[1]}(0) &= c_1 + c_2 v^{[1]}(0). \end{aligned} \tag{25}$$

Substituting these values of $y(0)$ and $y^{[1]}(0)$ into the first boundary condition of (11), we find

$$c_2 = \frac{1}{v(0) - av^{[1]}(0)} \int_0^{\omega} g_0(s)\sigma_0(s)\, ds. \tag{26}$$

Similarly from the second boundary condition of (11), we can find

$$\begin{aligned} c_1 = {}& \frac{1}{u(\omega) - bu^{[1]}(\omega)} \int_0^{\omega} g_1(s)\sigma_1(s)\, ds \\ &- \int_0^{\omega} \frac{v(s)}{W_s(u,v)} h(s)\, ds. \end{aligned} \tag{27}$$

Putting these values of c_1 and c_2 in (23), we get the formula (21), (22). □

Lemma 7. *Let condition (H1) hold. Then for the Wronskian of solution $u(x)$ and $v(x)$, the inequality $W_x(u,v) < 0$, $x \in J$ holds.*

Proof. Using the initial conditions (12), we can deduce from (5) for $u(x)$ and $v(x)$ the following equations:

$$\begin{aligned} u^{[1]}(x) &= 1 + \int_0^x q(s)u(s)\, ds, \\ u(x) &= a + \int_0^x \frac{1}{p(t)} dt \\ &\quad + \int_0^x \left[\int_s^x \frac{dt}{p(t)}\right] q(s)u(s)\, ds, \\ v^{[1]}(x) &= -1 - \int_x^{\omega} q(s)v(s)\, ds, \\ v(x) &= -b + \int_x^{\omega} \frac{1}{p(t)} dt \\ &\quad + \int_x^{\omega} \left[\int_x^s \frac{dt}{p(t)}\right] q(s)v(s)\, ds. \end{aligned} \tag{28}$$

From (28), by condition (H1) and Lemma 4, it follows that

$$\begin{aligned} u(x) &\geq a + \int_0^x \frac{dt}{p(t)} > 0, \quad u^{[1]}(x) \geq 1 > 0, \\ v(x) &\geq -b + \int_x^{\omega} \frac{dt}{p(t)} > 0, \quad v^{[1]}(x) \leq -1 < 0. \end{aligned} \tag{29}$$

Now from (3), we get $W_x(u,v) < 0$, $x \in J$. The proof is complete. □

From (21), (22), and Lemma 7, the following lemma follows.

Lemma 8. *Under condition (H1) the Green's function $G(x,s)$ of the BVP* (11) *is positive. That is, $G(x,s) > 0$ for $x, s \in J$.*

Let $C(J)$ denote the Banach of all continuous functions $y : I \to \mathbb{R}$ equipped with the form $\|y\| = \max\{|y(x)|; x \in J\}$, for any $y \in C(J)$. Denote $P = \{y \in C(J); y(x) \geqslant 0, y \in J\}$, then P is a positive cone in $C(J)$.

Let us set $A = \max_{0 \leqslant x,s \leqslant \omega} G(x,s)$, $B = \min_{0 \leqslant x,s \leqslant \omega} G(x,s)$, and by Lemma 8, obviously, $A > B > 0$, $x, s \in J$.

Define a mapping Φ in Banach space $C(J)$ by

$$(\Phi y)(x) = w(x) + \lambda \int_0^\omega G(x,s) f(s, y(s))\, ds + \sum_{i=0}^{m} G(x, x_i) I_i(y(x_i)), \quad x \in J, \tag{30}$$

where

$$w(x) = \frac{u(x)}{u(\omega) - bu^{[1]}(\omega)} \int_0^\omega g_1(s) y(s)\, ds + \frac{v(x)}{v(0) - av^{[1]}(0)} \int_0^\omega g_0(s) y(s)\, ds. \tag{31}$$

Lemma 9. *The fixed point of the mapping Φ is a solution of* (1).

Proof. Clearly, Φy is continuous in x for $x \in J$. For $x \neq x_k$,

$$(\Phi y)'(x) = w'(x) + \lambda \int_0^\omega \frac{\partial G}{\partial x} f(s, y(s))\, ds + \sum_{i=0}^{m} \frac{\partial G(x, x_i)}{\partial x} I_i(y(x_i)), \tag{32}$$

where

$$w'(x) = \frac{u'(x)}{u(\omega) - bu^{[1]}(\omega)} \int_0^\omega g_1(s) y(s)\, ds + \frac{v'(x)}{v(0) - av^{[1]}(0)} \int_0^\omega g_0(s) y(s)\, ds. \tag{33}$$

We have

$$(\Phi y)^{[1]}(x) = w^{[1]}(x) + \lambda \int_0^\omega p(x) \frac{\partial G}{\partial x} f(s, y(s))\, ds + \sum_{i=0}^{m} p(x) \frac{\partial G(x, x_i)}{\partial x} I_i(y(x_i)), \tag{34}$$

where

$$w^{[1]}(x) = \frac{u^{[1]}(x)}{u(\omega) - bu^{[1]}(\omega)} \int_0^\omega g_1(s) y(s)\, ds + \frac{v^{[1]}(x)}{v(0) - av^{[1]}(0)} \int_0^\omega g_0(s) y(s)\, ds. \tag{35}$$

We can easy get that

$$(\Phi y)(0) - a(\Phi y)^{[1]}(0) = \int_0^\omega g_0(s) y(s)\, ds,$$

$$(\Phi y)(\omega) - b(\Phi y)^{[1]}(\omega) = \int_0^\omega g_1(s) y(s)\, ds,$$

$$\begin{aligned}
\Delta(\Phi y)^{[1]}(x_k) &= p(x_k^+)(\Phi y)'(x_k^+) - p(x_k^-)(\Phi y)'(x_k^-) \\
&= p(x_k)\left[-\frac{u(x_k) v'(x_k)}{W_{x_k}(u,v)} + \frac{u'(x_k) v(x_k)}{W_{t_k}(u,v)}\right] \times I_k(y(x_k)) \\
&= -I_k(y(x_k)),
\end{aligned}$$

$$\begin{aligned}
\left(p(x)(\Phi y)'(x)\right)' &= \left[p(x) w'(x) + \lambda \int_0^\omega p(x) \frac{\partial G}{\partial x} f(s, y(s))\, ds + \sum_{i=0}^{m} p(x) \frac{\partial G(x, x_i)}{\partial x} I_i(y(x_i))\right]' \\
&= q(x) w(x) + \lambda q(x) \times \int_0^\omega G(x,s) f(s, y(s))\, ds - \lambda f(x, y(x)) + q(x) \times \sum_{i=0}^{m} G(x, x_i) I_i(y(x_i)) \\
&= q(x)(\Phi y)(x) - \lambda f(x, y(x)),
\end{aligned} \tag{36}$$

which implies that the fixed poind of Φ is a solution of (1). The proof is complete. □

Lemma 10. *Let $P_0 := \{y \in P; \min_{x \in J} y(x) \geq ((1 - M\omega)B/(1 - m\omega)A)\|y\|\}$, then P_0 is a cone.*

Proof. (i) For for all $y_1, y_2 \in P_0$ and for all $\alpha \geq 0$, $\beta \geq 0$, we have

$$\min(\alpha y_1) \geqslant \alpha \cdot \frac{(1 - M\omega) B}{(1 - m\omega) A} \|y_1\|, \quad \min(\beta y_2) \geqslant \beta \cdot \frac{(1 - M\omega) B}{(1 - m\omega) A} \|y_2\|. \tag{37}$$

Moreover

$$\min(\alpha y_1 + \beta y_2) \geqslant \frac{(1 - M\omega) B}{(1 - m\omega) A} (\alpha \|y_1\| + \beta \|y_2\|) \geqslant \frac{(1 - M\omega) B}{(1 - m\omega) A} \|\alpha y_1 + \beta y_2\|. \tag{38}$$

Thus $\alpha y_1 + \beta y_2 \in P_0$.

(ii) If $y \in P_0$ and $-y \in P_0$, we have

$$\min_{x \in J}(y(x)) \geqslant \frac{(1 - M\omega) B}{(1 - m\omega) A} \|y\|, \quad \min_{x \in J}(-y(x)) \geqslant \frac{(1 - M\omega) B}{(1 - m\omega) A} \|y\|. \tag{39}$$

It implies that $y = 0$. Hence P_0 is a cone. □

Defined a linear operator $A : C(J) \to C(J)$ by

$$(Ay)(x) = \int_0^{\omega} \phi(x,s)\, y(s)\, ds. \tag{40}$$

Then we have the following lemma.

Lemma 11. *If (H2) is satisfied, then*

(i) *A is a bounded linear operator, $A(P) \subset P$;*

(ii) *$(I - A)$ is invertible;*

(iii) *$\|(I - A)^{-1}\| \le 1/(1 - M\omega)$.*

Proof. (i)

$$\begin{aligned} A(\alpha y_1(x) + \beta y_2(x)) &= \int_0^{\omega} \phi(x,s)\left[\alpha y_1(s) + \beta y_2(s)\right] ds \\ &= \alpha (Ay_1)(x) + \beta (Ay_2)(x), \end{aligned} \tag{41}$$

for all $\alpha, \beta \in \mathbb{R}$, $y_1, y_2 \in C(J)$.

Using $\phi(x,s) \le M$, it is easy to see that $|(Ay)(t)| \le M\omega\|y\|$.

Let $y \in P$. Then $y(s) \ge 0$ for all $s \in J$. Since $\phi(t,s) \ge m \ge 0$, it follows that $(Ay)(x) \ge 0$ for each $x \in J$. So $A(P) \subset P$.

(ii) We want to show that $(I - A)$ is invertible, or equivalently 1 is not an eigenvalue of A.

Since $M < 1/\omega$, it follows from condition (H2) that $\|Ay\| \le M\omega\|y\| < \|y\|$.

So

$$\|A\| = \sup_{y \ne 0} \frac{\|Ay\|}{\|y\|} \leqslant M\omega < 1. \tag{42}$$

On the other hand, we suppose 1 is an eigenvalue of A, then there exists a $y \in C(J)$ such that $Ay = y$. Moreover, we can obtain that $\|Ay\|/\|y\| = 1$. So $\|A\| \geqslant 1$. Thus this assumption is false.

Conversely, 1 is not an eigenvalue of A. Equivalently, $(I - A)$ is invertible.

(iii) We use the theory of Fredholm integral equations to find the expression for $(I - A)^{-1}$.

Obviously, for each $x \in J$, $y(x) = (I - A)^{-1} z(x) \Leftrightarrow y(x) = z(x) + (Ay)(x)$.

By (40), we can get

$$y(x) = z(x) + \int_0^{\omega} \phi(x,s)\, y(s)\, ds. \tag{43}$$

The condition $M < 1/\omega$ implies that 1 is not an eigenvalue of the kernel $\phi(x,s)$. So (43) has a unique continuous solution y for every continuous function z.

By successive substitutions in (43), we obtain

$$y(x) = z(x) + \int_0^{\omega} R(x,s)\, z(s)\, ds, \tag{44}$$

where the resolvent kernel $R(x,s)$ is given by

$$R(x,s) = \sum_{j=1}^{\infty} \phi_j(x,s). \tag{45}$$

Here $\phi_j(x,s) = \int_0^{\omega} \phi(x,\tau)\phi_{j-1}(\tau,s) ds$, $j = 2, \ldots$ and $\phi_1(x,s) = \phi(x,s)$.

The series on the right in (45) is convergent because $|\phi(x,s)| \leqslant M < 1/\omega$.

It can be easily verified that $R(x,s) \le M/(1 - M\omega)$.

So we can get

$$(I - A)^{-1} z(x) = z(x) + \int_0^{\omega} R(x,s)\, z(s)\, ds. \tag{46}$$

Therefore

$$\begin{aligned} (I - A)^{-1} z(x) &\le z(x) + \frac{M}{1 - M\omega} \int_0^{\omega} z(s)\, ds \\ &\le \|z\| \left(1 + \frac{M\omega}{1 - M\omega}\right) = \frac{1}{1 - M\omega} \|z\|. \end{aligned} \tag{47}$$

So

$$\frac{\left\|(I - A)^{-1} z\right\|}{\|z\|} \le \frac{1}{1 - M\omega}. \tag{48}$$

Thus $\|(I - A)^{-1}\| \le 1/(1 - M\omega)$. This completes the proof of the lemma. □

Remark 12. Since $\phi(x,s) \ge m$ for each $(x,s) \in J$, it is easy to prove that $R(x,s) \ge m/(1 - m\omega)$.

3. Main Results

Consider the following boundary value problem (BVP) with impulses:

$$\begin{aligned} -\left(y^{[1]}(x)\right)' + q(x)\, y(x) &= \lambda f(x, y(x)), \\ & \qquad x \ne x_i,\ x \in J, \\ -\Delta\left(y^{[1]}(x_i)\right) &= I_i\left(y(x_i)\right), \quad i = 1, 2, \ldots, m, \\ y(0) - a y^{[1]}(0) &= \int_0^{\omega} g_0(s)\, y(s)\, ds, \\ y(\omega) - b y^{[1]}(\omega) &= \int_0^{\omega} g_1(s)\, y(s)\, ds. \end{aligned} \tag{49}$$

Denote a nonlinear operator $T : PC(J) \to PC(J)$ by

$$\begin{aligned} (Ty)(x) = \lambda &\int_0^{\omega} G(x,s)\, f(s, y(s))\, ds \\ &+ \sum_{i=0}^{m} G(x, x_i)\, I_i\left(y(x_i)\right). \end{aligned} \tag{50}$$

It is easy to see that solutions of (49) are solutions of the following equation:

$$y(x) = Ty(x) + Ay(x), \quad x \in J^{-1}. \tag{51}$$

According to Lemma 11, y is a solution of (51) if and only if it is a solution of

$$y(x) = (I - A)^{-1} T y(x). \tag{52}$$

It follows from (46) that y is a solution of (52) if and only if

$$y(x) = (Ty)(x) + \int_0^\omega R(x,s)(Ty)(s)\,ds. \tag{53}$$

So, the operator Φ can be written as

$$(\Phi y)(x) = (Ty)(x) + \int_0^\omega R(x,s)(Ty)(s)\,ds. \tag{54}$$

It satisfies the conditions of Theorem 1 with $E = C(J)$ and the cone $C = P_0$.

Let us list some marks and conditions for convenience.

The nonlinearity $f : J \times [0,\infty) \to [0,\infty)$ is continuous and satisfies the following.

(H3) There exist $L_1 > 0$ and $\alpha(x) \in P$, $r_1 \in \mathbb{R}$ with $r_1 \geqslant \sum_{i=0}^m I_i(y(x_i))/\lambda \int_0^\omega \alpha(s)ds$ such that

$$f(x,y) \leq \alpha(x)\left[y(1 - M\omega) - r_1\right] \tag{55}$$

for all $y \in (0, L_1]$, $x \in J$.

(H4) There exist $L_2 > L_1$ and $\beta(x) \in P$, $p_1 \in \mathbb{R}$ with $p_1 \leqslant \sum_{i=0}^m I_i(y(x_i))/\lambda \int_0^\omega \beta(s)ds$ such that

$$f(x,y) \geq \beta(x)\left[y(1 - m\omega) - p_1\right] \tag{56}$$

for all $y \in (L_2, \infty]$, $x \in J$.

Then, we can get the following theorem.

Theorem 13. *Assume (H1), (H2), (H3), and (H4) are satisfied. And*

$$(1 - m\omega) A^2 \int_0^\omega \alpha(s)\,ds \leqslant (1 - M\omega) B^2 \int_0^\omega \beta(s)\,ds, \tag{57}$$

then, if λ satisfies

$$\frac{(1 - m\omega) A}{(1 - M\omega) B^2 \int_0^\omega \beta(s)\,ds} \leqslant \lambda \leqslant \frac{1}{A \int_0^\omega \alpha(s)\,ds}. \tag{58}$$

The problem (49) *has at least one positive solution.*

Proof. First of all, we show that operator Φ is defined by (54) maps P_0 into itself. Let $y \in P_0$.

Then $(\Phi y)(x) \geq 0$ for all that $t \in J^{-1}$, and

$$\begin{aligned}(\Phi y)(x) \leqslant{}& \frac{\lambda A}{1 - M\omega} \int_0^\omega f(s, y(s))\,ds \\ &+ \frac{A}{1 - M\omega} \sum_{i=0}^m I_i(y(x_i)).\end{aligned} \tag{59}$$

Because from the formula (54), we have

$$\begin{aligned}(\Phi y)(x) &= (Ty)(x) + \int_0^\omega R(x,s)(Ty)(s)\,ds \\ &= \lambda \int_0^\omega G(x,s) f(s, y(s))\,ds \\ &\quad + \sum_{i=0}^m G(x, x_i) I_i(y(x_i)) \\ &\quad + \lambda \int_0^\omega R(x,s) \int_0^\omega G(x,\tau) f(\tau, y(\tau))\,d\tau\,ds \\ &\quad + \int_0^\omega R(x,s) \sum_{i=0}^m G(x, x_i) I_i(y(x_i))\,ds \\ &\leq \lambda \left(1 + \frac{M\omega}{1 - M\omega}\right) \int_0^\omega G(x,s) f(s, y(s))\,ds \\ &\quad + \sum_{i=0}^m G(x, x_i) I_i(y(x_i)) \\ &\quad + \frac{M\omega}{1 - M\omega} \sum_{i=0}^m G(x, x_i) I_i(y(x_i)) \\ &\leq \frac{\lambda A}{1 - M\omega} \int_0^\omega f(s, y(s))\,ds \\ &\quad + \frac{A}{1 - M\omega} \sum_{i=0}^m I_i(y(x_i)).\end{aligned} \tag{60}$$

Hence, inequality (59) is established.

This implies that

$$\|\Phi y\| \leqslant \frac{\lambda A}{1 - M\omega} \int_0^\omega f(s, y(s))\,ds + \frac{A}{1 - M\omega} \sum_{i=0}^m I_i(y(x_i)), \tag{61}$$

or equivalently

$$\int_0^\omega f(s, y(s))\,ds \geqslant \frac{1 - M\omega}{\lambda A} \|\Phi y\| - \frac{1}{\lambda} \sum_{i=0}^m I_i(y(x_i)). \tag{62}$$

On the other hand, it follows that

$$\begin{aligned}(\Phi y)(x) \geqslant{}& \frac{\lambda B}{1 - m\omega} \int_0^\omega f(s, y(s))\,ds \\ &+ \frac{B}{1 - m\omega} \sum_{i=0}^m I_i(y(x_i)).\end{aligned} \tag{63}$$

In fact, we have

$$\begin{aligned}(\Phi y)(x) &= \lambda \int_0^\omega G(x,s) f(s,y(s))\,ds \\ &\quad + \sum_{i=0}^m G(x,x_i) I_i(y(x_i)) \\ &\quad + \lambda \int_0^\omega R(x,s) \int_0^\omega G(x,\tau) f(\tau,y(\tau))\,d\tau\,ds \\ &\quad + \int_0^\omega R(x,s) \sum_{i=0}^m G(x,x_i) I_i(y(x_i))\,ds \\ &\geq \lambda \left(1 + \frac{m\omega}{1-m\omega}\right) \int_0^\omega G(x,s) f(s,y(s))\,ds \\ &\quad + \sum_{i=0}^m G(x,x_i) I_i(y(x_i)) \\ &\quad + \frac{m\omega}{1-m\omega} \sum_{i=0}^m G(x,x_i) I_i(y(x_i)) \\ &\geq \frac{\lambda B}{1-m\omega} \int_0^\omega f(s,y(s))\,ds + \frac{B}{1-m\omega} \sum_{i=0}^m I_i(y(x_i)).\end{aligned} \tag{64}$$

It follows from (62) that

$$\begin{aligned}(\Phi y)(x) &\geq \frac{\lambda B}{1-m\omega} \cdot \left[\frac{1-M\omega}{\lambda A}\|\Phi y\| - \frac{1}{\lambda}\sum_{i=0}^m I_i(y(x_i))\right] \\ &\quad + \frac{B}{1-m\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &= \frac{(1-M\omega)B}{(1-m\omega)A}\|\Phi y\| - \frac{B}{1-m\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &\quad + \frac{B}{1-m\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &= \frac{(1-M\omega)B}{(1-m\omega)A}\|\Phi y\|.\end{aligned} \tag{65}$$

So, we get

$$(\Phi y)(x) \geqslant \frac{(1-M\omega)B}{(1-m\omega)A}\|\Phi y\|. \tag{66}$$

This show that $\Phi y \in P_0$.

It is easy to see that Φ is the complete continuity.

We now proceed with the construction of the open sets Ω_1 and Ω_2.

First, let $y \in P_0$ with $\|y\| = L_1$. Inequality (59) implies

$$\begin{aligned}(\Phi y)(x) &\leq \frac{\lambda A}{1-M\omega}\int_0^\omega f(s,y(s))\,ds + \frac{A}{1-M\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &\leq \frac{\lambda A}{1-M\omega}\int_0^\omega \alpha(s)\left[y(s)(1-M\omega) - r_1\right]ds \\ &\quad + \frac{A}{1-M\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &= \lambda A \int_0^\omega \alpha(s) y(s)\,ds - \frac{\lambda A}{1-M\omega} r_1 \\ &\quad \times \int_0^\omega \alpha(s)\,ds + \frac{A}{1-M\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &= \lambda A \int_0^\omega \alpha(s) y(s)\,ds + \frac{A}{1-M\omega} \\ &\quad \times \left[\sum_{i=0}^m I_i(y(x_i)) - \lambda r_1 \int_0^\omega \alpha(s)\,ds\right].\end{aligned} \tag{67}$$

By condition (H3) and (58), we obtain

$$\begin{gathered}\sum_{i=0}^m I_i(y(x_i)) - \lambda r_1 \int_0^\omega \alpha(s)\,ds \leqslant 0, \\ \lambda A \int_0^\omega \alpha(s)\,ds \leqslant 1.\end{gathered} \tag{68}$$

So

$$(\Phi y)(x) \leqslant \lambda A \int_0^\omega \alpha(s)\,ds\,\|y\| \leqslant \|y\|. \tag{69}$$

Consequently, $\|\Phi y\| \leqslant \|y\|$.

Let $\Omega_1 := \{y \in C(J); \|y\| < L_1\}$. Then, we have $\|\Phi y\| \leqslant \|y\|$ for $y \in P_0 \cap \partial\Omega_1$.

Next, let $\widetilde{L_2} = \max\{2L_1, ((1-m\omega)A/(1-M\omega)B)L_2\}$ and set $\Omega_2 := \{y \in C(J); \|y\| < \widetilde{L_2}\}$.

For $y \in P_0$ with $\|y\| = \widetilde{L_2}$, we have

$$\begin{aligned}\min_{x\in J} y(x) &\geq \frac{(1-M\omega)B}{(1-m\omega)A}\|y\| = \frac{(1-M\omega)B}{(1-m\omega)A}\widetilde{L_2} \\ &\geq \frac{(1-M\omega)B}{(1-m\omega)A} \cdot \frac{(1-m\omega)A}{(1-M\omega)B} L_2 = L_2.\end{aligned} \tag{70}$$

It follows from (63) that

$$\begin{aligned}(\Phi y)(x) &\geq \frac{\lambda B}{1-m\omega}\int_0^\omega f(s,y(s))\,ds + \frac{B}{1-m\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &\geq \frac{\lambda B}{1-m\omega}\int_0^\omega \beta(s)\left[y(s)(1-m\omega) - p_1\right]ds \\ &\quad + \frac{B}{1-m\omega}\sum_{i=0}^m I_i(y(x_i))\end{aligned}$$

$$= \lambda B \int_0^\omega \beta(s) y(s) ds + \frac{B}{1 - m\omega} \times \left(\sum_{i=0}^m I_i(y(x_i)) - \lambda p_1 \int_0^\omega \beta(s) ds \right). \quad (71)$$

By condition (H4) and (58), we obtain

$$\sum_{i=0}^m I_i(y(x_i)) - \lambda p_1 \int_0^\omega \beta(s) ds \geqslant 0, \qquad \lambda B \geqslant \frac{(1 - m\omega) A}{(1 - M\omega) B \int_0^\omega \beta(s) ds}. \quad (72)$$

Since $y \in P_0$ we have $y(x) \geqslant ((1 - M\omega)B/(1 - m\omega)A)\|y\|$ for all $x \in J$. It follows from the above inequality that

$$(\Phi y)(x) \geqslant \frac{(1 - m\omega) A}{(1 - M\omega) B \int_0^\omega \beta(s) ds} \int_0^\omega \beta(s) ds \cdot \frac{(1 - M\omega) B}{(1 - m\omega) A} \|y\| = \|y\|. \quad (73)$$

Hence $\|\Phi y\| \geqslant \|y\|$ for $y \in P_0 \cap \partial\Omega_2$.

It follows from (i) of Theorem 1 that Φ has a fixed point in $P_0 \cap (\overline{\Omega}_2 \setminus \Omega_1)$, and this fixed point is a solution of (49).

This completes the proof. □

Next, with L_1 and L_2 as above, we assume that f satisfied the following.

(H5) There exist $\alpha^*(x) \in P$, $r_1^* \in \mathbb{R}$ with $r_1^* \leqslant \sum_{i=0}^m I_i(y(x_i))/\lambda \int_0^\omega \alpha^*(s) ds$ such that

$$f(x, y) \geq \alpha^*(x) [y(1 - m\omega) - r_1^*] \quad (74)$$

for all $y \in (0, L_1]$, $x \in J$.

(H6) There exist $\beta^*(x) \in P$, $p_1^* \in \mathbb{R}$ with $p_1^* \geqslant \sum_{i=0}^m I_i(y(x_i))/\lambda \int_0^\omega \beta^*(s) ds$ such that

$$f(x, y) \leq \beta^*(x) [y(1 - M\omega) - p_1^*] \quad (75)$$

for all $y \in (L_2, \infty]$, $x \in J$.

Theorem 14. *Assume (H1), (H2), (H5), and (H6) are satisfied. And*

$$(1 - m\omega) A^2 \int_0^\omega \beta^*(s) ds \leqslant (1 - M\omega) B^2 \int_0^\omega \alpha^*(s) ds, \quad (76)$$

then, if λ satisfies

$$\frac{(1 - m\omega) A}{(1 - M\omega) B^2 \int_0^\omega \alpha^*(s) ds} \leqslant \lambda \leqslant \frac{1}{A \int_0^\omega \beta^*(s) ds}. \quad (77)$$

The problem (49) *has at least one positive solution.*

Proof. Let Φ be a completely continuous operator defined by (54). Then Φ maps the cone P_0 into itself.

First, let $y \in P_0$ with $\|y\| = L_1$. Inequality (63) implies

$$\begin{aligned} (\Phi y)(x) &\geq \frac{\lambda B}{1 - m\omega} \int_0^\omega f(s, y(s)) ds + \frac{B}{1 - m\omega} \sum_{i=0}^m I_i(y(x_i)) \\ &\geq \frac{\lambda B}{1 - m\omega} \int_0^\omega \alpha^*(s) [y(s)(1 - m\omega) - r_1^*] ds + \frac{B}{1 - m\omega} \sum_{i=0}^m I_i(y(x_i)) \\ &= \lambda B \int_0^\omega \alpha^*(s) y(s) ds + \frac{B}{1 - m\omega} \times \left(\sum_{i=0}^m I_i(y(x_i)) - \lambda r_1^* \int_0^\omega \alpha^*(s) ds \right). \end{aligned} \quad (78)$$

By condition (H5) and (77), we obtain

$$\sum_{i=0}^m I_i(y(x_i)) - \lambda r_1^* \int_0^\omega \alpha^*(s) ds \geqslant 0, \qquad \lambda B \geqslant \frac{(1 - m\omega) A}{(1 - M\omega) B \int_0^\omega \alpha^*(s) ds}. \quad (79)$$

Hence

$$(\Phi y)(x) \geqslant \frac{(1 - m\omega) A}{(1 - M\omega) B \int_0^\omega \alpha^*(s) ds} \int_0^\omega \alpha^*(s) y(s) ds. \quad (80)$$

Since $y \in P_0$, we have $y(x) \geqslant ((1 - M\omega)B/(1 - m\omega)A)\|y\|$ for all $x \in J$. It follows from the above inequality that

$$(\Phi y)(x) \geqslant \frac{(1 - m\omega) A}{(1 - M\omega) B \int_0^\omega \alpha^*(s) ds} \int_0^\omega \alpha^*(s) ds \cdot \frac{(1 - M\omega) B}{(1 - m\omega) A} \|y\| = \|y\|. \quad (81)$$

Let $\Omega_1 := \{y \in C(J); \|y\| < L_1\}$. Then, we have $\|\Phi y\| \geqslant \|y\|$ for $y \in P_0 \cap \partial\Omega_1$.

Next, let $\widetilde{L_2} = \max\{2L_1, ((1 - m\omega)A/(1 - M\omega)B)L_2\}$ and set $\Omega_2 := \{y \in C(J); \|y\| < \widetilde{L_2}\}$.

Then for $y \in P_0$ with $\|y\| = \widetilde{L_2}$ for all $x \in J$, we have $\min_{x \in J} y(x) \geqslant L_2$. Inequality (59) implies

$$(\Phi y)(x) \leq \frac{\lambda A}{1 - M\omega} \int_0^\omega f(s, y(s)) ds + \frac{A}{1 - M\omega} \sum_{i=0}^m I_i(y(x_i))$$

$$\leq \frac{\lambda A}{1-M\omega}\int_0^{\omega}\beta^*(s)\left[y(s)(1-M\omega)-p_1^*\right]ds$$
$$+\frac{A}{1-M\omega}\sum_{i=0}^{m} I_i(y(x_i))$$
$$=\lambda A\int_0^{\omega}\beta^*(s)\,y(s)\,ds$$
$$+\frac{A}{1-M\omega}\left[\sum_{i=0}^{m} I_i(y(x_i))-\lambda p_1^*\int_0^{\omega}\beta^*(s)\,ds\right]. \quad (82)$$

By condition (H6) and (77), we obtain

$$\sum_{i=0}^{m} I_i(y(x_i))-\lambda p_1^*\int_0^{\omega}\beta^*(s)\,ds\leqslant 0,$$
$$\lambda A\int_0^{\omega}\beta^*(s)\,ds\leqslant 1. \quad (83)$$

So

$$(\Phi y)(x)\leqslant \lambda A\int_0^{\omega}\beta^*(s)\,ds\,\|y\|\leqslant\|y\|\leqslant 1. \quad (84)$$

Therefore $\|\Phi y\|\leqslant\|y\|$ with $\|y\|=\widetilde{L_2}$.

Then, we have $\|\Phi y\|\leqslant\|y\|$ for $y\in P_0\cap\partial\Omega_2$.

We see the case (ii) of Theorem 1 is met. It follows that Φ has a fixed point in $P_0\cap(\overline{\Omega}_2\setminus\Omega_1)$, and this fixed point is a solution of (49).

This completes the proof. □

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